ARITHMETIC GEOMETRY, NUMBER THOERY, AND COMPUTATION PROBLEM SESSION (AUGUST 20, 2018)

Problem 1 (Jordan Ellenberg). Compute the class of the *Ceresa cycle* (also known as Gross–Schoen class) among the curves in the database of nonhyperelliptic genus 3 curves computed by Andrew Sutherland (see arXiv:1806.06289).

To define the Ceresa cycle, choose a base point $P_0 \in X(K)$ on the curve X over K; map $X \to \mathrm{Jac}(X)$ by the Abel–Jacobi map $P \mapsto [P - P_0]$ or by its negative $[P_0 - P]$; the difference of these two images is homologically trivial, and so gives you a class. It shows up in many places, including recent work of Darmon–Rotger–Sols (see MR3074917 or this preprint)

In terms of Galois representations, it is a cycle in $H^1(\mathrm{Gal}_{\mathbb{Q}}, \bigwedge^3 H_1(X, \mathbb{Z}_\ell)(-1)/H_1(X, \mathbb{Z}_\ell))$; written in terms of Hodge theory, it is an iterated integral. This cycle is always trivial for hyperelliptic curves. Also see

`https://mathoverflow.net/questions/179003/`
`non-hyperelliptic-families-of-curves-with-trivial-ceresa-class-or-gross-schoen`

Problem 2 (Michael Stoll). Let $F \in \mathbb{Q}[x, y, z]$ be irreducible and let $X \subseteq \mathbb{A}^3$ be the (possibly singular) surface defined by F. Give an algorithm to detect if F is rational or unirational, or if F is of general type. (There is a Magma function that implicitly but not explicitly requires X to satisfy some assumptions, including that X is smooth.)

Problem 3 (Zeb Brady). For all $k \in \mathbb{Z}$, does there exist $x, y, z \in \mathbb{Z}$ such that $x^4 + y^4 - z^3 = k$? Over $\mathbb{Q}$, its projectivization is a K3 surface. Is there an algorithm that for each k computes whether or not there is a Brauer–Manin obstruction? Is there a k where there is no solution (maybe $k = 4$?)? Also see

`https://mathoverflow.net/questions/176314/`
`are-there-any-integers-which-cant-be-written-as-a-sum-of-two-fourth-powers-minu`

What other techniques from the resolution of Diophantine equations can be applied?

Problem 4 (Vladimir Dokchitser). Formulate an appropriate generalization of the Birch–Swinnerton-Dyer conjecture for $\tau(\chi)L(E, \chi, 1)/\Omega_+$ where E is an elliptic curve over $\mathbb{Q}$ (or over a number field) and χ is a 1-dimensional character of order > 2.

For example (due to Hanneke Wiersema), consider the cubic character χ of conductor 31 and the elliptic curves

$$\begin{aligned} E_1 &\colon y^2 + xy + y = x^3 - 55x + 154, \\ E_2 &\colon y^2 + xy + y = x^3 + 5x - 56. \end{aligned}$$

The conductors are $\mathrm{cond}(E_1) = \mathrm{cond}(E_2) = 8138 = 2 \cdot 13 \cdot 313$; their Cremona labels are `8138b1` and `8138c1`, respectively. For $i = 1, 2$ we have $E_i(K) = \{O\}$ trivial, $Ш(E_i/\mathbb{Q})$ and $Ш(E_i/K)$ trivial (assunming BSD), and trivial Tamagawa numbers everywhere (both over $\mathbb{Q}$ and over K); they also

Date: 20 August 2018.

have the same number of real components and the same number of points over $\mathbb{F}_3$. We compute $L(E_1, \chi, 1)\tau(\chi)/\Omega_+ = 1$ but $L(E_2, \chi, 1)\tau(\chi)/\Omega_+ = -1$. Explain the sign.

What other arithmetic invariants could be used to distinguish these two cases?

Is this related to Stark's conjecture? In Stark's conjecture, the sign is not specified. (Can it be specified?)

Problem 5 (David Roe). For $p > 2$, we have a presentation of $\mathrm{Gal}(\overline{\mathbb{Q}_p} \mid \mathbb{Q}_p)$ with generators σ, τ, x_0, x_1 as a profinite topological group with x_0, x_1 pro-p, the relation $\tau^\sigma = \tau^p$, and one other horrible relation.

Is there any connection between these relations and the ramification filtration? For example, if you fix a piece in the filtration, there is a quotient of the group for this piece, can you explicitly describe this quotient in terms of the generators and relations. (Maybe not? Maybe the presentation is not well-adapted to the filtration.)

The equicharacteristic case would still be interesting: there would still be an explicit description in terms of the profinite group.

Is the fact that this Galois group is the fundamental group of a perfectoid space helpful?

Problem 6 (Michael Seaman). Let K be an imaginary quadratic field, let $\psi\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ be a Hecke character of finite order and let

$$L(\psi, s) = \prod_{p} (1 - \psi(p)p^{-s})^{-1}$$

be its L-function. Let $L = \prod_{\sigma \in G} L(\psi^\sigma, s)$ be the product over Galois-conjugate characters, with a suitable change of variables made to yield an L-function over $\mathbb{Q}$. Find an explicit (CM) abelian variety A with $L(A, s) = L$. For example, we could take Hecke characters attached to Gauss sums or more generally Jacobi sums.

Some matches might be found by computing Euler factors and searching for matches in the LMFDB.

Problem 7 (Lian Duan). Let $E\colon y^2 = x^3 + A(t)x + B(t)$ be an elliptic curve over $k(t)$ (thought of as an elliptic surface over k), with k an algebraically closed field. Maybe take k to be characteristic zero. Let $E^{D(t)}\colon D(t)y^2 = x^3 + A(t)x + B(t)$ for $D(t) \in k(t)$. Does there exist a linear $D \in \mathbb{Q}[t]$ such that $\operatorname{rk} E^D(k(t)) > 0$?

Perhaps study the change of the root numbers of the twists?

There is an upper bound on the rank coming from the Picard number. There may be examples where there is sufficiently bad reduction at ∞ for which twisting only moves this bad reduction around, and then the answer might be no?

Problem 8 (Alex Bartel). Let E be a semistable elliptic curve over $\mathbb{Q}$ such that $\operatorname{rk} E(\mathbb{Q}) > 0$ and for an odd prime p that $E(\mathbb{Q})[p] = 0$ and $Ш(E)[p] = 0$. Then (it is a theorem that) for all $F \supseteq \mathbb{Q}$ Galois extensions with $\mathrm{Gal}(F \mid \mathbb{Q}) \simeq C_p \times C_p$, we have

$$\# \operatorname{Sel}_p(E/F) > \# \operatorname{Sel}_p(E/\mathbb{Q}).$$

The first few curves satisfying the hypotheses for all odd primes p are the elliptic curves with Cremona labels are `91b3`, `123a2`, `141a1`, `142a1`.

The conjectures of Mazur–Rubin imply that, if $p > 5$, then for E satisfying the hypotheses of the theorem and for all but finitely many F as above, $Ш(E/F)[p] \neq 0$. Can this be shown unconditionally, e.g., by using visibility of $Ш$, or by computing ℓ-Selmer groups for some $\ell \neq p$?

Problem 9 (Anwesh Ray). Let $f \in S_2(\Gamma_0(N))$ be a weight 2 eigenform of level N. Let $p \geq 3$ be prime with $\gcd(p, N) = 1$. Then we have an attached Galois representation $\rho_f\colon \mathrm{Gal}_{\mathbb{Q},S} \to GL_2(\mathcal{O}_K)$ unramified outside pN and where $\mathcal{O}_K$ is the ring of Hecke coefficients. Consider the reduction $\overline{\rho}_f\colon \mathrm{Gal}_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_q)$. Suppose that f is supersingular, i.e., $v_p(a_p(f)) \geq 1$.

Let

$$\mathcal{F} := \{g \in S_2(\Gamma_0(M)) \text{ eigenform, supersingular} : \gcd(p, M) = 1, \overline{\rho}_g \simeq \overline{\rho}_f\}.$$

Study the slopes

$$\mathrm{Slopes}(\mathcal{F}) = \{v_p(a_p(g)) : g \in \mathcal{F}\}.$$

We believe that $\mathbb{Z}_{\geq 1} \subseteq \mathrm{Slopes}(\mathcal{F})$.

Is there an algorithm that will (eventually) compute this set of slopes?