SUPERSINGULAR MAIN CONJECTURES, SYLVESTER’S CONJECTURE
AND GOLDFELD’S CONJECTURE

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ABSTRACT. We formulate and prove a Rubin-type main conjecture for imaginary quadratic fields $K$ in which a prime $p$ is inert or ramified, relating the characteristic ideal of a certain $\Lambda$-torsion quotient of the Galois group of the maximal abelian pro-$p$-extension of the $\mathbb{Z}_p^{\oplus 2}$-extension of $K$ unramified outside $p$, to a supersingular Katz-type anticyclotomic $p$-adic $L$-function. A key component of this work is the development of a new interplay between Iwasawa theory and relative $p$-adic Hodge theory, in order to extend the method of Coleman power series and construct measures from norm-compatible systems of local units in the height 2 setting. As an application, we prove rank 0 and 1 converse theorems for supersingular CM elliptic curves $E/\mathbb{Q}$: “for $r = 0, 1$, if $\text{corank}_{\mathbb{Z}_p}\text{Sel}_p^{\infty}(E/\mathbb{Q}) = r$ then $\text{ord}_{s=1}L(E/\mathbb{Q}, s) = r$”. We hence get results on two classical problems of arithmetic. First, we prove the 1879 conjecture of Sylvester on primes expressible as the sum of two rational cubes: “If $p$ is a prime with $p \equiv 4, 7, 8 \pmod{9}$, then there exist $x, y \in \mathbb{Q}$ such that $x^3 + y^3 = p$”. Moreover, we show that the rank part of BSD holds for these elliptic curves. Second, by combining our $p$-converse theorem with the Selmer distribution results of Smith, we prove that 100% of squarefree integers congruent to 5, 6, 7 (mod 8) are congruent numbers, thus settling the congruent number problem in 100% of cases, and verifying the rank part of BSD and Goldfeld’s celebrated conjecture on quadratic twists for the congruent number family.

1. INTRODUCTION

Certain special cases of the Birch and Swinnerton-Dyer conjecture have long been known to have connections with classical problems in number theory. For example, the rank part of the conjecture for the cubic twists family $x^3 + y^3 = p$, $p$ a prime which is 4, 7, 8 (mod 9), is Sylvester’s conjecture on primes expressible as the sum of two rational cubes. These questions often involve “exceptional cases” of elliptic curves with CM, which are of supersingular bad reduction at primes for which it is natural to study descent. Hence these curves less amenable to classical methods of Iwasawa theory.

In this article, we prove converse theorems to the well-known theorem of Gross-Zagier and Kolyvagin in such exceptional cases, through establishing a suitable version of a $\Lambda$-adic version of the Birch and Swinnerton-Dyer conjecture, often known as a “Heegner point Main Conjecture” (see Section 6.1). Converse theorems in this vein were first developed by Skinner [55], and are also consequences of Zhang’s work on Kolyvagin’s conjecture [67], though the big Galois image hypotheses of both of these works rule out the CM case. In [3], the ordinary CM case was treated when $p \geq 5$. A special case of our supersingular CM converse theorem implies Sylvester’s conjecture (Theorem 7.4), while another special case, combined with work of Smith [57], shows that 100% of squarefree integers congruent to 5, 6, 7 (mod 8) are congruent numbers and establishes Goldfeld’s conjecture on quadratic twists [22] for the congruent number family. The key step for establishing this Heegner point Main Conjecture (a “supersingular $GL_2$ main conjecture”) is to reduce it to a supersingular analogue of a Rubin-type Main Conjecture for imaginary quadratic fields $K$ involving elliptic units (a “supersingular $GL_1$ main conjecture”), and an essential for this reduction is the $p$-adic Waldspurger formula established by the author in [34] Chapter 9 (see also [33], and for the $p$ split in $K$ case see [3], [5], [6], and [37]). Proving the latter Main Conjecture in this supersingular setting (i.e. when $p$ is inert or ramified in $K$) involves overcoming several fundamental obstacles,
among which include identifying the correct torsion $\Lambda$-modules for which to formulate the main conjecture, and constructing the appropriate “Katz-type” $p$-adic $L$-function.

Substantial progress toward the latter step was already made by the author in [33] and [34] and, in which supersingular Rankin-Selberg and Katz-type $p$-adic $L$-functions in the relevant framework were introduced. Among the various results toward establishing the Main Conjecture in Section 4.12 we show that the Katz-type $p$-adic $L$-function constructed in loc. cit. is an element of the relevant Iwasawa algebra, and in fact is the restriction of a anticyclotomic Katz-type $p$-adic measure. In order to construct the anticyclotomic Katz-type $p$-adic measure, we adapt the method of Coleman power series in the height 2 setting, combining it with $p$-adic Hodge theory in a fundamentally new way in order to find the “correct” power series ring for constructing appropriate measures. Perhaps the largest conceptual leap made in this step is to work with coefficients in period rings appearing from relative $p$-adic Hodge theory.

Finally, we point out that actually proving the Rubin-type Main Conjecture involves studying the usual fundamental exact sequence from class field theory, and reducing it to the Elliptic units Main Conjecture, which is the assertion of a certain equality of characteristic ideals involving elliptic units and the maximal unramified pro-$p$ abelian extension of the $\mathbb{Z}_p^{\oplus 2}$-extension of $K$. Rubin’s strategy for proving the Elliptic units Main Conjecture involves the Euler system of elliptic units, which however entails introducing certain assumptions on the prime $p$ inconvenient for our applications. In order to remove these assumptions, we instead show that the $\mu$-invariant of the restriction of our Katz-type $p$-adic $L$-function to the cyclotomic line is 0, which by an additional argument and work of Johnson-Leung-Kings [28] on equivariant main conjectures, gives the Elliptic units Main Conjecture.

1.1. Outline of the paper, main results and proofs. We give a brief overview of the paper.

The first half of the paper concerns the $GL_1/K$ setting, where $K$ is an imaginary quadratic field. Let $L = K(f)$ for some ideal $(f,p) = 1$, and let $L_\infty/L$ denote the compositum of $L$ and the $\mathbb{Z}_p^{\oplus 2}$-extension $K_\infty$ of $K$. Fix an embedding $\overline{Q} \subset \overline{Q}_p$, and for any algebraic extension $F$ of $Q$, let $F_p$ denote the induced $p$-adic completion of $F$. Let $\Lambda = \mathbb{Z}_p[\text{Gal}(L_\infty/K)]$, and for any $\mathbb{Z}_p$-algebra, let $\Lambda_R = \Lambda \otimes_{\mathbb{Z}_p} R$. From class field theory, one gets an exact sequence of the form

$$0 \to \mathcal{E} \to \mathcal{U} \xrightarrow{\text{rec}} \mathcal{X} \to \mathcal{Y} \to 0,$$

where $\mathcal{E}$ is the usual tower of global units associated with $L_\infty/L$, $\mathcal{U}$ is the tower of principal semilocal units associated with $L_p,\infty/L_p$, $\mathcal{X}$ is the Galois group of the maximal pro-$p$ abelian extension of $L_\infty$ unramified outside $p$, and $\mathcal{Y}$ is the Galois group of the maximal pro-$p$ abelian extension of $L_\infty$ unramified everywhere. Letting $\mathcal{C} \subset \mathcal{E}$ denote a rank-1 free suitable module of elliptic units, we can descend the above exact sequence to an exact sequence

$$0 \to \mathcal{E}/\mathcal{C} \to \mathcal{U}/\mathcal{C} \xrightarrow{\text{rec}} \mathcal{X} \to \mathcal{Y} \to 0.$$

In the “ordinary” case, i.e. when $p$ splits $K$, all modules in the above exact sequence are $\Lambda$-torsion, and one can naturally formulate Iwasawa main conjectures. Namely, Coleman (see [53], Chapter I-II) produces a $\Lambda$-linear map $\mathcal{U} \to \Lambda_T(\overline{F}_p)$ with pseudonull cokernel sending $\mathcal{C}$ to a Katz $p$-adic $L$-function $L_p$. Let $\Delta = \text{Gal}(L_\infty/K_\infty)$, and suppose that $p \nmid \#\Delta$. Then for each nontrivial character $\chi$ on $\Delta$, Rubin ([45]) proved that the isotypic component $L_{p,\chi}$ generates the characteristic ideal of the isotypic component $(\mathcal{X} \otimes_{\mathbb{Z}_p} W(\overline{F}_p))_\chi$ by using the Euler system of elliptic units to show $\chi_{A,\chi}(\mathcal{E}/\mathcal{C})_\chi = \chi_{A}\chi(\mathcal{Y})_\chi$ under certain assumptions on $p$. This Rubin-type main conjecture has immediate applications to the arithmetic of certain CM elliptic curves, see [45], Section 11).

In the “supersingular” case when $p$ is inert or ramified in $K$, several complications arise. First, the middle two terms of the above exact sequence are not $\Lambda$-torsion, meaning there is no immediate formulation of a main conjecture. One of the key innovations of this paper is to use $p$-adic Hodge...
theory, namely the de Rham comparison theorem for Lubin-Tate groups and the Cartier duality pairing, in order to construct a Coleman-type $\Lambda$-linear map $\mu : U \hookrightarrow \Lambda$ (see (11)). As we must deal with height 2 Lubin-Tate formal groups in the supersingular CM setting, we need to use new ideas from relative $p$-adic Hodge theory, including working with the period rings $\mathcal{O}_B^{+}_{dR}$ from [50] and the closely related ring $\mathcal{O}_\Delta$ from [34]. The correct coordinate for constructing the Katz-type measures arises from the “de Rham fundamental period” of loc. cit. We do this locally in Section 3, and semilocally in the beginning of Section 4. We see $\mu$ has a kernel of $\Lambda_{\mathcal{O}_\Delta}$-rank $[L_p : Q_p] - 1$. In Section 4, we show that $\mathcal{L} := \mu(\mathcal{C})$ gives rise to a Katz-type $p$-adic $L$-function interpolating critical Hecke $L$-values, analogous to the ordinary case.

Denoting $U' := \ker(\mu)$ and $X' := \text{rec}(U')$, one can show that $(U \otimes_{Z_p} \mathcal{O}_\Delta)/(U', C \otimes_{Z_p} \mathcal{O}_\Delta)$ and $X \otimes_{Z_p} \mathcal{O}_\Delta/X'$ are $\Lambda_{\mathcal{O}_\Delta}$-torsion. From (1), we then get an exact sequence of torsion $\Lambda_{\mathcal{O}_\Delta}$-modules

$$0 \to (E/\mathcal{C}) \otimes_{Z_p} \mathcal{O}_\Delta \to (U \otimes_{Z_p} \mathcal{O}_\Delta)/(U' \otimes_{Z_p} \mathcal{O}_\Delta, C \otimes_{Z_p} \mathcal{O}_\Delta) \xrightarrow{\text{rec}} X \otimes_{Z_p} \mathcal{O}_\Delta \to Y \otimes_{Z_p} \mathcal{O}_\Delta \to 0.$$ 

Hence these modules admit natural Rubin-type main-conjectures (Conjecture 4.34 and Conjecture 4.35) that their $\Lambda_{\mathcal{O}_\Delta}$-characteristic ideals are equal, which one can again approach using the Euler system of elliptic units. However, to avoid certain restrictive assumptions on $p$ imposed by Rubin’s methods, we invoke the results of Johnson-Leung-Kings [28] in order to prove an equivariant $\Lambda$-main conjecture stating that $\det_\Lambda(E/\mathcal{C}) = \det_\Lambda(Y)$. The results of loc. cit. assume that the total $\mu$-invariant of $Y$ is 0 at $p$. In order to prove this that this is always the case, we show that the total $\mu$-invariant of $\mathcal{L}$ is 0 by employing a method introduced by Sinnott (in the classical Kubota-Leopoldt $GL_1/Q$ setting), and extended by Robert ([42]) to the $GL_1/K$ setting. The exact sequence (2) along with precise index calculations of Galois groups following Coates-Wiles [11] then gives the desired vanishing of the total $\mu$-invariant of $Y$, and hence the results of ohnson-Leung-Kings apply.

This gives the supersingular Rubin-type main conjecture, which is the first main result of this paper (Theorem 4.55 and Corollary 4.56).

In Section 5, we give some immediate applications of the supersingular Rubin-type main conjecture towards rank 0 BSD for supersingular elliptic curves with complex multiplication. By studying the dual Selmer group $X \otimes_{Z_p} \mathcal{O}_L/I_p/X'$ using Wiles’s explicit reciprocity law, we prove a rank 0 $p$-converse theorem (Theorem 5.13), which says that “if $E/Q$ has CM by $K$, $\text{corank}_{Z_p} \text{Sel}_{p}^\infty(E/Q) = 0 \implies \text{ord}_{s=1} L(E/Q, s) = 0$”. This in particular applies to rank 0 members of the congruent number family, and invoking Selmer distribution results of Smith, we show that 100% of the curves $y^2 = x^3 - d^2x$ with squarefree $d \equiv 1, 2, 3 \pmod{8}$ have analytic rank 0.

The second half of the paper concerns the $GL_2/Q$ setting in the case of CM elliptic curves, and is motivated primarily by the desire to establish a rank 1 $p$-converse theorems, and apply it to the classical sums of cubes problem (Sylvester’s conjecture) and the congruent number problem. We follow the general strategy of formulating a Perrin-Riou type Heegner point main conjecture, and showing (in the CM setting) that it is equivalent to the Rubin-type main conjecture proven in the first half of the paper. This philosophy of reducing “difficult” main conjectures to more amenable “Greenberg-type” main conjectures was first realized by Skinner in his pioneering paper on $p$-converse theorems [55].

Finally, in Section 7, we prove our rank 1 $p$-converse theorem (Theorem 7.3): “if $E/Q$ has CM by $K$, $\text{corank}_{Z_p} \text{Sel}_{p}^\infty(E/Q) = 1 \implies \text{ord}_{s=1} L(E/Q, s) = 1$”. We can then apply this theorem to prove Sylvester’s conjecture on sums of cubes (Corollary 7.4) and Goldfeld’s conjecture for the congruent number family (Corollary 7.7).

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2. Review of the Coleman map

2.1. Notation. For the rest of this paper, $K/Q$ will denote an imaginary quadratic field. Let $D \in \mathbb{Z}_{<0}$ denote the fundamental discriminant of $K$. Fix an algebraic closure $\overline{Q}$ of $Q$, and henceforth view all number fields as embedded in $\overline{Q}$.

**Definition 2.1.** Henceforth, fix an algebraic closure $\overline{Q}_p$ of $Q_p$, let $C_p$ denote its $p$-adic completion, and fix embeddings

\[ i : \overline{Q} \hookrightarrow C, \quad i_p : \overline{Q} \hookrightarrow C_p. \]

Let $p$ be the prime ideal of $\mathcal{O}_K$ fixed by (3). Given any field $L \subset \overline{Q}_p$, let $L_p$ denote its $p$-adic completion in $\overline{Q}_p$. Given a local field $L_p$, let $k(L_p)$ denote its residue field.

2.2. Relative Lubin-Tate groups. We recall some notions concerning relative Lubin-Tate groups, following [53 Chapter I.1.1], to which we refer the reader for more details. Let $L_p/K_p$ be an unramified extension of local fields, and let $k(L_p)$ denote the residue field associated with $L_p$. Let $\hat{K}^{ur}$ denote the $p$-adic completion of the maximal unramified extension of $K_p$. Let

\[ \phi : \hat{K}^{ur}_p \to \hat{K}^{ur}_p \]

denote the Frobenius automorphism corresponding to $K_p/Q_p$. Let $q$ denote the order of the residue field $k(K_p)$ of $K_p$, so that $\phi$ lifts the $q$-power Frobenius. Let $v : \hat{K}^{ur}_p \to \mathbb{Z}$ denote the normalized valuation on $\hat{Q}_p^{ur}$, i.e. with $v(p) = 1$. Henceforth, let $F$ denote a relative Lubin-Tate formal group law (more precisely, the Lubin-Tate formal $\mathcal{O}_{K_p}$-module) of height $h := [K_p : Q_p]$ with respect to the unramified extension $L_p/K_p$, and with special $\phi$-linear endomorphism $f$ lifting the Frobenius $\phi$ on $k(K_p)$. In particular, there is an isomorphism

\[ [\cdot]_f : \mathcal{O}_{K_p} \xrightarrow{\sim} \text{End}_{\mathcal{O}_{L_p}}(F). \]

We recall some properties of the special $\phi$-linear endomorphism $f$. Fix an element $\xi \in K_p^\times$ with $v(\xi) = d := [L_p : K_p]$, and fix $\pi' \in L_p$ with $\text{Nm}_{L_p/K_p}(\pi') = \xi$. Then

\[ f \in \text{Hom}(F, F^\phi), \]

is a homomorphism of formal groups over $\mathcal{O}_{L_p}$, and so in particular an element of $\mathcal{O}_{L_p}[X]$ where $X$ is the formal parameter on $F$. Here for $n \in \mathbb{Z}_{\geq 0}$, $F^{\phi^n}$ denotes the formal group law obtained by applying $\phi^n$ to the coefficients of $F$. Moreover, $f$ satisfies

\[ f(X) \equiv \pi'X \pmod{X^2}, \quad f(X) \equiv X^q \pmod{\pi'\mathcal{O}_{L_p}} \]

Let $\phi^n f$ denote the $\phi$-linear endomorphism of $F^{\phi^n}$ obtained by applying $\phi^n$ to the coefficients of the $\phi$-linear endomorphism $f$ of $F$, and note that $F^{\phi^n} = F^{\phi \circ f}$.

Let $F[f^n] \subset F(\mathcal{O}_{C_p})$ denote the subgroup of $f^n$-torsion points on $F$, and let

\[ L_{p,n} = L_p(F[f^n]). \]

Let

\[ \text{Nm}_n : L_{p,n} \to L_{p,n-1} \]

denote the norm map. Henceforth, let $p_n$ denote the maximal ideal of $\mathcal{O}_{L_{p,n}}$, and let $p_\infty = \bigcup_n p_n$. Since $L_p/K_p$ is inert we have $p_0 = p\mathcal{O}_{L_p}$. Following the notation of [53 Chapter I.1.1], let

\[ f^n = \phi^{-1} f \circ \phi^{-2} f \circ \cdots \circ \phi f \circ f. \]
Let $F[f^n]$ denote the kernel of the $n$-fold composition $f^n$, and note that $F[f^\infty]$ is the $p$-divisible group associated with the formal group $F$. Let

$$T_f F := \lim_{\phi^{-nf}} F_{\phi^{-nf}}[(\phi^{-nf})^n].$$

Let $p$ denote the prime ideal of $O_{K_p}$ above $p$. By Lubin-Tate theory (see [53, Proposition I.1.7]), there exist isomorphisms $\alpha_n : O_{K_p}/p^n \overset{\sim}{\to} F_{\phi^{-nf}}[(\phi^{-nf})^n]$ for all $n \in \mathbb{Z}_{\geq 0}$ such that we have the following commutative diagram:

$$\begin{array}{ccc}
O_{K_p}/p^{n+1} & \alpha_{n+1} & \rightarrow & F_{\phi^{-(n+1)f}}[(\phi^{-(n+1)f})^{n+1}] \\
\downarrow & & & \downarrow \phi^{-(n+1)f} \\
O_{K_p}/p^n & \alpha_n & \rightarrow & F_{\phi^{-nf}}[(\phi^{-nf})^n]
\end{array}$$

(4)

This implies that abstractly, there exists an isomorphism

$$\alpha_{\infty} = \lim_{n} \alpha_n : O_{K_p} \overset{\sim}{\to} T_f F.$$

We henceforth refer to such an isomorphism as a $(p^\infty)$-level structure.

Let $G := \text{Gal}(L_p,\infty/L_p)$. By [53, Proposition I.1.8], we have an isomorphism

$$\kappa : G \overset{\sim}{\to} O_{K_p}^\times, \quad \forall x \in F[f^n], \sigma(x) = [\kappa(\sigma)]_f(x).$$

By the remark after equation (9) in Chapter I.3.3 of loc. cit., composing $\kappa$ by the local Artin symbol $O_{K_p}^\times \rightarrow G, \sigma \mapsto \sigma^{-1}$.

By [59, Section 2.2, Proof of Proposition 1], identifying the category of connected formal groups over complete local Noetherian rings with the category of connected $p$-divisible groups over complete local rings, we have a non-canonical isomorphism

$$F \cong \text{Spf}(O_{L_p}[[X]]).$$

When $F$ is a Lubin-Tate group, its $p$-divisible group $F[f^\infty]$ is connected. Hence we can and will often freely identify the formal $O_{K_p}$-module $F$ with its $p$-divisible group $F[f^\infty]$.

Henceforth, denote

$$\mathcal{U} := \lim_{\overset{\rightarrow}{N \in \mathbb{N}}} O_{L_{p,n}}^\times,$$

and let

$$\mathcal{U} := \lim_{\overset{\rightarrow}{N \in \mathbb{N}}} O_{L_{p,n}}^{\times,1}$$

where $O_{L_{p,n}}^{\times,1}$ denotes principal units.

2.3. The Coleman map and Tsuji’s reformulation.

**Theorem 2.2** (Coleman, see Theorem I.2.2 of [53]). Let $\beta = (\beta_n) \in \mathcal{U}$. Fix a level structure $\alpha_{\infty} = \lim_{n} \alpha_n : O_{K_p} \overset{\sim}{\to} T_f F$.

There exists a unique $g_\beta \in O_{L_p}[[X]]$ such that

$$\phi^{-ng_\beta(\alpha_n)} = \beta_n$$

for all $n \geq 0$. This defines a map

$$\text{Col}_{(F,\alpha_{\infty})} : \mathcal{U} \rightarrow O_{L_p}[[X]]^{\times, N_f = \phi} \subset O_{L_p}[[X]]^\times$$
where $N_f$ is Coleman’s norm operator attached to $f$ (see [53, Chapter I.2.1]). Moreover, for any $\sigma \in \text{Gal}(L_{p,\infty}/L_p)$, we have
\begin{equation}
\text{Col}_{(F,\alpha,\infty)}(\sigma \beta) = \text{Col}_{(F,\alpha,\infty)}(\beta) \circ [\kappa(\sigma)]_f.
\end{equation}

Tsuji has the following “coordinate-free” formulation of Coleman’s theorem.

**Theorem 2.3** (Tsuji’s formulation of Coleman’s theorem, Theorem 4.1 of [61]). Fix a level structure $\alpha_\infty : \mathcal{O}_{K_p} \xrightarrow{\sim} T_f F$.

Then there exists a natural $\text{Gal}(\overline{L}_p/L_p)$-equivariant isomorphism
\begin{equation}
\text{Col}_{(F,\alpha,\infty)} : \mathcal{U} \xrightarrow{\sim} \mathcal{O}_{F[f^\infty]}^\times (F[f^\infty])^{N_f = \phi}, \quad \beta \mapsto g_\beta.
\end{equation}

In particular, we have an inclusion
\[\mathcal{O}_{F[f^\infty]}^\times (F[f^\infty])^{N_f = \phi} \subset \mathcal{O}_{F[f^\infty]}^\times (F[f^\infty])\]
and so applying (8) to (9), we recover (7).

The universal formal group $F^{\text{univ}} \to \text{LT}$ also inherits an action of $\mathcal{O}_{K_p}^\times$ and $\phi$ from the universal property (see [61, Section 1]), and norm operator $N_f^{\text{univ}}$.

**Proposition 2.4.** The specialization $F \to F^{\text{univ}}$ induces an isomorphism of $\mathbb{Z}_p[\text{Gal}(L_p(F[p^\infty])/L_p)]$-modules
\begin{equation}
\Gamma(F^{\text{univ}})^{\times, N_f^{\text{univ}} = \phi} \xrightarrow{\sim} \Gamma(F)^{\times, N_f = \phi}.
\end{equation}
We will often denote the preimage of $g_\beta$ by $g_\beta^{\text{univ}}$.

### 3. Construction of measure on the local Galois group

**Definition 3.1.** Henceforth, given a profinite group $G$, let $\mathbb{Z}_p[G]$ denote its completed group algebra, and for any $\mathbb{Z}_p$-algebra $R$, let $\Lambda(G, R) := \mathbb{Z}_p[G] \hat{\otimes}_{\mathbb{Z}_p} R = R[[G]]$.

In this section, we prove the following theorem, which gives a construction of a 1-variable (anticyclotomic) measure map.

**Theorem 3.2** (Definition 3.40). Assume that $F$ has height 2, and fix a $p^\infty$-level structure $\alpha : \mathcal{O}_{K_p} \xrightarrow{\sim} T_f F$. Let $\mathcal{O}_{K_p} = \Delta \times \Gamma$, let $\Gamma_- = \Gamma/(1 + q\mathbb{Z}_p)$, where $q = 4$ if $p = 2$ and $q = p$ if $p > 2$. Assume further $p = 2, 3$. Let $\chi : \Delta \to \mathbb{T}_p^\times$ be a non-trivial character. Then there is a $\Lambda(\Gamma_-, \mathcal{O}_{C_p})[1/p]$-equivariant map
\begin{equation}
\mu : \mathcal{U}_{\chi}^{1-} \to \Lambda(\Gamma_-, \mathcal{O}_{C_p})[1/p].
\end{equation}
(See Section 3.7 for precise definitions of the notation.)

**Remark 3.3.** Moreover, the twist of $\mu$ by the Lubin-Tate character satisfies an explicit reciprocity law on elliptic units. See (44).

**Remark 3.4.** The assumption $p = 2, 3$ is made for technical convenience, as the $\mathbb{Z}_p$-free part of $\mathcal{O}_{K_p}^\times$ is contained in $1 + p\mathcal{O}_{K_p}$ in this case. We are mostly interested in this idiosyncratic case for arithmetic applications. The general form of Theorem 3.2 will appear in future work.

The proof of this theorem will actually take up much of this section. The essential idea is to find a suitable coordinate $q_{\text{DR}}$ for constructing a map into power series in $q_{\text{DR}} - 1$, which are identified with measures using the Amice transform. The coordinate $q_{\text{DR}}$ will come from two ingredients: mapping the universal Lubin-Tate group $F^{\text{univ}} \to \text{LT}$ to a constant family $\mathcal{C}_{m,LT}$ of tori over LT in a suitable sense to get a coordinate $Q$, and using relative $p$-adic Hodge theory to construct $q_{\text{DR}}$. 

\[\text{6}\]
Let under the universal principal polarization.\[\text{denote the image of } \alpha\]

Note that since \(\langle O \rangle\) to an indefinite quaternion algebra split at \(p\), the universal false elliptic curve \(A\) a principal polarization of formal groups \(F\).

Henceforth, fix a \(\text{Definition 3.7.}\)

Let \(\hat{F}_{p,LT}\) be the dual \(\langle \hat{\alpha}_{1,n}, \alpha_{2,n} \rangle = \zeta_{p^n}\).

\(\text{Assumption 3.6.}\) Henceforth, assume that \(p\) is ramified in \(K\). Then for any uniformizer \(\pi\) of \(O_{K_p}\), we have \(\pi^2 O_{K_p} = pO_{K_p}\), and hence \(F[\pi]\) is a finite flat group scheme over \(O_{L_p}\) of order \(p\). We will also denote \(\hat{F}\) by \(\hat{F}\). Also, henceforth we let \((1, \zeta_p, \zeta_{p^2}, \ldots)\) denote the compatible sequence of \(p\)-th-power roots of unity given by \(\langle \hat{\alpha}_{1,n}, \alpha_{2,n} \rangle = \zeta_{p^n}\).

\(\text{Definition 3.7.}\) Henceforth, fix a \(Z_p\)-linear basis \((\alpha_1, \alpha_2) : Z_p^2 \xrightarrow{\sim} T_p F\), which determines a point \(y = (F, \alpha_1, \alpha_2) \in LT_\infty\). On \(LT_\infty\), we have a universal \(Z_p\)-basis \((\alpha_1^{\text{univ}}, \alpha_2^{\text{univ}}) : \hat{Z}_{p,LT_\infty}^\oplus \xrightarrow{\sim} T_p F_{\text{univ}} := R_1 \hat{\rho}_* \hat{Z}_{p,F_{\text{univ}}} |_{LT_\infty}\).

We make the further requirement that \(\alpha_2 : Z_p \hookrightarrow T_p F \xrightarrow{\alpha^{-1}} O_{K_p}\) is the natural inclusion \(Z_p \hookrightarrow O_{K_p}\), and that \(\langle \alpha_1 \mod p \rangle = F[\pi]\).

Let \(\hat{F}[p^n] = \text{Hom}_{\text{groupschemes}}(F[p^n], \mu_{p^n})\) denote the Cartier dual of \(F[p^n]\), and let \(\hat{F}[p^n] = \lim_{\text{\tiny \eta}} \hat{F}[p^n]\) be the dual \(p\)-divisible group, and let \(\hat{F}\) be the formal group associated with the connected \(p\)-divisible group \(\hat{F}[p^n]\). Fix a principal polarization \(F_{\text{univ}}[p^n] \cong \hat{F}_{\text{univ}}[p^n]\), which by [59, Proposition 1] gives a principal polarization of formal groups \(F_{\text{univ}} \cong \hat{F}_{\text{univ}}\). In our application, this principal polarization will be fixed by the identification \(\hat{A}_{\text{univ}} \cong \hat{A}_{\text{univ}}\) induced by the canonical polarization on the universal false elliptic curve \(A_{\text{univ}} \rightarrow Y\), \(Y\) an appropriate Shimura curve over \(Q\) corresponding to an indefinite quaternion algebra split at \(p\), and where \(A_{\text{univ}}\) is the formal group of \(A_{\text{univ}}\). Let \(\hat{\alpha}\) denote the image of \(\alpha\) under the principal polarization, and let \(\hat{\alpha}_{\text{univ}}\) denote the image of \(\hat{\alpha}_{\text{univ}}\) under the universal principal polarization.

\(\text{Definition 3.8.}\) Let \(z \in K_p\) be defined by \(\langle [1/z] \rangle_f (\alpha_2) = \alpha_1\).

Note that since \(\langle \alpha_1 \mod p \rangle = F[\pi], 1/z \in \pi O_{K_p}^\times\) and in particular \(1/z\) is a local uniformizer of \(O_{K_p}\).

\(\text{1This is made mainly for convenience, namely to use the “canonical subgroup” } F[\pi].\) Many aspects of the subsequent methods apply also to the \(p\) inert case, and this is the subject of future work.
By [59 Proposition 1], since $F_{\text{univ}}^{\text{univ}}$ and $\hat{G}_{m,LT} = \hat{G}_m \times_{\text{Spec}(\mathbb{Z}_p)} \text{LT}$ have Noetherian rings of definition (as $\mathcal{O}_{\text{LT}}(\text{LT}) = \mathcal{O}_{\mathbb{P}[T]}$, it is Noetherian), the above étale map extends to an étale map of adic generic fibers pulled back to the pro-étale cover $\text{LT}$

\[
\hat{\mathcal{O}}_{\hat{G}_{m,LT},(\text{LT})} \cong \hat{\mathcal{O}}_{\text{LT}}[Q - 1].
\]

**Convention 3.9.** The sections $\alpha^{\text{univ}} : \text{LT}_n \to F_{\text{LT}_n}^{\text{univ}}$ coalesce to give a section

\[
e := \lim_{\to} \alpha_n^{\text{univ}} : \text{LT}_\infty = \lim_{\to} \text{LT}_n \to \lim_{\to} F_n^{\text{univ}} = F_\infty^{\text{univ}}
\]

of the universal map $\rho : F_\infty^{\text{univ}} \to \text{LT}_\infty$. For the remainder of this discussion, given a point $y \in \text{LT}_\infty$, we will use the subscript “$y$” to denote stalks at $y \in \text{LT}_\infty$ of sheaves in the pro-étale topos of $\text{LT}_\infty$ and stalks at $e(y) \in F_\infty^{\text{univ}}$ of sheaves in the pro-étale topos of $F_\infty^{\text{univ}}$. The correct situation will be clear from context. Similarly, we will let “$(y)$” denote the image in the fiber at $y$ or $e(y)$.

\[
\hat{\mathcal{O}}_{\text{LT}_y}^+: = \{ f \in \hat{\mathcal{O}}_{\text{LT}_y} : |f(y)| \leq 1 \},
\]

and similarly with $\Omega_{\text{LT}_y}$, $\hat{\mathcal{O}}_{F_\infty^{\text{univ}}}^+$, etc.

### 3.2. A brief review of some relative $p$-adic Hodge theory

Recall the period sheaf $\mathcal{O}\mathbb{B}_d^+_\text{dR}$, equipped with a natural connection and filtration, as defined in [50 Section 6]. We have a projection $\hat{\vartheta} : \mathcal{O}\mathbb{B}_d^+_\text{dR} \to \hat{\mathcal{O}}$.

**Definition 3.10.** We have a relative Fontaine $2\pi i$ on $\text{LT}_\infty$,

\[
t := \log[[\langle \alpha_{1,1}^{\text{univ}}, \alpha_{2,1}^{\text{univ}} \rangle, \langle \alpha_{1,2}^{\text{univ}}, \alpha_{2,2}^{\text{univ}} \rangle, \ldots]] \in \mathbb{B}_d^{\text{univ}}(\text{LT}_\infty) \subset \mathcal{O}\mathbb{B}_d^{\text{univ}}(\text{LT}_\infty).
\]

Then we have the reduced period sheaf

\[
\mathcal{O}_\Delta, \text{LT}_\infty := \mathcal{O}\mathbb{B}_d^{\text{univ}}(\text{LT}_\infty)/(t),
\]

as in [34]. A few remarks on how to conceptually view this period sheaf. Informally viewing

\[
\mathcal{O}\mathbb{B}_d^+ = \lim_n \left( \mathcal{O} \otimes \mathbb{B}_d^+ \right)/(\ker \vartheta)^n,
\]

where $\vartheta$ is given by $\mathcal{O} \to \hat{\mathcal{O}}$ on the first tensor factor and by the standard $\vartheta : \mathbb{B}_d^+ \to \hat{\mathcal{O}}$ on the second tensor factor (which has kernel generated by $t$), one can informally view

\[
\mathcal{O}_\Delta = \lim_n \left( \mathcal{O} \otimes \hat{\mathcal{O}} \right)/(\Delta)^n
\]

, where $\Delta$ is the diagonal ideal generated by elements $x \otimes 1 - 1 \otimes x$. Hence $\mathcal{O}_\Delta$ in this optic is simply the structure sheaf of a formal neighborhood of the diagonal. If one were to replace $\mathcal{O}\mathbb{B}_d^+$ with $\mathcal{O}\mathbb{B}_\text{cris}$, one would instead obtain the structure sheaf of a PD thickening of a formal neighborhood of the diagonal. We will only need relative de Rham period sheaves in our discussion.

As $t \in \ker(\vartheta)$, we get a projection $\vartheta : \mathcal{O}_\Delta, \text{LT}_\infty \to \hat{\mathcal{O}}_{\text{LT}_\infty}$. Recalling the universal map $\rho : F_\infty^{\text{univ}} \to \text{LT}$, we get a section $\rho^* t \in \mathbb{B}_d^{\text{univ}}(F_\infty^{\text{univ}})$, and we define $\mathcal{O}_\Delta, F_\infty^{\text{univ}} := \mathcal{O}\mathbb{B}_d^{\text{univ}}(F_\infty^{\text{univ}})/(\rho^* t)$. 8
Definition 3.11. By construction (see [50] Section 6), there are natural inclusions $\mathcal{O} \subset \mathcal{O}_{\text{dr}}^+, \mathcal{B}^+_{\text{dr}} \subset \mathcal{O}_{\text{dr}}^+$. The former is compatible with connections, and the latter gives the sheaf of horizontal sections of the natural connection $\nabla$ on $\mathcal{O}_{\text{dr}}^+$. Reducing modulo $t$, we get a natural inclusion

$$j: \mathcal{O}_{LT} = \mathcal{B}^+_{\text{dr},LT}/(t) \hookrightarrow \mathcal{O}_{\Delta,LT}.$$

This gives the sheaf of horizontal sections for the natural connection (also denote $\nabla$) on $\mathcal{O}_{\Delta,LT}$. For brevity, we often denote the image of a section $f$ under this map by $\bar{f} = j(f)$.

Definition 3.12. Let

$$z_{\text{dr}} \in \mathcal{O}_{\text{dr},LT}(LT)$$

be the de Rham fundamental period of $[34]$. The period $z_{\text{dr}}$ is a coordinate on $\mathbb{P}(\mathcal{O}_{\text{dr},LT})$, and measures the position of the Hodge line bundle in the trivialized relative de Rham cohomology

$$\rho_* \Omega^1_{\text{univ}/LT} \subset H^1_{\text{DR}}(F^\text{univ})_{LT} \xrightarrow{\iota_{\text{dr}}} T_p \hat{F}^\text{univ} \otimes_{\bar{z}_{p,LT}} \mathcal{O}_{\text{dr},LT} \cong \mathcal{O}_{\text{dr},LT}(-1)^{\otimes 2};$$

where

$$H^1_{\text{DR}}(F^\text{univ}) := R^1 \rho_* (0 \to \mathcal{O}_{\text{univ}} \to \Omega^1_{F^\text{univ}} \to 0)$$

is the universal de Rham cohomology, a rank 2 vector bundle over $LT$ and equipped with its Gauss-Manin connection

$$\nabla: H^1_{\text{DR}}(F^\text{univ}) \to H^1_{\text{DR}}(F^\text{univ}) \otimes_{\mathcal{O}_LT} \Omega^1_{LT},$$

and $\iota_{\text{dr}}$ is the relative comparison map of $[50]$, and is compatible with connections and natural filtrations. We then define the reduced period

$$z_{\text{dr}} := z_{\text{dr}} \pmod{t} \in \mathcal{O}_{\Delta,LT}(LT).$$

(cf. [34] Chapter 5).

Definition 3.13. Let $\langle \cdot , \cdot \rangle: T_p \hat{F}^\text{univ} \times F^\text{univ}[p^\infty] \to \mu_{p,LT} := \lim_{\gamma_{\mathfrak{n}}} \mu_{p^{\mathfrak{n}},LT}$ denote the universal Weil pairing. Define $\epsilon_{\text{univ}}^p$ by

$$\epsilon_{\text{univ}}^p := \langle \alpha^\text{univ}_2, \alpha^\text{univ}_1 \rangle.$$

Note that the universal Weil pairing extends to a pairing

$$\langle \cdot , \cdot \rangle: T_p \hat{F}^\text{univ} \otimes_{\bar{z}_{p,LT}} \mathcal{O}_{\Delta,LT} \times F^\text{univ}[p^\infty] \to \mu_{p,LT} \otimes \bar{z}_{p,LT} \mathcal{O}_{\Delta,LT}.$$

3.3. The coordinate $Q_{\text{dr}}$. This will be constructed by “exponentiating $Q$ by $z_{\text{dr}}$”, where $z_{\text{dr}}$ is the de Rham fundamental period (see [34]). It turns out that $z_{\text{dr}}$ is the “correct” exponent to yield integral $Q_{\text{dr}}$-series from integral $X^\text{univ}$-series, analogously to the ordinary case (where $LT$ is replaced with an ordinary residue disc, or equivalently the deformation space of the étale $p$-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$) and where $z_{\text{dr}}$ is Dwork’s divided power function (whose exponential is the Serre-Tate coordinate on the ordinary residue disc).

Definition 3.14. Note that we have a map $\rho^*: \mathcal{O}_{LT} \to \mathcal{O}_{\hat{F}^\text{univ}}$ which induces a map $\rho^*: \mathcal{O}_{\Delta,LT} \to \mathcal{O}_{\hat{F}^\text{univ}}$. As $z_{\text{dr}} - \bar{z}_{\text{dr}} \in \mathcal{O}_{\Delta,LT}$, we get $\rho^*(z_{\text{dr}} - \bar{z}_{\text{dr}}) \in \mathcal{O}_{\hat{F}^\text{univ}}$ where as shorthand we let $\bar{z}_{\text{dr}} := \bar{\vartheta}(z_{\text{dr}})$ in the notation of Definition 3.11. We define

$$Q_{\text{dr}} := \exp(\rho^*(z_{\text{dr}} - \bar{z}_{\text{dr}}) \log(Q)) \in \hat{F}^\text{univ}(F^\text{univ})[\rho^*(z_{\text{dr}} - \bar{z}_{\text{dr}})] \subset \mathcal{O}_{\hat{F}^\text{univ}}$$

which is well-defined since $\rho^*(z_{\text{dr}} - \bar{z}_{\text{dr}}) \in \ker(\vartheta)$. It is easily seen that $\vartheta(Q_{\text{dr}} - 1) = 0$, and for brevity we denote $dQ_{\text{dr}} = \nabla(Q_{\text{dr}})$.

---

In loc. cit., $z_{\text{dr}}^+$ was defined on the global infinite-level modular curve $\mathcal{Y}$ using the universal level structure on the latter and the relative de Rham comparison of Scholze [39]. Here, we define $z_{\text{dr}}$ on $LT$ on the local tower $LT$, applying the de Rham comparison to the universal formal group $F^\text{univ} \to LT$, and the steps of the construction easily carry over from loc. cit.
\textbf{Remark 3.15.} Note that we used the “\( \ker \vartheta \)-adic topology” on \( \mathcal{O}_{\Delta,F_{\text{univ}}} \) essentially in order to define \( dQ_{dR} \), which allows us to raise to the \((z_{dR} - \bar{z}_{dR})\)-th power. Taking arbitrary powers in the usual \( p \)-adic topology in \( \mathcal{O}_{\Delta,F_{\text{univ}}} \) does not result in analytic sections in general.

3.4. \( Q_{dR} \)-expansions at the CM point \( y \). The purpose of this section is to convert integral \( X_{y}^{\text{univ}} \)-expansions at a CM point \( y \) into integral \( Q_{dR} \)-expansions. The key point is to take advantage of the natural Frobenius lifts on \( F_{\text{univ}}^{\text{univ}} \) which exist in small neighborhoods of the CM point \( y = (F,\alpha_1,\alpha_2) \), and which induces computable actions on the stalks of \( X_{y}^{\text{univ}} \) and \( Q_{dR} \) at \( y \). The reason this Frobenius lift exists is because \( F \) has a canonical subgroup \( \langle \alpha_1 \, (\text{mod} \, p) \rangle = F[\pi] \), and division by this canonical subgroup gives a lift of Frobenius modulo \( \pi' \). On the other hand, \( Q_{dR} \), which can informally be viewed as \( Q_{dR}^{\text{univ}} - \bar{z}_{dR} \), is acted on by division by the “pseudo-canonical” subgroup \( \langle \alpha_1^{\text{univ}} \, (\text{mod} \, p) \rangle \) (in the terminology of [10]) in the following way: \( Q = (\alpha_2^{\text{univ}})^{*}q \) is sent to \( Q \) since the line parametrized by \( \alpha_2^{\text{univ}} \) is preserved by the division, and \( z_{dR} - \bar{z}_{dR} \) is sent to \( p(z_{dR} - \bar{z}_{dR}) \), since the isogeny multiplies the Hodge-de Rham period \( z_{dR} \) by \( p \). Hence, \( Q_{dR} \mapsto Q_{dR}^{p} \), and we may view \( Q_{dR} \) as the correct variable for “linearizing” the division by \( \langle \alpha_1^{\text{univ}} \, (\text{mod} \, p) \rangle \) (and hence also the Frobenius lift in a neighborhood of \( y \)). It is this key property of \( Q_{dR} \) that makes the below Dieudonné-Dwork type argument (Theorem 3.20) for showing integrality of \( Q_{dR} \)-expansions work, which is in turn key to our construction of measures.

First, we show that there is a natural map from arbitrary \( X_{y}^{\text{univ}} \)-expansions to \( Q_{dR,y} \)-expansions (all viewed inside the ambient period ring \( \mathcal{O}_{\Delta,F_{\text{univ}}} \)). By [50], we have the characterization

\begin{equation}
\hat{\mathcal{O}}_{\text{LT},y}^{+} = \{ f \in \hat{\mathcal{O}}_{\text{LT},y} : |f(y)| \leq 1 \},
\end{equation}

and similarly with \( \Omega_{\text{LT}}^{+}, \hat{\mathcal{O}}_{F_{\text{univ}}}^{+} \), etc.

\textbf{Proposition 3.16.} We have

\[ \vartheta \left( \frac{dQ_{dR,y}}{dX_{y}^{\text{univ}}} \right) \in \hat{\mathcal{O}}_{\text{LT},y}^{\times}. \]

\textbf{Proof.} This follows from the fact \( dQ_{dR,y} \) and \( dX_{y}^{\text{univ}} \) both give generators at \( y \). For the former, note that

\[ \vartheta(dQ_{dR}) = \vartheta(d \exp(\rho^{*}(z_{dR} - \bar{z}_{dR}) \log(Q))) = \vartheta(Q_{dR}(d\rho^{*}z_{dR} \cdot \log(Q) + \rho^{*}(z_{dR} - \bar{z}_{dR})d\log(Q))) = \vartheta(d\rho^{*}z_{dR} \log(Q)), \]

which is a generator by [33] Theorem 4.36, (4.50)]. \( \square \)

Recall that \( F_{\text{univ}} \) is a formal scheme over the formal scheme LT, and in particular the stalk map \( \mathcal{O}_{\text{LT}}(\text{LT}) \to \hat{\mathcal{O}}_{\text{LT},y}^{+} \) factors through \( \mathcal{O}_{\text{LT}}(\text{LT}) \to \hat{\mathcal{O}}_{\text{LT},y}^{+} \). Using change of variables for \( X_{y}^{\text{univ}} \) to \( Q_{dR,y} - 1 \), we have a natural sequence of maps

\begin{equation}
\mathcal{O}_{F_{\text{univ}}}(\text{univ}) = \mathcal{O}_{\text{LT}}(\text{LT})[X_{y}^{\text{univ}}] \to \hat{\mathcal{O}}_{\text{LT},y}^{+}[X_{y}^{\text{univ}}] \subset \hat{\mathcal{O}}_{F_{\text{univ}},y}^{+}[Q_{dR},y - 1] \subset \mathcal{O}_{\Delta,\text{F}_{\text{univ}},y},
\end{equation}

where the second inclusion is given by

\[ g \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \vartheta \left( \left( \frac{d}{dQ_{dR,y}} \right)^{n} \right)(Q_{dR,y} - 1)^{n}, \]

and indeed this is injective by Proposition 3.16. We wish to show that the second inclusion actually lands in \( \hat{\mathcal{O}}_{F_{\text{univ}},y}^{+}[Q_{dR,y} - 1] \), which follows from the next Theorem establishing the integrality of the \( Q_{dR,y} - 1 \)-expansion of \( X_{y}^{\text{univ}} \).
Definition 3.17. Denote the image of an element $g$ under the above inclusion (from [15])
\[
\mathcal{O}_{F_{\infty}^{univ},y}^{+} \subset \hat{O}_{F_{\infty}^{univ},y}[Q_{dR, y} - 1]
\]
by $g(Q_{dR, y} - 1)$.

We first note the following lemma relating the Frobenius operators on $F_{\infty}^{univ}$ and on $\hat{G}_{m,LT_{\infty}}$. Recall that the Frobenius automorphism $\phi : \mathcal{O}_{L_{p}} \to \mathcal{O}_{L_{p}}$ induces automorphisms of LT and $F_{\infty}^{univ}$. First, we introduce a “pseudo-Frobenius operator” on $F_{\infty}^{univ}$.

We will briefly need the following global setting in order to package together Lubin-Tate deformation spaces $LT_{\infty}$ for varying characteristic $p$ formal groups. Let $Y$ be a quaternionic Shimura curve corresponding to a quaternion algebra split at $\infty$ which we base change to $\mathbb{Q}$, but which we base change to $\mathbb{Q}_{p}$, take the adicification, and finally base-change this adic space to $Spa(L_{p}, \mathcal{O}_{L_{p}})$. Then let as in [49]
\[
Y_{\infty} := \lim_{\longleftarrow} Y_{n} \in Y_{\text{pro\acute{e}t}}
\]
where $Y_{n} \to Y$ is the cover parametrizing full $(\mathbb{Z}/p^{n})[\mathbb{Z}^{2}]$-level structure. Then we have an identification of the infinite-level supersingular locus $Y_{\infty}^{ss} = \bigsqcup Y_{LT_{\infty}}$, i.e. $Y_{\infty}^{ss}$ is a finite union of infinite-level $LT_{\infty}$. For simplicity, assume that the tame level of $Y$ is neat (cf. [10]), so that there is a universal principally polarized (false) elliptic curve $A \to Y$ (in the case where $D$ is the totally split quaternion algebra, $A$ is indeed just an elliptic curve) with tame level structure. Let $\hat{A} = \hat{A}_{1} \times \hat{A}_{2}$ denote the formal completion at along the special fiber, where $\hat{A}_{1}$ has dimension 1 and $\hat{A}_{2}$ has dimension 0 if the quaternion algebra is totally split and has dimension 1 otherwise. Then letting $A_{\infty} := \hat{A}_{1}|_{Y_{\infty}^{ss}}$, and we get an identification $A_{\infty}|_{LT_{\infty}} = F_{\infty}^{univ}$. Continue to denote the universal $\mathbb{Z}_{p}[\mathbb{Z}^{2}]$-level structure by $(\alpha_{1}^{\text{univ}}, \alpha_{2}^{\text{univ}})$; these restrict the the same $\mathbb{Z}_{p}[\mathbb{Z}^{2}]$-level structure on each $LT_{\infty} \subset Y_{\infty}^{ss}$.

Definition 3.18. Recall that $Y_{\infty}^{ss}$ has a natural $GL_{2}(\mathbb{Q}_{p})$-action ([49], [10]), where we let $GL_{2}(\mathbb{Q}_{p})$ act on the right as in loc. cit. Denote the matrix
\[
\Phi = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.
\]
By the universal property of $A_{\infty} \to Y_{\infty}^{ss}$ (classifying height 2 formal groups $F'$ together with trivialization $T_{\phi}F' \cong \mathbb{Z}_{p}[\mathbb{Z}^{2}]$), we also have an action of $GL_{2}(\mathbb{Q}_{p})$ on $F_{\infty}^{univ}$. We now describe how $\Phi$ acts on $A_{\infty}$. Let $C = \langle \alpha_{1}^{\text{univ}}, (\text{mod } p) \rangle \subset A_{\infty}$ which is a finite flat group scheme over $Y_{\infty}^{ss}$ of order $p$. Then we have a degree-$p$ isogeny
\[
\lambda : \hat{A}_{1} \to \hat{A}_{1}/C.
\]
Recall our fixed point $y = (F, \alpha_{1}, \alpha_{2}) \in LT_{\infty} \subset Y_{\infty}^{ss}$ which we assumed satisfies $\langle \alpha_{1} \text{ (mod } p) \rangle = F[\pi] = F[f]$. Hence the induced map $\lambda(y) : F \to F/C(y)$ on the fiber at $y$ is equal to $f$ up to an isomorphism $F/C(y) \cong F^{\Phi}$. Rewriting $A_{\infty} = (\hat{A}_{1}, \alpha_{1}^{\text{univ}}, \alpha_{2}^{\text{univ}})$, we have by the definition of the $GL_{2}(\mathbb{Q}_{p})$-action in loc. cit.
\[
A_{\infty} \cdot \Phi = (\hat{A}_{1}, \alpha_{1}^{\text{univ}}, \alpha_{2}^{\text{univ}}') \cdot \Phi = (\hat{A}_{1}^{\text{univ}}/C, \alpha_{1}^{\text{univ}}, \alpha_{2}^{\text{univ}}') =: A_{\infty}/C,
\]
where $\alpha_{1}^{\text{univ}}' = \frac{1}{p} \lambda(\alpha_{1}^{\text{univ}})$ (taking the division inside $T_{\phi}(\hat{A}_{1}/C) \otimes \mathbb{Z}_{p}$) and $\alpha_{2}^{\text{univ}}' = \lambda(\alpha_{2}^{\text{univ}})$.

We note that by the universal property of $A_{\infty} \to Y_{\infty}^{ss}$ we get a classifying map
\[
\Phi_{Y_{\infty}^{ss}} : Y_{\infty}^{ss} \to Y_{\infty}^{ss}
\]
such that
\[
A_{\infty}/C = A_{\infty} \times_{Y_{\infty}^{ss}, \Phi_{Y_{\infty}^{ss}}} Y_{\infty}^{ss}.
\]
In fact, $\Phi_{Y_{\infty}^{ss}} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ in terms of the $GL_2(\mathbb{Q}_p)$-action on $Y_{\infty}^{ss}$. Subsequently, $(\alpha_{1}^{univ^{'}} , \alpha_{2}^{univ^{'}})$, a pair of sections $Y_{\infty}^{ss} \to A_{\infty}/C$ of $F_{\infty}^{univ}/C \to Y_{\infty}^{ss}$, is the base change along $\Phi_{Y_{\infty}^{ss}} : Y_{\infty}^{ss} \to Y_{\infty}^{ss}$ of $(\alpha_{1}^{univ} , \alpha_{2}^{univ})$ (viewed as a pair of sections of $A_{\infty} \to Y_{\infty}^{ss}$). For brevity, we will simply use $\Phi$ to denote $\Phi_{Y_{\infty}^{ss}}$. No confusion should arise, as one can view $\Phi$ as the $GL_2(\mathbb{Q}_p)$-action of $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ on either $A_{\infty}$ or $Y_{\infty}^{ss}$, and the situation will be clear from context. Similarly, we let $z_{dR}$ be the Hodge-de Rham period of $[34]$ Chapter 4.8, and $z_{dR} = z_{dR} \pmod{t} \in \mathcal{O}_{\Delta,Y_{\infty}^{ss}}$, which restricts on $LT_{\infty}$ to $z_{dR}$ above. Moreover, we also let $X^{univ}$ denote the formal parameter of $\hat{A}_1$ restricting to the above $X^{univ}$ on $LT_{\infty}$. From this, we get a corresponding section $Q_{dR} = \exp(\rho^*(z_{dR} - \bar{z}_{dR})) \in \mathcal{O}_{\Delta,A_{\infty}}(A_{\infty})$ which restricts on $F^{univ}_{\infty}$ to the above $Q_{dR}$.

We also have a map

$$\hat{\alpha}_2^{univ} : A_{\infty} \to \hat{\mathbb{G}}_{m,Y_{\infty}^{ss}}$$

where $\hat{\mathbb{G}}_{m,Y_{\infty}^{ss}}$ is the formal multiplicative group over $Y_{\infty}^{ss}$. This restricts to the map $\hat{\alpha}_2^{univ} : F^{univ}_{\infty} \to \hat{\mathbb{G}}_{m,LT_{\infty}}$.

We have the following commutative diagram:

$$
\begin{array}{ccc}
A_{\infty} & \xrightarrow{\Phi} & A_{\infty}/C = A_{\infty} \times_{Y_{\infty}^{ss},\Phi} Y_{\infty}^{ss} \\
\downarrow{\rho} & & \downarrow{\rho} \\
Y_{\infty}^{ss} & \xrightarrow{\Phi} & Y_{\infty}^{ss}
\end{array}
$$

(16)

where $[p]$ is the usual multiplication by $p$ on $\hat{\mathbb{G}}_{m}$ and $\nu$ is the natural projection. (Recall fiber products exist in the proétale topology by $[50]$ Section 3.)

By the above discussion, we have maps

$$\Phi^* : \mathcal{O}_{A_{\infty}}(A_{\infty}) \to \mathcal{O}_{A_{\infty}}(A_{\infty}),$$

and similarly with the global sections of $\mathcal{O}_{Y_{\infty}^{ss}}, \hat{\mathcal{O}}_{A_{\infty}}, \hat{\mathcal{O}}_{Y_{\infty}^{ss}}$, etc.

**Lemma 3.19.** We have

$$\Phi^* \circ \nu^*(Q) = Q, \quad \Phi^* \circ \nu^*(\rho^*(z_{dR} - \bar{z}_{dR})) = p\rho^*(z_{dR} - \bar{z}_{dR}).$$

**Proof.** Recall $Q = (\hat{\alpha}_2^{univ})^*q$. Hence for the first identity, we need to check that the isogeny $A_{\infty} \to A_{\infty}$ induced by $\nu \circ \Phi$ fixes $\hat{\alpha}_2^{univ}$. By the definition of the $GL_2(\mathbb{Q}_p)$-action, we see that this isogeny sends $\hat{\alpha}_2^{univ}$ to

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & p \end{array} \right)^{\gamma} \hat{\alpha}_2^{univ} = \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) \hat{\alpha}_2^{univ} = \alpha_2^{univ},$$

where $\gamma^{\vee} = \gamma^{-1}\det(\gamma)$ (the contragredient).

For the second identity, one simply calculates using the $GL_2(\mathbb{Q}_p)$ action on the de Rham fundamental period (see, for example, $[34]$ Chapter 4.8)).

**Theorem 3.20.** We have

$$X^{univ}_y(Q_{dR,y} - 1) \in \hat{\mathcal{O}}_{F^{univ}_{\infty},y}^+[Q_{dR,y} - 1].$$

**Proof.** This will follow from a Dieudonné-Dwork-style argument (cf. $[16]$ Lemma 1]). Write

$$X^{univ}_y(Q_{dR,y} - 1) = \sum_{n=0}^{\infty} a_n(Q_{dR,y} - 1)^n$$

for
where
\[ a_n = \vartheta \left( \frac{1}{n!} \left( \frac{d}{dQ_{\text{dr}}} \right)^n X_{\text{univ}}^{\text{univ}} \right) \in \hat{O}_{A_{\infty}}(A_{\infty}), \]
and for brevity we also use \( a_n \) to denote the image of \( a_n \) in the stalk at \( y \in \text{LT}_\infty \) for the remainder of this proof. Hence \( \Phi^* \circ \nu^* \) acts on the \( a_n, X_y^{\text{univ}} \) and \( Q_{\text{dr}, y} \). Applying \( \vartheta \) we see that \( a_0 = X_y^{\text{univ}} \in \hat{O}_{F_{\infty}^{\text{univ}, y}}^+ \) (in fact, applying \( \vartheta \) we see that \( a_0(y) = \hat{X}_y^{\text{univ}}(y) = \hat{X} \)).

Recall our Frobenius-linear Frobenius lift \( f : F \to F^\varphi \). After possibly base-changing \( F, \text{LT}, \) and \( F^{\text{univ}} \) to \( \mathcal{O}_{K_p} W(\mathbb{F}_p) \), we may assume that \( F \) is isomorphic to a Lubin-Tate group with Frobenius-linear Frobenius lift \( X^p + \pi' X \) where \( \pi' \) is a uniformizer of \( \mathcal{O}_{L_p} \). Hence, without loss of generality and working over \( \mathcal{O}_{K_p} W(\mathbb{F}_p) \), we assume \( f(X) = X^p + \pi' X \). Viewing
\[ \Phi^* \circ \nu^* : \mathcal{O}_{A_{\infty}}(A_{\infty}) \to \mathcal{O}_{A_{\infty}}(A_{\infty}), \]
write
\[ \Phi^* \circ \nu^*(X^{\text{univ}}) = \sum_{m=0}^\infty b_m(X^{\text{univ}})^m. \]

Following our convention in this proof for the \( a_n \), we denote the image of \( b_m \in \mathcal{O}_{\mathcal{Y}_{\infty}^s}(\mathcal{Y}_{\infty}^{\text{ss}}) \) in the stalk at \( y \) also by \( b_m \). As noted before, we have on the fiber at \( y = (F, \alpha_1, \alpha_2) \) of \( \Phi \) recovers the Frobenius lift of \( F \)
\[ \sum_{m=0}^\infty b_m(y)(X^{\text{univ}}(y))^m = X^p + \pi' X = f(X). \]

In particular, by \((14)\) we have for \( m \neq 1, p \),
\[ b_m \in \bigcap_{i=0}^\infty (\pi')^i \hat{O}_{\text{LT}, \infty, y}. \]
as well as
\[ b_1 \in \pi' \hat{O}_{\text{LT}, \infty, y}, \quad b_p - 1 \in \bigcap_{i=0}^\infty (\pi')^i \hat{O}_{\text{LT}, \infty, y}. \]

In particular we have
\((18)\)
\[ \Phi^* \circ \nu^*(X_y^{\text{univ}}(Q_{\text{dr}, y} - 1)) = \sum_{m=0}^\infty b_m(X_y^{\text{univ}}(Q_{\text{dr}, y} - 1))^m \equiv (X_y^{\text{univ}})(Q_{\text{dr}, y} - 1)^p + b_1 X_y^{\text{univ}}(Q_{\text{dr}, y} - 1) \pmod{\pi' \hat{O}_{F_{\infty}^{\text{univ}, y}}^+[Q_{\text{dr}, y} - 1]} \]
for some \( b_1 \in \pi' \hat{O}_{F_{\infty}^{\text{univ}, y}}^+[Q_{\text{dr}, y} - 1] \).

Hence, in all we have
\[ X_y^{\text{univ}}(Q_{\text{dr}, y} - 1)^p + b_1 X_y^{\text{univ}}(Q_{\text{dr}, y} - 1) \equiv \Phi^* \circ \nu^*(X_y^{\text{univ}}(Q_{\text{dr}, y} - 1)) \]
\[ \pmod{\pi' \hat{O}_{F_{\infty}^{\text{univ}, y}}^+[Q_{\text{dr}, y} - 1]} \]
where we write \( \Phi^*(a_n) = (\Phi \circ \nu)^*(a_n) \) for short hand.

We show that \( a_n \in \hat{O}_{F_{\infty}^{\text{univ}, y}}^+ \) by induction on \( n \). By the above, this is true for \( n = 0, 1 \). Now assume \( n \geq 1 \) and \( a_0, \ldots, a_{n-1} \in \hat{O}_{F_{\infty}^{\text{univ}, y}}^+ \). Truncating the above congruence to terms of degree less
than or equal to \( n \) in \( Q_{\text{dR},y} - 1 \), we get
\[
(b_1 + pa_0^{n-1})a_n(Q_{\text{dR},y} - 1)^n \equiv \Phi^*(a_n)p^n(Q_{\text{dR},y} - 1)^n \pmod{(p, (Q_{\text{dR},y} - 1)^{n+1})\hat{O}_{F_{\text{univ},y}}^+[Q_{\text{dR},y} - 1]}. 
\]
By induction hypothesis, we have \( \Phi^*(a_i) \equiv a_i^p \pmod{\pi'\hat{O}_{F_{\text{univ},y}}^+} \) for \( 0 \leq i \leq n - 1 \), and so clearing terms we are left with
\[
(b_1 + pa_0^{n-1})a_n(Q_{\text{dR},y} - 1)^n \equiv \Phi^*(a_n)p^n(Q_{\text{dR},y} - 1)^n \pmod{(p, (Q_{\text{dR},y} - 1)^{n+1})\hat{O}_{F_{\text{univ},y}}^+[Q_{\text{dR},y} - 1]},
\]
and hence
\[
(1 + p/b_1a_0^{n-1})a_n = \Phi^*(a_n)p^n/b_1 \pmod{\hat{O}_{F_{\text{univ},y}}^+}.
\]
Since \( 1 + p/b_1a_0^{n-1} \in (\hat{O}_{F_{\text{univ},y}}^+)\) (for example, by (14)), this gives \( a_n \in \hat{O}_{F_{\text{univ},y}}^+ \). This completes the induction.

**Corollary 3.21.** The inclusions of (15) factor through
\[
O_{F_{\text{univ}}}^+(\hat{O}_{\text{dR},y}^+) \to \hat{O}_{\text{LT},\infty,y}^+\llbracket X_y \rrbracket \subset \hat{O}_{F_{\text{univ},y}}^+[Q_{\text{dR},y} - 1] \to \hat{O}_{F_{\text{univ},y}}^+[Q_{\text{dR},y} - 1] \subset \hat{O}_{\Delta,F_{\text{univ},y}}^+.
\]

### 3.5. The coordinate \( q_{\text{dR}} \).
Note that the section \( e : \text{LT}_\infty \to F_{\text{univ}}^+ \) is an inverse limit of open immersions \( \alpha_{2,n}^\text{univ} : \text{LT}_n \to F_n^{\text{univ}} \) by the universal property, and so in particular is an étale morphism in the proétale site. Hence we have a natural identification of proétale stalks induced by \( e^* \):
\[
e^* : \hat{O}_{\text{LT},\infty,y}^{(+)} \cong \hat{O}_{\Delta,y}^{(+)}.
\]

**Definition 3.22.** Henceforth, let
\[
q_{\text{dR}} := e^*Q_{\text{dR},y} \in \hat{O}_{\text{LT},\infty,y}.
\]
Note that the \( q_{\text{dR}}(y) \) (the image of \( q_{\text{dR}} \) under the fiber map \( \hat{O}_{\text{LT},\infty,y} \to \hat{O}_{\text{LT},\infty,y}(y) \)) is 1. By definition of \( e \), note that \( q_{\text{dR}} = (a_{2,n}^\text{univ})^*Q_{\text{dR},y} \) for all \( n \geq 0 \). Note that \( q_{\text{dR}} \) is invertible in \( \hat{O}_{\Delta,y} \), as \( \vartheta(q_{\text{dR}}) = 1 \) (and \( \ker\vartheta \) belongs to the maximal ideal of \( \hat{O}_{\Delta,y} \)).

We define
\[
\zeta_{p^n}^\text{univ} := (j\alpha_{1,n}^\text{univ} + e^*)Q_{\text{dR},y} \in \hat{O}_{\Delta,\text{LT},\infty,y}.
\]
By the previous paragraph, we could also define the above as \( (\alpha_{1,n}^\text{univ})^*Q_{\text{dR},y} \). The notation is suggestive, as we can view
\[
(\zeta_{p^n}^\text{univ})^j := \frac{(\zeta_{p^n}^\text{univ})^j q_{\text{dR}}}{q_{\text{dR}}} \in \hat{O}_{\Delta,\text{LT},\infty,y}
\]
as periods (belonging to the period ring \( \hat{O}_{\Delta,y} \)) arising from evaluations of the coordinate \( Q_{\text{dR},y} \) at the sections \( \alpha_{1,n}^\text{univ} + \alpha_{2,n}^\text{univ} \). Since \( j\alpha_{1,n}^\text{univ} = j\alpha_{1,n}^\text{univ} \), \( (\zeta_{p^n}^\text{univ})^p = \zeta_{p^{n-1}}^\text{univ} \) and \( (\zeta_{p^n}^\text{univ})^j \) is indeed the \( j \)th power of \( \zeta_{p^n}^\text{univ} \).

**Definition 3.23.** Applying \( e \) to the latter half of (19), we get
\[
O_{F_{\text{univ}}}^+(\hat{O}_{\text{dR},y}^+) \to \hat{O}_{\text{LT},\infty,y}^+\llbracket X_y \rrbracket \subset \hat{O}_{\Delta,\text{LT},\infty,y}(\hat{O}_{\text{dR},y} - 1) \subset \hat{O}_{\Delta,\text{LT},\infty,y}(\hat{O}_{\text{dR},y} - 1) \subset \hat{O}_{\Delta,\text{LT},\infty,y}.
\]

**Proposition 3.24.** We have
\[
(X_{\text{univ}}^+[j\alpha_{1,n}^\text{univ}])^j q_{\text{dR}} - 1) = X_{\text{univ}}(q_{\text{dR}}(\zeta_{p^n}^\text{univ})^j - 1).
\]
Proof. This follows from (20).

3.6. A “half” Coleman map on the anticyclotomic specialization. Let

$$U^{1,-} = U^1 \otimes_{\Lambda(\mathcal{O}_{K_p}, \mathbb{Z}_p)} \Lambda(\mathcal{O}_{K_p}^\times / \mathbb{Z}_p^\times, \mathbb{Z}_p).$$

For any $\beta = (\beta_n) \in U^1$, where $\beta_n \in L_p(F[p^n])$, so that $\beta_{2n} \in L_p(F[p^n])$, let $L_p(F[p^n])^+$ be the fixed field of $(\mathbb{Z}/p^n)^\times \subset (\mathcal{O}/p^n)^\times \cong \text{Gal}(L_p(F[p^n])/L_p)$ and let

$$\beta_n = N_{L_p(F[p^n])/L_p(F[p^n])}^{(2)}(\beta_{2n}).$$

By compositional properties of the norm, we have

$$\beta^- := \lim_{\rightarrow} \beta_n \in \lim_{\rightarrow} \mathcal{O}_{L_p(F[p^n])}^{1,-};$$

where the superscript “1” denotes principal units. In fact the above construction gives a natural map

$$U^{1,-} \to \lim_{\rightarrow} \mathcal{O}_{L_p(F[p^n])}^{1,}, \quad \bar{\beta} \mapsto \beta^-$$

where for any element $\bar{\beta} \in U^{1,-}$, $\beta \in U^1$ is any element mapping to $\bar{\beta}$ under $U^1 \to U^{1,-}$, and $\beta^-$ corresponds to $\beta$ as above.

The goal of this section is to produce a “half Coleman map” from $U^{1,-}$ to power series in $q_{\text{dR}}-1$, which will comprise one step of our map into measures. It construction involves using the coordinate $q_{\text{dR}}$ to “halve” the usual construction of Coleman power series in order to produce a map on the anticyclotomic specialization of $U^{1,-}$; using the local reciprocity law, one can view this anticyclotomic specialization as being parametrized by half of the Lubin-Tate tower $\text{Gal}(L_p(F[p^\infty])/L_p)$.

Remark 3.25. Recall that $(\alpha_n) \in T_f F$ is our chosen $f$-compatible $\mathcal{O}_{K_p}$-basis (i.e. with $\phi^{-n} f(\alpha_n) = \alpha_{n-1}$), and $(\alpha_n^{\text{univ}})$ is our universal $f$-compatible basis. We emphasize that we do not necessarily have $\alpha_{2n} = \alpha_{2,n}$, and in fact this is only so when $f = [p]$. Similarly for the universal versions $\alpha_n^{\text{univ}}$ and $\alpha_{2,n}^{\text{univ}}$.

Before beginning our construction, we note the following elementary lemma.

Lemma 3.26. Let $\pi$ be a uniformizer of $\mathcal{O}_{K_p}$. We have the following equality of subsets of the quotient group $\frac{1 + p \mathcal{O}_{K_p}}{1 + p \mathbb{Z}_p}$:

$$\frac{(1 + \pi)(1 + p \mathcal{O}_{K_p})}{1 + p \mathbb{Z}_p} = \frac{1 + \pi(1 + p \mathbb{Z}_p)}{1 + p \mathbb{Z}_p}.$$

Proof. This is a straightforward calculation following from the fact that $\frac{1 + p \mathcal{O}_{K_p}}{1 + p \mathbb{Z}_p} \cong 1 + p \mathbb{Z}_p$.

Definition 3.27. Recall that $z \in K_p$ is defined by

$$[1/z] f(\alpha_{2,n}) = \alpha_{1,n},$$

so that $1/z \in \pi \mathcal{O}_{K_p}^\times$. As it will come up frequently in the ensuing discussion, we let

$$\alpha_{2,n} := [1 + 1/z] f(\alpha_{2,n}) = \alpha_{1,n} + \alpha_{2,n}, \quad (\alpha_{2,n}^{\text{univ}})' = [1 + 1/z] f(\alpha_{2,n}^{\text{univ}}) = \alpha_{1,n}^{\text{univ}} + \alpha_{2,n}^{\text{univ}}.$$

Theorem 3.28. Let $y = (F, \alpha_1, \alpha_2) \in \text{LT}_\infty$. There is a natural $\mathbb{Z}_p$-linear map

$$\text{Col}^- : U^{1,-} \to \mathring{\hat{\mathcal{O}}}_{\text{LT}_\infty, y}[q_{\text{dR}} - 1] \times [1/p], \quad \beta^- \mapsto g^{\text{univ}}_\beta^-$$

where $g^{\text{univ}}_\beta^-$ is characterized by the following: for all $n \geq 1$,

$$\phi^{-n} g^{\text{univ}}_\beta^- (s^{\text{univ}} - 1)(y) = \beta_n^-.$$
Moreover, as in equation (3) of p. 14 of loc. cit, we will show that

\[ h_n(X^\text{univ}) \in \hat{\mathcal{O}}_{L_{\infty},y}[X^\text{univ}] \times \text{ with } h_n(\alpha_{2n}^\text{univ})(y) = \beta_{2n}; \text{ for example, we can take } h_n = \phi^{-n}g_{2n}^\text{univ}. \]

Since \( \alpha_{2n}^\text{univ} \) and \( \alpha_{2n}^\text{univ} \) are both \( (\mathcal{O}_{K_p}/p^n) \)-generators of \( F^\text{univ}[p^n] \), they differ by \([\lambda]_{\text{univ}} \) for a unit \( \lambda \in \mathcal{O}_{K_p}^\times \). Hence replacing \( h_n \) with \( h_n \circ [\lambda(1 + 1/z)^{-1}]_{\text{univ}} \), we may assume without loss of generality that \( h_n((\alpha_{2n}^\text{univ})')(y) = \beta_{2n} \). Define

\[ h_n^+(X) = \prod_{j=0, (j,p)=1}^{p^n-1} h_n([j]_f(X)), \]

so that \( h_n^+(\alpha_{2n}^\text{univ}) = \beta_{2n} \). We will now need the following norm operator.

**Definition 3.29.** Define a norm operator \( N_p \) on with respect to the operator

\[ [p](q_{\text{DR}} - 1) = (q_{\text{DR}})^p - 1, \]

so that for any \( H(q_{\text{DR}} - 1) \in \hat{\mathcal{O}}_{L_{\infty},y}[q_{\text{DR}} - 1], \)

\[ N_p H((q_{\text{DR}})^p - 1) = N_p H([p](q_{\text{DR}} - 1)) = \prod_{j=0}^{p-1} H(q_{\text{DR}}(\zeta_{p^n}^j - 1)). \]

Now as in equation (2) of p. 14 of loc. cit., we have for all \( 1 \leq m \leq n \), using Lemma 3.26

\begin{equation}
(22) \quad (N_{\phi^{-n}[p]}h_n^+)X^{\text{univ}}(\zeta_m^m - 1) = (N_{\phi^{-n}[p]}h_n^+)\phi^{-n}[p])\zeta_m^m - 1
\end{equation}

\[ = \prod_{j=1}^{p^n-m} h_n^+(X^{\text{univ}}(\zeta_m^m - \zeta_m^m)^j - 1)) \]

\[ = \prod_{\lambda \in ((\mathcal{O}_{K_p}/p^n)^\times)/((\mathcal{O}_{K_p}/p^n)^\times)/(Z/p^n)^\times} h_n^+([\lambda]_{f}((\alpha_{2n}^\text{univ})')) \]

\[ = \beta_{-n}. \]

Moreover, as in equation (3) of p. 14 of loc. cit., we will show that

\begin{equation}
(23) \quad \frac{N_{\phi^{-n}[p]}h_n^+(X^{\text{univ}}(q_{\text{DR}} - 1))}{\phi^{-m}N_{\phi^{-n}[p]}h_n^+} = N_{\phi^{-n}[p]}(N_{\phi^{-n}[p]}h_n^+) \equiv 1 \quad (\text{mod } p'\hat{\mathcal{O}}_{L_{\infty},y}[q_{\text{DR}} - 1])
\end{equation}

where we recall that \( p' \) is the prime ideal of \( \mathcal{O}_{L_p} \). The first equality follows from standard properties of the norm operator (cf. Proposition I.2.1 (ii) of loc. cit.). We prove the following lemmas about \( N_p \).

**Lemma 3.30.** For any \( H(q_{\text{DR}} - 1) \in \hat{\mathcal{O}}_{L_{\infty},y}[q_{\text{DR}} - 1], \)

\[ N_{\phi^{-n}[p]}H(q_{\text{DR}} - 1) \equiv H^\phi(q_{\text{DR}} - 1) \quad (\text{mod } p'\hat{\mathcal{O}}_{L_{\infty},y}[q_{\text{DR}} - 1]) \]

**Proof of Lemma 3.30.** We have

\[ N_{\phi^{-n}[p]}H((q_{\text{DR}})^p - 1) = N_{\phi^{-n}[p]}H([p](q_{\text{DR}} - 1)) = \prod_{j=0}^{p-1} H(q_{\text{DR}}(\zeta_{p^n}^j - 1)) \equiv H^\phi((q_{\text{DR}})^p - 1) \quad (\text{mod } (\zeta_{p^n}^n - 1)\mathcal{O}_{L_{\infty},\zeta_{p^n}^n, y}[q_{\text{DR}} - 1]) \]
where \( LT_{\infty,\mathcal{C}_{\text{univ}}}^+ \rightarrow LT_{\infty} \) is the étale torsor trivializing \( \mu_{p,LT_{\infty}} \), and \( LT_{\infty,\mathcal{C}_{\text{univ}},y}^+ \) is the proétale stalk at the fiber above \( y \). However, the right-hand side of the above term clearly belongs to \( \tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1] \) since it is invariant under the action of \( \text{Gal}(LT_{\infty,\mathcal{C}_{\text{univ}}}^+/LT_{\infty}) \). Thus the above congruence holds modulo

\[ (\zeta_{\text{univ}}^n - 1)O_{LT_{\infty,\mathcal{C}_{\text{univ}},y}}^+[q_{\text{dR}} - 1] \cap \tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1] = p\tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1], \]

which gives \( [24] \).

**Lemma 3.31.** For any \( H(q_{\text{dR}} - 1) \in \tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1] \),

\[ H(q_{\text{dR}} - 1) \equiv 1 \pmod{(p')^i\tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1]} \]

\[(26) \implies N_{\phi^{-n}[p]}H(q_{\text{dR}} - 1) \equiv 1 \pmod{(p')^{i+1}\tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1]} \]

*Proof of Lemma [3.33]* As on p. 13 of loc. cit., we prove this inductively. For \( i = 1 \) the claim is clear. Now for \( i \geq 1 \), we have

\[ N_{\phi^{-n}[p]}H((q_{\text{dR}})^p - 1) \equiv N_{\phi^{-n}[p]}H \circ \phi^{-n}[p](q_{\text{dR}} - 1) \equiv H(q_{\text{dR}} - 1)^p \]

\[ \equiv 1 \pmod{(p')^i(z_{\text{univ}}^n - 1)O_{LT_{\infty,\mathcal{C}_{\text{univ}},y}}^+[q_{\text{dR}} - 1]} \]

since

\[ H(q_{\text{dR}} - 1) \equiv N_{\phi^{-n}[p]}H(q_{\text{dR}} - 1) \equiv 1 \pmod{(p')^i\tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1]} \]

by induction hypothesis. However, since \( N_{\phi^{-n}[p]}H(q_{\text{dR}} - 1) \in \tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1] \), then from \( [25] \) we have

\[ N_{\phi^{-n}[p]}H((q_{\text{dR}})^p - 1) \equiv 1 \pmod{(p')^{i+1}\tilde{O}_{LT_{\infty,\mathcal{C}_{\text{univ}},y}}^+[q_{\text{dR}} - 1]} \].

\[ \square \]

Now \( [23] \) follows from \( [24] \) and \( [26] \). Letting \( g_n(q_{\text{dR}} - 1) = N_{\phi^{-n}[p]}^n(X_{\text{univ}}^n(q_{\text{dR}} - 1)) \), \( [22] \) and \( [23] \) imply that for all \( 1 \leq m \leq n \),

\[ (\phi^{-m}g_n)(\zeta_{\text{univ}}^n - 1)(y)/\beta_m^+ \equiv 1 \pmod{(p')^{n-m+1}} \]

Since \( \tilde{O}_{LT_{\infty},y}^+[q_{\text{dR}} - 1] \) is compact, this implies that \( \{g_n\} \) has limit point, which we call \( g_{\beta^-}^\text{univ} \). Then by continuity, \( \phi^{-n}g_{\beta^-}(\zeta_{\text{univ}}^n - 1) = \beta_m^- \) for all \( 1 \leq n \), and the Weierstrass preparation theorem shows that this uniquely determines \( g_{\beta^-}^\text{univ} \) and in fact \( g_{\beta^-}^\text{univ} = \lim_{n \to \infty} g_n \).

\[ \square \]

**Definition 3.32.** Given \( \lambda \in (1 + p\mathcal{O}_{K_p})/(1 + p\mathbb{Z}_p) \subset \mathcal{O}^\times_{K_p}/\mathbb{Z}_p^\times \), let \( \tilde{\lambda} \in 1 + p\mathbb{Z}_p \) be such that

\[ (1 + 1/\lambda)(1 + p\mathbb{Z}_p) = (1 + 1/\tilde{\lambda})(1 + p\mathbb{Z}_p) \]

as in Lemma 3.26 with \( \pi = 1/\tilde{\lambda} \). The following corollary is proved in the same way as in loc. cit., adapting the proof for the norm operator \( N_{[p]} \).

We let \([\cdot]\) denote the standard \( \mathbb{Z}_p \)-module structure on \( \hat{\mathcal{C}}_{m,LT} \), so that

\[ [\lambda](q_{\text{dR}} - 1) = (q_{\text{dR}})^\lambda - 1. \]
Proof. (1) and (2) are clear and follow from the argument of loc. cit., \textit{mutatis mutandis}. For (3), let \( \tilde{\lambda} \in 1 + p\Zp \) be as in Definition 3.32. In other words, we have
\[
\langle (1 + 1/z) f(\alpha_2) \rangle = \langle [\tilde{\lambda} - 1]/(1 + 1/z) f(\alpha_2) \rangle.
\]
Note that \( [1 + 1/z] f(\alpha_2) = \alpha_1 + \alpha_2 = \alpha'_2 \) by definition, and so we could rewrite the above equation as
\[
\langle \lambda f(\alpha'_2) \rangle = \langle [\tilde{\lambda} - 1]'(\alpha'_2) \rangle.
\]
Now (3) follows from the construction of \( g_{\beta^-}. \)

3.7. Stabilized logarithm of the half Coleman map.

\textbf{Definition 3.34.} Given \( H(q_{\text{dR}} - 1) \in \hat{O}_{\LT_{\infty,y}}[q_{\text{dR}} - 1][1/p] \), we define
\[
\tilde{H}(q_{\text{dR}} - 1) := H(q_{\text{dR}} - 1) - \frac{1}{p} \sum_{j=0}^{p-1} H(q_{\text{dR}}(\zeta_1^{\text{univ}})^j - 1).
\]
Further, given \( \beta^- \in U_{1}^- \), define
\[
G_{\beta^-}(q_{\text{dR}} - 1) := \log(g_{\beta^-}(q_{\text{dR}} - 1)).
\]

\textbf{Proposition 3.35.} We have
\[
G_{\beta^-}(q_{\text{dR}} - 1) \in \hat{O}_{\LT_{\infty,y}}[q_{\text{dR}} - 1] \cdot \frac{1}{\pi},
\]
where \( \pi \) is a local uniformizer of \( \mathcal{O}_K \).

\textbf{Proof.} By properties of the half Coleman map, we have
\[
(g_{\beta^-}^{\text{univ}})^p(q_{\text{dR}} - 1) \equiv (g_{\beta^-}^{\text{univ}})^p \circ [p](q_{\text{dR}} - 1) = N[p]g_{\beta^-}^{\text{univ}} \circ [p](q_{\text{dR}} - 1)
\]
\[
= \prod_{j=0}^{p-1} g_{\beta^-}^{\text{univ}}(q_{\text{dR}}(\zeta_1^{\text{univ}})^j - 1) \pmod{p'\hat{O}_{\LT_{\infty,y}}[q_{\text{dR}} - 1]},
\]
and hence
\[
pG_{\beta^-}(q_{\text{dR}} - 1) \equiv \sum_{j=0}^{p-1} G_{\beta^-}(q_{\text{dR}}(\zeta_1^{\text{univ}})^j - 1) \pmod{p'\hat{O}_{\LT_{\infty,y}}[q_{\text{dR}} - 1]}.
\]
Dividing by \( p \) gives the claim.

\textbf{Definition 3.36.} We define a map
\[
(27) \quad \mu_{q_{\text{dR}}} : U_{1}^- \otimes_{\Zp} \hat{O}_{\LT_{\infty,y}}[1/p] \to \hat{O}_{\LT_{\infty,y}}[q_{\text{dR}} - 1][1/p]
\]
by
\[
\mu_Q(\beta^-) = G_{\beta^-}(q_{\text{dR}} - 1) \forall \beta^- \in U_{1}^-.
\]
and extending \( \otimes_{\Zp} \hat{O}_{\LT_{\infty,y}}[1/p] \)-linearly.
So far $\mu_{q_{\text{dir}}}$ is only $\Lambda(\mathcal{O}_{K_p}^\times /\mathbb{Z}_p^\times, \hat{\mathcal{O}}_{LT,\infty,\beta}^+)[1/p]$-linear with respect to the “non-standard action” on its image

$$(\lambda \cdot G_{\beta^-})(q_{\text{dir}} - 1) = G_{\kappa^{-1}(\lambda)(\beta^-)}(q_{\text{dir}} - 1)$$

for any $\lambda \in \mathcal{O}_{K_p}^\times /\mathbb{Z}_p^\times$ and arbitrary lift $\lambda \in \mathcal{O}_{K_p}^\times$ of $\lambda$. However, we wish to interpret the target as inside the Iwasawa algebra on $\mathbb{Z}_p^\times$, which would inherit the standard action for $\lambda \in \mathbb{Z}_p^\times$.

$$[\lambda]^*G_{\beta^-}(q_{\text{dir}} - 1) = G_{\beta^-}([\lambda](q_{\text{dir}} - 1)) = G_{\beta^-}((q_{\text{dir}})^{\lambda} - 1),$$

that induces the standard Iwasawa module structure. We will find an isomorphic $\mathcal{O}_{K_p}^\times$-module structure on the source of $\mu_{q_{\text{dir}}}$ such that precomposing $\mu_{q_{\text{dir}}}$ with this induced isomorphism will make the resulting map $\Lambda(\mathcal{O}_{K_p}^\times /\mathbb{Z}_p^\times, \hat{\mathcal{O}}_{LT,\infty,\beta}^+)[1/p]$-equivariant with respect to the standard Iwasawa module structure.

### 3.8. Isotypic components

For any character $\chi : \Delta \to \overline{\mathbb{Q}}^\times_p$, we have that $\chi$ is always valued in $\Delta$ and hence in $\mathcal{O}_{K_p}$. For a $\Lambda(\mathcal{O}_{K_p}^\times, \mathcal{O}_{K_p})$-module $M$, let

$$M_\chi = M \otimes_{\Lambda(\mathcal{O}_{K_p}, \mathcal{O}_{K_p})} \Lambda(\Gamma, \mathcal{O}_{K_p})$$

denote the $\chi$-isotypic component of $M$, where the structure map for the tensor product is given by $\chi : \Delta \to \mathcal{O}_{K_p}^\times$. By [23], we have that for $\chi \neq 1$, $U_1^{\chi}$ is free of $\Lambda(\Gamma, \mathcal{O}_{K_p})$-module of rank 2.

**Definition 3.37.** Henceforth, fix $\chi \neq 1$ and fix a basis $\beta_1, \beta_2$ of $U_1^{\chi}$.

We have a natural map

$$M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \bigoplus_{\chi : \Delta \to \overline{\mathbb{Q}}^\times_p} U_1^{\chi} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad x \mapsto x_\chi := \frac{1}{#\Delta} \sum_{\sigma \in \Delta} \chi^{-1}(\sigma)x^\sigma.$$

**Definition 3.38.** Using the structure theorem of finitely generated modules over PIDs, we have an isomorphism $\mathcal{O}_{K_p}^\times = \Delta \times \Gamma$ where $\Delta$ is a finite abelian group and $\Gamma \cong \mathbb{Z}_p^{\oplus 2}$. Using the $p$-adic logarithm, we have $\Delta = (\mathcal{O}_{K_p}/\pi^\epsilon)^\times$ where

$$\epsilon = \left\lfloor \frac{1}{\text{ord}_p(\pi)(p-1)} \right\rfloor + 1.$$

Moreover, from the norm we have an exact sequence

$$1 \to \Gamma' := \Gamma^{\text{Nm}_{K_p/\mathbb{Q}_p}=1} \to \Gamma \xrightarrow{\text{Nm}_{K_p/\mathbb{Q}_p}} \Gamma_+ := 1 + q\mathbb{Z}_p \to 1$$

where $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. Define

$$\Gamma_- := \Gamma/(1 + q\mathbb{Z}_p) \cong 1 + p\mathbb{Z}_p.$$

(Note that for $p = 2$, the above is indeed isomorphic to $1 + 2\mathbb{Z}_2$ since $q = 4$.) Moreover,

$$\Delta/(\mathbb{Z}/q)^\times \cong \begin{cases} \mathbb{Z}/p & p = 2, 3 \\ 1 & p > 3. \end{cases}$$
3.9. The measure map. Let $G_{\beta,-}(q_{\text{dR}} - 1)$ denote the image of $G_{\beta,-}(q_{\text{dR}} - 1)$ in the isotypic component $\mathcal{O}_{LT,\infty}^+[q_{\text{dR}} - 1][1/p]$. Using $(\alpha_1, \alpha_2) : \mathbb{Z}_p^2 \xrightarrow{\sim} T_p F$ and the $\mathcal{O}_K^\times$-action, we get an embedding $\mathcal{O}_K^\times \subset GL_2(\mathbb{Z}_p)$ and $\Delta/(\mathbb{Z}/q)^\times \times \Gamma_- = \mathcal{O}_K^\times / \mathbb{Z}_p^\times \subset PGL_2(\mathbb{Z}_p)$.

**Definition 3.39.** Assume that $\epsilon \geq 2$, which implies $p = 2, 3$. Then $\Gamma = 1 + p^e \mathcal{O}_K \subset 1 + p \mathcal{O}_K$. Recall our basis $\beta_1, \beta_2$ of $\mathcal{U}_1$. Fix an isomorphism

$$i : \Gamma_- \cong 1 + p\mathbb{Z}_p.$$ 

Hence we can use $i$ to view $\Gamma_- \cong 1 + p\mathbb{Z}_p \subset \hat{\mathcal{O}}_p$. We will construct measures on $\mathbb{Z}_p^\times$ from power series in $q_{\text{dR}} - 1$, and hence measures on $\Gamma_-$. We now define a map for $j = 1, 2$,

$$G_{\kappa^{-1}(\gamma),(\beta_j),\chi}^{-} \mapsto (G_{\beta_j,\chi}) \circ [i(\gamma)],$$

and extend linearly to get an automorphism $\xi$ of $\Lambda(1 + p\mathbb{Z}_p, \hat{\mathcal{O}}_{LT,\infty}^+[y])[1/p]$-modules on the image of $\mu_{q_{\text{dR}}}^\times$.

**Definition 3.40 (Measure map).** Now define the $\Lambda(1 + p\mathbb{Z}_p, \hat{\mathcal{O}}_{LT,\infty}^+[y])[1/p]$-linear map

$$\mu := \xi \circ \mu_{q_{\text{dR}}} : \mathcal{U}_1^{-} : \hat{\mathcal{O}}_{LT,\infty}^+[1/p] \to \hat{\mathcal{O}}_{LT,\infty}^+[Q - 1][1/p] \subset \Lambda(1 + p\mathbb{Z}_p, \hat{\mathcal{O}}_{LT,\infty}^+[y])[1/p].$$

The source is a free $\Lambda(1 + p\mathbb{Z}_p, \hat{\mathcal{O}}_{LT,\infty}^+[y])[1/p]$-module of rank 2, with basis $\beta_1, \beta_2$, and the target clearly has $\Lambda(1 + p\mathbb{Z}_p, \hat{\mathcal{O}}_{LT,\infty}^+[y])[1/p]$-rank at most 1. In particular, $\mu$ has a kernel which we call $\mathcal{M}$. Let $\mathcal{W}$ denote its cokernel. Using the fiber specialization map at the point $y = (F, \alpha_1, \alpha_2)$

$$\hat{\mathcal{O}}_{LT,\infty}^+[y] \to \hat{\mathcal{O}}_{LT,\infty}^+[y] =: \mathcal{O}_{L_{p,\infty}},$$

where $\hat{\mathcal{O}}_{LT,\infty}^+[y]$ denotes residue field, we get a map

$$\mu(y) : \mathcal{U}_1^\times : \hat{\mathcal{O}}_{LT,\infty}^+[y] \to \Lambda(1 + p\mathbb{Z}_p, \mathcal{O}_{L_{p,\infty}})[1/p].$$

**Remark 3.41.** While the definition of $\mu$ may at first seem unnatural, as it depends on choices of $\beta_1, \beta_2$, $(\alpha_1, \alpha_2)$, and coset representatives, we shall see that it still satisfies expected explicit reciprocity laws on elliptic units $\mathcal{C}$. That is, $\mu(\mathcal{C})$ will still be related to logarithms of elliptic units, and twists of $\mu$ will be related to $L$-values.

**Remark 3.42.** Perhaps a conceptual way to view the role of $q_{\text{dR}}$ is that it gives a “$\mathbb{Z}_p$-linearized parameter” the $\Gamma_-$-action.

3.10. Extending $\mu$.

**Definition 3.43.** Recall that $\text{Gal}(L_p/K_p) = \langle \phi \rangle$, where $\phi^d = 1$, and $\mathcal{G} = \text{Gal}(L_p(F[p^\infty])/L_p)$. Fix a decomposition

$$\tilde{\mathcal{G}} := \text{Gal}(L_{p,\infty}/K_p) = \mathcal{G} \times \langle \phi \rangle.$$

The above direct product decomposition exists by ramification theory (note that $L_{p,\infty}/L_p$ is totally ramified, whereas $L_p/K_p$ is unramified), and the fact that $\text{Gal}(L_{p,\infty}/K_p)$ is abelian. Let

$$\tilde{\Gamma}_- := \Gamma_- \times \langle \phi \rangle.$$ 

Let

$$\tilde{\Delta} = \Delta \times \langle \phi \rangle,$$

and let $\tilde{\chi} : \tilde{\Delta} \to \overline{\mathcal{O}_p}^\times$ be a character whose restriction to $\Delta$ is equal to $\chi$. 

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Note that there is a natural action of \( \tilde{G} \) on \( U^1 \), and let
\[
U^1_{\chi} = U^1 \otimes_{\Lambda(\tilde{G}, \mathbb{Z}_p), \tilde{\chi}} \Lambda(\Gamma, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]),
\]
where the structure map is given by \( \tilde{\chi} : \Delta \to \mathbb{Q}_p \). Since \( \tilde{\chi} \) extends \( \chi \), there is a natural isomorphism of \( \Lambda(\Gamma_-, \mathcal{O}_K, [\tilde{\chi}]) \)-modules
\[
U^1_{\chi} \otimes_{\Lambda(\mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}])} \Lambda(\Gamma, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]) \cong U^1_{\chi} \otimes_{\Lambda(\mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}])} \Lambda(\Gamma, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]).
\]
We can thus extend the map \( (11) \) to a \( \Lambda(\Gamma_-, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]) \)-equivariant map
\[
\mu : U^1_{\chi} \otimes_{\Lambda(\mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}])} \Lambda(\Gamma, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]) \to \Lambda(\Gamma_-, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]),
\]
in the following way. For any open set \( U \subset \Gamma_- \) such that \( \delta U \subset \Gamma_- \) for some \( \delta \in \langle \phi \rangle \), we define
\[
\mu(\beta)(U) = \mu(\kappa^{-1}(\delta^{-1})(\beta))(\delta U).
\]

**Definition 3.44.** Henceforth, for \( \beta \in U \), we often will adopt the notation
\[
\mu_\beta := \mu(\beta) \in \Lambda(\Gamma_-, \mathcal{O}_K, \mathcal{O}_K, [\tilde{\chi}]).
\]
We adopt analogous notation for all restrictions and extensions of \( \mu \).

### 4. The Main Conjectures

In this section, we formulate and then prove Rubin-type Main Conjectures for imaginary quadratic fields in which \( p \) is inert or ramified. The key is to use the kernel of \( \mu_{\mathcal{O}_L} \) as a \( \Lambda \)-adic local condition to define the relevant torsion \( \Lambda_{\mathcal{O}_L} \)-module that will appear in the Main Conjecture (Conjecture 4.35).

#### 4.1. Construction of the measure on the global Galois group.

**Definition 4.1.** Let \( \mathfrak{f} \subset \mathcal{O}_K \) be an integral ideal prime to \( p \). Suppose that \( w_\mathfrak{f} = 1 \). Let \( L := K(\mathfrak{f}) \), the ray class field of conductor \( \mathfrak{f} \) over \( K \). Henceforth fix an elliptic curve \( A \) such that

1. \( A \) has CM by \( \mathcal{O}_K \),
2. \( A \) is defined over \( K(\mathfrak{f}) \),
3. \( A^{\text{tors}} \subset A(K^{ab}) \),
4. \( A \) has good reduction at all primes of \( L \) not dividing \( \mathfrak{f} \).

As remarked in \([39] \text{ p. 4}\), the existence of such an elliptic curve is proven in \([54] \text{ p. 216}\) and \([53] \text{ Chapter II, Lemma 1.4}\). Let \( \mathcal{O}_{K,p} \) denote the localization of \( \mathcal{O}_K \) at \( p \) (i.e., inverting all elements outside of \( p \)). Then let \( R \) be the integral closure of \( \mathcal{O}_{K,p} \) in \( L \). We then fix a minimal generalized Weierstrass model of \( E \) over \( R \),
\[
y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.
\]
We let
\[
\omega_A = \frac{dx}{2y + a_1 x + a_3} \in \Omega^1_{A/R}
\]
be the usual holomorphic invariant differential attached to the above Weierstrass model. The pair \( (A, \omega_A) \) determines a unique \( \mathcal{O}_K \)-lattice with
\[
\theta_{\infty, A} : \mathbb{C}/L \cong A(\mathbb{C}).
\]
Here, for all \( z \in \mathbb{C} \setminus L \), \( \theta_{\infty, A}(z) \) is the unique point with coordinates
\[
x(z) := \wp_L(z) - b_2/12, \quad y(z) := \wp'(z) - a_1 x(z) - a_3)/2
\]
where \( b_2 := a_1^2 + 4a_2 \). Let \( \hat{A} \) be the formal group of \( A \), with parameter \( t := -x/y \). We note that the function field of \( E/\overline{\mathbb{Q}} \) is \( \overline{\mathbb{Q}}(\wp_L, \wp_L') \), and we denote the natural map \( \overline{\mathbb{Q}}(\wp_L, \wp_L') \to \mathbb{C}_p[t] \) (taken
with respect to the embedding \( i_p \) fixed in (3) obtained by expanding a rational function on \( E \) in terms of the formal parameter \( t \) by
\[
f \mapsto \hat{f}.
\]

**Definition 4.2.** Retain the notation of Definition [4.1]. Fix an ideal \( \mathfrak{f} \subset \mathcal{O}_K \); we do not impose \( \mathfrak{f} \neq 1 \) or \((\mathfrak{f}, p) = 1 \) unless otherwise specified. Given any ideal \( \mathfrak{c} \subset \mathcal{O}_K \), let \( K(\mathfrak{c}) \) be the ray class field over \( K \) of conductor \( \mathfrak{c} \). Henceforth let \( L = K(\mathfrak{f}) \), and for \( 0 \leq n \leq \infty \) let \( L_n = K(\mathfrak{f})(A[p^n]) \). As before, let \( A/L \) be an elliptic curve with complex multiplication by \( \mathcal{O}_K \), and such that \( L(A_{\text{tors}})/K \) is abelian. Equivalently, we have
\[
\psi_{A/L} = \varphi \circ \text{Nm}_{L/K}
\]
where \( \psi_{A/L} : \hat{A}^\times_L \to \mathbb{C}^\times \) is the Hecke character associated with \( A/L \) by the theory of complex multiplication and \( \varphi : \hat{A}^\times_K \to \mathbb{C}^\times \) is some Hecke character of infinity type \((1, 0)\).

**Definition 4.3.** Given any algebraic Hecke character \( \chi : \hat{A}^\times_K \to \mathbb{C}^\times \), we let \( \hat{\chi} : \hat{A}^\times_K \to \mathbb{C}^\times \) denote its \( p \)-adic avatar. Given any \( p \)-adic algebraic Hecke character \( \chi : \hat{A}^\times_K \to \mathbb{C}^\times \), we let \( \hat{\chi} \) denote its complex avatar.

**Definition 4.4.** Let
\[
\begin{align*}
G_n &:= \text{Gal}(L_n/K), \quad G_n^+ := \text{Gal}(L(\mu_{p^n})/K), \quad G_\infty := \text{Gal}(L_\infty/K), \quad G_\infty^+ := \text{Gal}(L(\mu_{p^\infty})/K).
\end{align*}
\]
Let \( \hat{G}_\infty \) denote the group of (continuous) \( p \)-adic characters on \( G_\infty \), and similarly for the other above groups. When we wish to emphasize the dependence on the auxiliary conductor \( \mathfrak{f} \), we will write \( G_\infty(\mathfrak{f}) \), and similarly for other groups. Let
\[
\Phi_n := L_n \otimes_K K_p \cong \bigoplus_{\mathfrak{p} \mid p} L_n, \quad R_n = \mathcal{O}_{L_n} \otimes_K \mathcal{O}_{K_p} \cong \bigoplus_{\mathfrak{p} \mid p} \mathcal{O}_{L_n, \mathfrak{p}}
\]
where \( \mathfrak{p} \mid p \) runs over prime ideals of \( \mathcal{O}_{L_n} \) above \( p \). (Recall that \( K_p = K_p \) with respect to our fixed embeddings \([\mathfrak{f}]\).) For simplicity let \( \Phi_0 = \Phi \). Note that we have norm maps \( \text{Nm}_n : \Phi_n \to \Phi_{n-1} \) and \( \text{Nm}_n : R_n \to R_{n-1} \). Let
\[
\mathbb{U} := \lim_{\to \text{Nm}_n} R_n^\times.
\]

**Definition 4.5.** In the notation of (33), let \( R^1_n \) denote the pro-\( p \) part of \( R_n^\times \), i.e. the semi-local principal units. Then let
\[
\mathbb{U}^1 := \lim_{\to \text{Nm}_n} R^1_n.
\]

**Definition 4.6.** Note that we still have a natural isomorphism \( \text{Gal}(L_\infty/L(A[p^n])) \cong \Gamma \). Let \( \hat{\chi} : \text{Gal}(L(A[p^n])/K) \to \mathbb{C}^\times \) be any non-trivial character. We wish to construct a semilocal version of the map \( \mu : \mathcal{U}_K^{1, -} \otimes \mathcal{O}_{K_p}[x] / \hat{\chi} \hat{\mathcal{O}}_{L_\infty, y} [1/p] \to \Lambda(\Gamma, \hat{\mathcal{O}}_{L_\infty, y} [1/p]) \).

**Theorem 4.7.** Assume \((\mathfrak{f}, p) = 1 \). There is a \( \Lambda(\Gamma, \hat{\mathcal{O}}_{L_\infty, y} [1/p]) \)-equivariant map
\[
\mu_{\text{glob}} : \mathcal{U}^1 \otimes \mathcal{O}_{K_p}[x] / \hat{\mathcal{O}}_{L_\infty, y} [1/p] \to \Lambda(\text{Gal}(L/K) \times \Gamma, \hat{\mathcal{O}}_{L_\infty, y} [1/p])
\]
\[
\mu_{\text{glob}} : \mathcal{U}^1 \otimes \mathcal{O}_{K_p}[x] / \hat{\mathcal{O}}_{L_\infty, y} [1/p] \to \Lambda(\text{Gal}(L/K) \times \Gamma, \hat{\mathcal{O}}_{L_\infty, y} [1/p])
\]
\[
\mu_{\text{glob}} : \mathcal{U}^1 \otimes \mathcal{O}_{K_p}[x] / \hat{\mathcal{O}}_{L_\infty, y} [1/p] \to \Lambda(\text{Gal}(L/K) \times \Gamma, \hat{\mathcal{O}}_{L_\infty, y} [1/p])
\]

**Proof.** This follows from Theorem 4.2, applied to \( F = (\hat{A}, [\mathfrak{p}]) \) over \( L_p = L_\mathfrak{p} \) for any prime \( \mathfrak{p} \mid p \) of \( L \) (varying the embeddings \((3)\)), pulling back the resulting map \( \mu \) via the natural projection (induced from \( i_p \) from (3))
\[
\mathbb{U}^1 \to \mathcal{U}^1.
\]
To extend from \( \Gamma_\infty \) to \( \text{Gal}(L/K) \times \Gamma_\infty \), we use the analogous construction as in Definition 31. □
When we wish to emphasize \( f \) in the definition of \( \mathbb{U} \), we will write \( \mathbb{U}(f) \) for \( \mathbb{U} \). When we wish to emphasize \( \tilde{\chi} \) in the definition of \( \mu_{\text{glob}} \) we will write \( \mu_{\text{glob}}(\tilde{\chi}) \). We denote the image of \( \mu_{\text{glob}} \) under the fiber specialization \( \hat{\Omega}_{\text{LT}, \infty, y}^+ \rightarrow \mathcal{O}_{L_p, \infty}^+ \) by \( \mu_{\text{glob}}(y) \) or \( \mu_{\text{glob}, f}(y) \).

**Proposition 4.8** (cf. Proposition III.1.3 of [53]). There is a unique way to extend \( \mathbb{U} \) to a map

\[
\mu_{\text{glob}, f} : \mathbb{U}(f)^{1, -}_{\tilde{\chi}} \otimes_{O_K[\tilde{\chi}]} \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p] \rightarrow \Lambda(\text{Gal}(K(f)/K) \times \Gamma_{-}, \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p])
\]

for any \( f \in O_K \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{U}(f)^{1, -}_{\tilde{\chi}} \otimes_{Z_p} \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p] & \xrightarrow{\mu_{\text{glob}, f}^{\otimes_{Z_p}}} & \Lambda(\text{Gal}(K(f)/K) \times \Gamma_{-}, \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p]) \\
\downarrow \text{inclusion} & & \downarrow \text{cores} \\
\mathbb{U}(g)^{1, -}_{\tilde{\chi}'} \otimes_{Z_p} \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p] & \xrightarrow{\mu_{\text{glob}, g}^{\otimes_{Z_p}}} & \Lambda(\text{Gal}(K(g)/K) \times \Gamma_{-}, \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p]) \tilde{\chi}' \\
\downarrow \text{norm} & & \downarrow \text{res} \\
\mathbb{U}(f)^{1, -}_{\tilde{\chi}} \otimes_{Z_p} \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p] & \xrightarrow{\mu_{\text{glob}, f}^{\otimes_{Z_p}}} & \Lambda(\text{Gal}(K(f)/K) \times \Gamma_{-}, \hat{\Omega}_{\text{LT}, \infty, y}^+[1/p]) \tilde{\chi}
\end{array}
\]

for all \( f, g \) with \( (g, p) = 1 \) (see [53] Lemma III.1.2), and \( \chi' : \text{Gal}(K(\mathfrak{g}p^\infty)/K) \rightarrow \overline{\mathbb{Q}}_p^\times \) factoring through \( \tilde{\chi} : \text{Gal}(K(\mathfrak{f}p^\infty)/K) \rightarrow \overline{\mathbb{Q}}_p^\times \).

**Proof.** One uses the same diagrams and argument as in [53] Lemma III.1.2 and Proposition III.1.3], except that the index of the image (i.e. \( w_f \)) of the norm map in the proof of Proposition 1.3 may be non-trivial, and hence it is necessary to invert \( p \) to make it surjective. In particular, \( \mu_{\text{glob}}(f) \) is explicitly defined by choosing \( g \in O_K \) with \( [g, (g, p) = 1 \) and \( w_g = 1 \) so that \( \mu_{\text{glob}}(g) \) is already defined, and letting

\[
\mu_{\text{glob}, f} := \frac{1}{\# \text{Gal}(K(\mathfrak{g}p^\infty)/K(\mathfrak{f}p^\infty))} \text{res} \circ \mu_{\text{glob}, g} \circ \text{inclusion}.
\]

\[ \square \]

**Definition 4.9.** Henceforth, let

\[
\mathcal{M} := \ker(\mu_{\text{glob}}),
\]

which is a finitely generated \( \Lambda(G, \mathcal{O}_{L_p, \infty}) \)-module.

**Definition 4.10** (Elliptic functions). We recall the elliptic functions of Robert [42], following [39] Section 3. Suppose \( L \subset L' \) are two lattices in \( \mathbb{C} \) such that \( ([L' : L], 6) = 1 \), and let \( \wp_L \) be the Weierstrass elliptic function attached to \( L \). Then the Robert elliptic function ([42]) is

\[
\psi(z; L, L') := \delta(L, L') \prod_{\rho \in \mathcal{Z}(L, L')} (\wp_L(z) - \wp_L(\rho))^{-1},
\]

where \( \delta(L, L') \) is the canonical 12th root of \( \Delta(L)[L':L]/\Delta(L') \) defined in [43]. As recalled in [39] p.3, these functions satisfy the usual distribution and norm compatibility relations:

\[
\psi(z; L', a^{-1}L') = \prod_{\rho} \psi(z + \rho; L, a^{-1}L),
\]

\[
\text{Nm}_{K^{(m'/q')}/K^{(m')}}(\psi(\rho; L, a^{-1}L)) \overset{w_{m'/q'}}{=} \begin{cases} 
\psi(\rho; q^{-1}L, a^{-1}q^{-1}L) & q \not| m' \\
\psi(\rho; q^{-1}L, a^{-1}q^{-1}L)^{1-(q, K(m'))^{-1}} & q | m'
\end{cases}
\]

where \( m'' = m' \).
Here, given an integral ideal $n \subset O_K$, $w_n$ denotes the number of roots of unity in $O_K^\times$ which are congruent to $1 \pmod{n}$. Let $m \neq O_K$ be a proper integral ideal with $(m, p) = 1$, and let $\Omega \in \mathbb{C}$ be a primitive $m$-torsion point of $\mathbb{C}/L$, i.e. with $m = \Omega^{-1} L \cap O_K$. For any ideal $b$ with $(b, 6mfp) = 1$, we define

\begin{equation}
\psi_{\Omega, b}(z) := \psi(z + \Omega; L, b^{-1}L).
\end{equation}

If we further fix an integral ideal $a$ such that $(a, 6mfp) = 1$ and $(b, 6amfp) = 1$, we define

\begin{equation}
\psi_{\Omega, a, b}(z) := \psi(z + \Omega; a^{-1}L, a^{-1}b^{-1}L).
\end{equation}

**Proposition 4.11** (Proposition 3.1 of [39]). We have

\[ \hat{\psi}_{\Omega, b}(t), \hat{\psi}_{\Omega, a, b}(t) \in \mathcal{O}_H[t]^\times \]

where $H = L(A[m])$. As recalled in loc. cit., these elliptic units satisfy the appropriate norm-compatibility properties. We now recall the following definition.

**Definition 4.12** (Elliptic units). We define distinguished norm-compatible systems of Robert elliptic units with respect to a lattice $L \subset O_K$ by

\[ \xi_b := (i_p(\psi(\Omega; p^nL, b^{-1}p^nL)))_{n \in \mathbb{Z}_{\geq 0}}, \quad \xi_{a, b} := (i_p(\psi(\Omega; a^{-1}p^nL, b^{-1}a^{-1}p^nL)))_{n \in \mathbb{Z}_{\geq 0}} \in \mathbb{U}^1. \]

Recall our fixed embedding $i_p : \mathbb{U} \hookrightarrow \mathbb{C}_p$ from (3). We then get an induced map $\hat{\iota}_p : \Phi = \bigoplus_{p \mid \mathfrak{m}} L_{p^\infty} \hookrightarrow \mathbb{C}_p$ which maps one $L_{p^\infty}$ isomorphically onto a subfield $L_p \subset \mathbb{C}_p$, and maps all the other $L_{p^\infty}$'s to 0. We let $\hat{\mu}_0$ denote the image of $\hat{\mu}$ under $\iota_p$.

Note that we have distinguished elements from (4.12)

\[ \xi_b, \xi_{a, b} \in \mathbb{U}. \]

Let $\xi^1 = \xi^1_{a, b} \in \mathbb{U}^1$ denote their projections onto semilocal principal units.

4.2. A twist of $\mu_{\text{glob}}$. We introduce the following twist of $\mu_{\text{glob}}$. Recall that $X$ is the parameter of $F = A$, and consider the operator

\[ D = \frac{d}{\omega_0} = \frac{d}{\log'_F(X) dX} : \mathcal{O}_F \to \mathcal{O}_F \]

along with its universal version

\[ D_{\text{univ}} = \frac{d_{\text{univ}}}{\omega_{0, \text{univ}}} = \frac{d_{\text{univ}}}{\log'_{F_{\text{univ}}}(X) dX} : \mathcal{O}_{F_{\text{univ}}} \to \mathcal{O}_{F_{\text{univ}}}. \]

Recall that $\varphi$ denotes the type $(1, 0)$ character attached to $A$. Then the $p$-adic avatar $\hat{\varphi}$ can be viewed as a Galois character $\hat{\varphi} : \text{Gal}(L_{\infty}/K) = \text{Gal}(L_{\infty}/L_\ell) \times \text{Gal}(L_\ell/K) \to \mathbb{Q}_p^\times$, so that we have $\hat{\Delta} = \text{Gal}(L_\ell/K)$.

**Definition 4.13**. Let $\tilde{\chi}_A$ denote the restriction of $\hat{\varphi}$ to $\hat{\Delta}$. Then let $\mathbb{U}^1(\hat{\varphi}^{-1})$ denote $\mathbb{Z}_p$ module $\mathbb{U}^1$ with $G_\infty$-action twisted by $\varphi^{-1}$, and consider the rank-2 $\Lambda(\Gamma_\infty, \mathcal{O}_{L_p}[\chi])$-module $(\mathbb{U}^1(\hat{\varphi}^{-1}))^{\tilde{\chi}}$.

The decomposition subgroup of $G_\infty$ specified by the choice of embedding $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ is identified with $\text{Gal}(L_p(F[p^\infty])/K_p)$ via Lubin-Tate theory, and $\hat{\varphi}|_{\text{Gal}(L_p(F[p^\infty])/K_p)}$ is identified with $\kappa : \text{Gal}(L_p(F[p^\infty])/K_p) \to \mathcal{O}^\times_{K_p}$. We henceforth let $\chi_F$ denote the restriction of $\kappa$ to $\text{Gal}(L_p(F[p^\infty])/K_p)$. 

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We can also consider the twisted Iwasawa module $U^1(\kappa^{-1})$, of which $U^1(\varphi^{-1})$ is the semi-local version.

Let $\beta^-(\kappa^{-1})$ denote $\beta^-$ viewed inside $U^1(\kappa^{-1})$. Using $Dh_m$ in place of $h_m$ in our construction of $g_{\beta^-}$, we then get a twisted half Coleman map $\beta^-(\kappa^{-1}) \mapsto Dg_{\beta^-}(\kappa^{-1})$. Then using $D^{\text{univ}}g_{\beta^-}/g_{\beta^-}^{\text{univ}}$ in place of $\log(g_{\beta^-}^{\text{univ}})$, we get a $\Lambda(\Gamma_-, \mathcal{O}_{C_p})[1/p]$-equivariant map $D\mu_{qhdr}$ which is the corresponding twist of $\mu_{qhdr}$, and

$$D\mu := \xi \circ D\mu_{qhdr} : (U^1(\kappa^{-1}))_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \to \Lambda(\Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$$

(here $\tilde{\chi}$ is any nontrivial character on $\text{Gal}(\mathbb{A}(p^\infty))/K$ and need not necessarily be $\tilde{\chi}_{\Lambda}$). The semilocal version of this map, repeating the analogous process as in the construction of (34) from (11), gives a $\Lambda(\Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$-equivariant map

$$D\mu_{\text{glob}} : (U^1(\varphi^{-1}))_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \to \Lambda(\Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p].$$

As before, the source by (23) is a free $\Lambda(\Gamma_-, \mathcal{O}_{C_p})[1/p]$-module of rank $[\mathcal{O}_L : \mathbb{Z}_p]$, whereas the target is free of rank 1. Hence the kernel has rank $[\mathbb{Z}_p : \mathbb{Z}_p] - 1$. As before, we denote the image of these maps under the fiber specialization $\hat{O}^{\varepsilon}_{LT\otimes,y} \to \hat{O}^{\varepsilon}_{LT\otimes,y}(y) = \mathcal{O}_{L_p}^{\text{uni}}$ by $D\mu(y)$, and $D\mu_{\text{glob}}(y)$.

We have the following analogue of Proposition 4.14.14

**Proposition 4.14** (cf. Proposition III.1.3 of [53]). There is a unique way to extend (34) to a map

$$D\mu_{\text{glob}} : \mathcal{U}(\mathcal{f})_{\tilde{\chi}} \otimes_{\mathcal{O}_{K_p}} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \to \Lambda(\text{Gal}(\mathcal{K}(\mathcal{f})/K) \times \Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$$

for any $\mathcal{f} \subset \mathcal{O}_K$ such that the following diagram is commutative

$$\begin{array}{c}
\mathcal{U}(\mathcal{f})_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \\
\downarrow \text{inclusion} \\
\mathcal{U}(\mathcal{g})_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \\
\downarrow \text{Nm} \\
\mathcal{U}(\mathcal{f})_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \\
\downarrow \text{core reduction} \\
\Lambda(\text{Gal}(\mathcal{K}(\mathcal{f})/K) \times \Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]
\end{array}$$

for all $\mathcal{f} \mid \mathcal{g}$ with $(\mathcal{g}, p) = 1$ (see [53] Lemma III.1.2), and $\tilde{\chi}' : \text{Gal}(\mathcal{K}(\mathcal{f}^p)/K) \to \mathbb{Q}_p^\times$ factoring through $\tilde{\chi} : \text{Gal}(\mathcal{K}(\mathcal{f}^p)/K) \to \mathbb{Q}_p^\times$.

We also have twisted analogues of local units

$$\xi_b^1(\varphi^{-1}), \quad \xi_{a,b}^1(\varphi^{-1}) \in \mathcal{U}(\mathcal{f})(\varphi^{-1}).$$

4.3. **Surjectivity of $D\mu$.** In this section we prove that $D\mu$ is surjective.

**Proposition 4.15.** Suppose $\chi \neq 1$. The map

$$D\mu : \mathcal{U}^1(\kappa^{-1})_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \to \Lambda(\text{Gal}(\mathcal{L}_p/K_p) \times \Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$$

is a surjective map of $\Lambda(\text{Gal}(\mathcal{L}_p/K_p) \times \Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$-modules. Similarly, supposing $\tilde{\chi} \neq 1$,

$$D\mu_{\text{glob}} : \mathcal{U}^1(\varphi^{-1})_{\tilde{\chi}} \otimes_{\mathbb{Z}_p} \hat{O}^{\varepsilon}_{LT\otimes,y}[1/p] \to \Lambda(\text{Gal}(\mathcal{L}/K) \times \Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$$

is a surjective map of $\Lambda(\text{Gal}(\mathcal{L}/K) \times \Gamma_-, \hat{O}^{\varepsilon}_{LT\otimes,y})[1/p]$-modules.
Proof. This follows from the proof of [53 Theorem 1.3.7]. For $D\mu$, we see from the discussion before section I.3.8 of loc. cit. that $D\mu$ gives a map $j$ which fits into the following sequence, exact except possibly at the third term,

$$0 \to U_{\chi}^{-1}(\kappa^{-1}) \hat{\otimes}_{L_{\infty},\varphi}[1/p] \xrightarrow{D\mu} \Lambda(\text{Gal}(L_p/K_p) \times \Gamma_-, \hat{\otimes}_{L_{\infty},\varphi}[1/p]) \xrightarrow{j} \hat{\otimes}_{L_{\infty},\varphi}[1/p]_{\chi} \to 0,$$

where $\text{Gal}(L/K) \times \Gamma_-$ acts trivially on $\hat{\otimes}_{L_{\infty},\varphi}[1/p]$ in the fourth term. Here, we also used the fact that the formal group law $[\cdot](q_{\text{DR}} - 1)$ is the formal group law on the universal torus $\hat{G}_{m, L_{\infty}}$, and so the term “$N$” in loc. cit. is $\infty$, and the twist corresponding to “(1)” in loc. cit. is canceled out by the twist $\kappa^{-1}$ in the second term of the exact sequence in $D\mu$. However, as $\chi$ is the restriction of $\kappa$ to $\text{Gal}(L_p(F[p^\ell]) / K_p)$, $\chi$ is nontrivial, and so $\hat{\otimes}_{L_{\infty},\varphi}[1/p]_{\chi} = 0$, and for the surjectivity claim we are reduced to showing that the above sequence is exact at the third term. However, now one can apply the argument of Section 3.8-3.13 verbatim, as $[\cdot]$ is given by the group law on $\hat{G}_{m, L_{\infty}}$. This proves the surjectivity of $D\mu_{\text{glob}}$ immediately follows from the surjectivity of $D\mu$. \hfill $\square$

4.4. Interpolation. In this section, we will study interpolation properties of $D\mu_{\text{glob}}$ applied to twisted elliptic units $\xi_{n,b}(\varphi^{-1})$, relating them to $L$-values. However, this interpolation property differs from the ordinary case ([53 Theorem II.4.14]) in that it is only confined to 1 Hodge-Tate weight and finite order twists along the anticyclotomic tower, as opposed to the situation concerning all Hodge-Tate weights and finite order twists along the $\mathbb{Z}^{\leq 2}$-extension as in loc. cit. This nuance is related to the fact that the logarithmic derivatives $\frac{\partial f}{\partial s}$ are only the holomorphic parts of the $p$-adic Mass-Shimura derivatives of [34], which are more directly related to $L$-values. In the ordinary case, these $p$-adic Maass-Shimura derivatives are “completely holomorphic” and simply the classical Atkin-Serre operators, which are directly related to $L$-values by work of Katz ([29]).

Definition 4.16. Let $K[p^n] \subset K(p^n)$ denote the ring class extension of $K$ of conductor $p^n$. Let $L[p^n] = LK[p^n]$ and $L(p^n) = LK(p^n) = L(A[p^n])$. Then $\text{Gal}(L(p^n)/L) \cong (\mathcal{O}_K/p^n)^\times$ and $\text{Gal}(L[p^n]/L) \cong (\mathcal{O}_K/p^n)^\times / (\mathbb{Z}/p^n)^\times \cong \Delta/\mathbb{Z}/p^n \times \Gamma_{-, n}$. For any $n \geq \varepsilon$

$$\mathbb{G}_n = \text{Gal}(L[p^n]/K) \cong \text{Gal}(L(A[p^n])/K) \times \Gamma_{-, n}.$$

Suppose $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ has finite order, and so has a finite-order $p$-adic avatar $\hat{\chi}$ which factors through $\hat{\chi} : \Gamma_- \to \mathbb{Q}_p^\times$. Let $\Gamma_{-, n} = \Gamma_- / \Gamma_{p^n}^\times$ and suppose that $n \in \mathbb{Z}_{\geq 0}$ is maximal so that $\hat{\chi}$ factors through $\hat{\chi} : \Gamma_{-, n} \to \mathbb{Q}_p^\times$. In fact, since $\chi$ is finite order it is $\mathbb{Q}$-valued, and using $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ and $i_\infty : \mathbb{Q} \hookrightarrow \mathbb{C}$ we may identify $\hat{\chi} = \hat{\lambda}$.

Definition 4.17. Recall that for any $\lambda \in (1 + p\mathcal{O}_{K_p})/(1 + p\mathbb{Z}_p)$, we defined $\hat{\lambda} \in 1 + p\mathbb{Z}_p$ by

$$(1 + 1/z)\lambda \hat{f}(\alpha_2) = \langle \hat{\lambda}^{-1} \rangle (1 + 1/z)\hat{f}(\alpha_2)).$$

Moreover, for any $a \in (\mathbb{Z}/q)^\times$, one checks that

$$\langle a \rangle (1 + 1/z)\hat{f}(\alpha_2) = \langle \hat{\lambda} \rangle (1 + 1/z)\hat{f}(\alpha_2))$$

for some unique $\lambda \in (1 + \pi\mathcal{O}_{K_p})/(1 + p\mathcal{O}_{K_p})$. We again write $a = \hat{\lambda}$. We denote the inverse of our correspondence $\lambda \mapsto \hat{\lambda}$ by $a \mapsto \overline{a}$.

Definition 4.18. Finally, recall that $q = 4$ if $p = 2$ and $q = p$ if $p > 2$. For any $a \in (\mathbb{Z}/q^{pn})^\times$, let $a = (a_0, a_1)$ in the coordinates given by the decomposition $(\mathbb{Z}/q^{pn})^\times \cong (\mathbb{Z}/q)^\times \times (1 + q\mathbb{Z}_p)/(1 + q^{pn}\mathbb{Z}_p)$. Using this, define the Gauss sum as

$$\tau(\psi) = \frac{1}{q^{pn}} \sum_{a \in (\mathbb{Z}/q^{pn})^\times} \psi(\overline{a}) (\zeta_{q^{pn}}^{\text{univ}})^{-a}.$$
Lemma 4.19 (cf. [53] Lemma II.4.8). Suppose $\psi \in \hat{G}_n$ is of finite order. Recall $\chi = \varphi|_{\text{Gal}(L[p^n]/K)}$. Choose ideals $\mathfrak{c} \subset \mathcal{O}_K$ prime to $\mathfrak{p}$ whose Artin symbols $\sigma_\mathfrak{c}$ represent $G_n = \text{Gal}(L(A[p^n])/K) = \text{Gal}(K(\mathfrak{p}^n)/K)$. Then we have for any $\psi \in \hat{G}_n$

$$D \mu_{\text{glob},\beta^-}(\psi) = \frac{\tau(\psi)}{\# \Delta} \sum_\mathfrak{c} (\varphi \psi)(\mathfrak{c}^{-1}) D \mu_{\sigma_\mathfrak{c},\beta^-}(\zeta_{\text{univ}}^{\psi} - 1).$$

Proof. This is the same calculation as in [53] Lemma II.4.8, using (41) which implies

$$\chi((1 + 1/z)i^{-1}(a)) = \chi([a]'(1 + 1/z)).$$

Definition 4.20. Recall our fixed CM elliptic curve $A$ from Definition 4.1. From the isomorphism $\theta_{\infty,A} : \mathbb{C}/L \cong A(\mathbb{C})$, we get a differential $2\pi idz \in \Omega^1_{A/\mathbb{C}}$ from the standard differential $2\pi idz$ on $\mathbb{C}$. Define $\Omega_{\infty} \in \mathbb{C}^\times$ by

$$\Omega_{\infty} : 2\pi idz = \omega_0.$$

Definition 4.21. Given any Hecke character $\epsilon$ of infinity type $(k,j)$ of conductor $f$, and let $n \in \mathbb{Z}_{>0}$ be maximal such that $p^n|f$. Supposing that $w_f = 1$, by [53] Chapter II.4.7-II.4.8, there exists a Hecke character $\varphi$ of type $(1,0)$, so that $\epsilon = \varphi^k \varphi | \chi$, where $\chi$ is a finite-order character. Let $q \subset \mathcal{O}_K$ be any ideal with $(q,pf) = 1$, and $(qL/K) = (p^n, L/K)$. We define

$$G(\epsilon) := \frac{\varphi^k(p)^n \chi(q) \tau(\chi)}{\# \Delta}.$$

Theorem 4.22 (cf. Theorem II.4.11 of [53]). Suppose $w_f = 1$, $(p,f) = 1$. In the situation of Definition 4.1, we have for any finite order Hecke character $\psi : \mathbb{A}_K^\times / K^\times \rightarrow \mathbb{C}^\times$ and conductor dividing $\mathfrak{p}p^\infty$, and whose $p$-adic avatar $\hat{\psi}$ has restriction $\hat{\psi}|_{\mathcal{O}_{K_p}}$ factoring through $\Gamma_-$, we have

$$i_p^{-1} \left( D_{\mu_{\text{glob}}}((\xi_{b}^1)) \right)(y) = i_p^{-1} \left( G(\varphi \psi) \Omega_{\infty}^{-k} 12(k-1)! \left( 1 - \frac{(\varphi \psi)(p)}{N(p)} \right) ((\varphi \psi)(b) - N(b)L_{\mathfrak{m}}(\varphi^{-1} \psi^{-1}, 0) \right),$$

where $N(m)$ denotes the positive generator of the ideal $Nm_{K/Q}(m)$, and $L_m$ denotes the $L$-function with Euler factors dividing $m$ removed. The same formula holds with $D_{\mu_{\text{glob}}}$ replaced by $D_{\mu_{\text{glob}}}(y)$.

Proof. One can verify, using the analogous interpolation to (20) for $Dg_{\beta^-}/g_{\beta^-}$ that the half Coleman power series $Dg_{\beta^-}$ for the anticyclotomic specialization of the elliptic unit $\beta^- = \xi_{b}^1$ is the anticyclotomic specialization of the fundamental theta series as in [53] Proposition II.4.9]. This is the same series of calculations as in loc. cit., combined with Lemma 4.19. The factor of $\varphi(p)^n \chi(q)$ in $G(\varphi \psi)$ comes from the $\phi^{-n}$ in the defining property (20) of $g_{\beta^-}$. \hfill $\Box$

We now define a special measure obtained from dividing $\mu_{\text{glob}}(\xi_{b}^1)$ by an appropriate twisting measure associated with $b$.

Definition 4.23. Recall $b,f \subset \mathcal{O}_K$ as chosen in Definition 4.1 with $(f,p) = 1$ (but not necessarily $w_f = 1$), and $L = K(f)$. Recall that $\sigma_b = (b, L_{\infty}/K)$ denotes the Artin symbol. Define the pseudomeasure

$$\mu_{\text{glob}}(f) := \frac{1}{12}(\sigma_b - N\mathfrak{b})^{-1} \mu_{\text{glob}}(\xi_{b}^1),$$

where $N\mathfrak{b}$ denotes the positive generator of $Nm_{K/Q}(\mathfrak{b})$.\hfill $\Box$
Proposition 4.24 (cf. Theorem II.4.12 of [53]).

1. Suppose $f \not= O_K$. In fact, under the above assumptions, the pseudomeasure $\mu_{\text{glob}}(f)$ is a measure, i.e.

$$\mu_{\text{glob}}(f) \in \Lambda(\text{Gal}(K(f)/K) \times \Gamma_-, \hat{\mathcal{O}}^+_{LT,\infty,y})[1/p].$$

2. If $f|g$ and $\mu_{\text{glob}}(g)$ is the measure on $\text{Gal}(K(f)/K) \times \Gamma_-$ induced by $\mu_{\text{glob}}(g)$, then

$$\tilde{\mu}_{\text{glob}}(g) = \prod_{v|g, v \not= f} (1 - \sigma_v^{-1})\mu(f).$$

3. If $f = (1) = O_K$, then for any $\sigma \in \text{Gal}(K(f)/K) \times \Gamma_-$,

$$(1 - \sigma)\mu_{\text{glob}}(1) \in \Lambda(\text{Gal}(K(f)/K) \times \Gamma_-, \hat{\mathcal{O}}^+_{LT,\infty,y})[1/p].$$

Proof. This is given by essentially the same argument as in [53, Proof of Theorem II.4.12] (and Proposition 4.3 if $w_1 \neq 1$). First note that (47) follows from (44). The point (3) is follows from (46) because $\mu_{\text{glob}}(1) = (1 - \sigma_v^{-1})^{-1}\mu_{\text{glob}}(\nu)$ for any $v$. Hence it suffices to prove (46).

Let

$$\mu_b := \mu_{\text{glob}}(\xi_b^1) \in \Lambda(\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_-, \hat{\mathcal{O}}^+_{LT,\infty,y}),$$

and let

$$\delta_b := (\sigma_b - b) \in \Lambda(\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_-, \hat{\mathcal{O}}^+_{LT,\infty,y}).$$

Then by (44), we have

$$\delta_a \cdot \mu_b = \delta_b \cdot \mu_a.$$  

Essentially, one shows that the twisting measures $\delta_b$ have greatest common divisor 1 among all $b \subset O_K$ as in Definition 4.1.

First, following loc. cit., let $K_{\infty}/K$ be the the anticyclotomic $\Z_p$-extension of $K$, say with Galois group $\Gamma'_-$, and an isomorphism $\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_- := \Delta_0 \times \Gamma'_-$. Note that $\hat{\mathcal{O}}^+_{LT,\infty,y} \Gamma'_-$ satisfies the Weierstrass preparation theorem since $\hat{\mathcal{O}}^+_{LT,\infty,y}$ is a complete local ring. Fix a character $\theta \in \Delta_0$. Then $\theta(\delta_b) = \theta(\sigma_b|\Delta_0) \cdot (\sigma_b|\Gamma'_-)$. One can show that the greatest common divisor of the $\theta(\sigma_b)$ is 1. Since $\mu_{p,\infty} \subset L_{p,\infty}$, the argument on pp. 77-78 in loc. cit. shows that the greatest common divisor of all the $\delta(\sigma_b)$ divides $\theta(\sigma)$ for any $\sigma \in \Delta_0$. But $\delta(\sigma)$ is a unit, this gives the assertion.

Now since $\theta(\delta_b)$ have greatest common divisor 1, applying $\delta$ to (48), there must exist $\mu_\theta \in \hat{\mathcal{O}}^+_{LT,\infty,y} \Gamma'_-$ such that $\theta(\delta_b) \cdot \mu_\theta = \theta(\mu_b)$ for any $b$. Letting $e_\theta = \frac{1}{\#\Delta_0} \sum_{\sigma \in \Delta \sigma \in \Delta} \theta(\sigma)\sigma^{-1}$ denote the projector corresponding to $\theta \in \Delta_0$, we then see that $\mu = \sum_{\theta \in \Delta_0} \mu_\theta\epsilon_\theta \in \Lambda(\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_-, \hat{\mathcal{O}}^+_{LT,\infty,y})$, and $\delta_b \cdot \mu = \mu_b$. In particular $\#\Delta_0 \cdot \mu \in \Lambda(\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_-, \hat{\mathcal{O}}^+_{LT,\infty,y})$. However, proceeding by the same argument as on p. 78 of loc. cit., one concludes that in fact $\mu \in \Lambda(\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_-, \hat{\mathcal{O}}^+_{LT,\infty,y})[1/p].$

Hence $\mu = \mu_b/\delta_b \in \Lambda(\text{Gal}(K(f)K_{\infty}/K_{\infty}) \times \Gamma'_-, \hat{\mathcal{O}}^+_{LT,\infty,y})[1/p]$, and is independent of $b$. When $(p,6) = 1$, this gives the Proposition. If $p = 2, 3$, then it is known that $\xi_b \in \mathbb{U}^1$ is a $20^\text{th}$ power, and so $\mu_b$ is divisible by 12, and so we can repeat the above argument for $\mu_b/12$ to conclude.

Definition 4.25 (Iwasawa module of elliptic units). For every $n \in \mathbb{Z}_{\geq 0}$, let $C_n = C_{f^b}$ be the group generated by the $\xi_b$, $(b,6p) = 1$, and by the roots of unity in $L_n$. Assume $f = m \not= (1)$ so that the $\xi_b$ are in fact units. Let $\overline{C}_n$ denote the closure of $C_n$ in $R_n^\times$, and let $\overline{C}_n^1$ denote the projection on the principal part of $R_n^\times$. Then put

$$\overline{C}_f := \lim_{\longleftarrow n} (\overline{C}_n) \subset \mathbb{U}^1.$$
\[ \mathcal{C}(f) = \prod_{\mathfrak{g} \mid f} \mathcal{C}_\mathfrak{g} \]

where \( \mathfrak{g} \subset \mathcal{O}_K \) runs over integral ideals dividing \( f \).

**Definition 4.26.** Consider the submodules \( \overline{\mathcal{C}}(f) \subset \mathcal{U}^1 \), \( \overline{\mathcal{C}}(f)(\varphi^{-1}) \subset \mathcal{U}^1(\varphi^{-1}) \), where the bar line denotes \( p \)-adic closure. Let \( \chi \) denote the restriction of \( \hat{\varphi} : \text{Gal}(L(A[p^\infty]) / K) \to \hat{\mathbb{Q}}_p^\times \) to the direct factor \( \text{Gal}(L(A[p^\infty]) / K) \). We henceforth define the \( \Lambda \)-module

\[ (L_{p,f}) := \mu_{\text{glob}}(\overline{\mathcal{C}}(f) \otimes_{\mathbb{Z}_p} \hat{\mathcal{O}}_{L,\infty, y}^+[1/p] \chi). \]

We denote by \( (L_{p,f}(y)) \) the image of \( L_{p,f} \) under the fiber specialization map \( \hat{\mathcal{O}}_{L,\infty, y}^+ \to \hat{\mathcal{O}}_{L,\infty, y}^+(y) = \mathcal{O}_{L,\infty} \). Similarly, we define

\[ (D_{L_{p,f}}) := D_{\mu_{\text{glob}}}(\overline{\mathcal{C}}(f) \otimes_{\mathbb{Z}_p} \hat{\mathcal{O}}_{L,\infty, y}^+[1/p] \chi) \]

and \( (D_{L_{p,f}}(y)). \) One should think of \( D_{L_{p,f}}(y) \) as being characterized by the interpolation (44). On the other hand, \( L_{p,f}(y) \) is characterized by the \( p \)-adic Kronecker limit formula as a defining interpolation property (which is however not immediately related to \( L \)-values).

4.5. The \( p \)-adic Kronecker limit formula. We now state and prove special value formulas for our \( p \)-adic \( L \)-function. This is related to the special value formula (in the Eisenstein case of [34, Chapter 9, Theorem 9.10]). We emphasize that for the measure \( \mu_{\mathfrak{C}} \), we can view this \( p \)-adic Kronecker limit formula as a defining interpolation property (which is however not immediately related to \( L \)-values).

**Definition 4.27.** Suppose that \( f \subset \mathcal{O}_K \) is an integral ideal with \( (f,p) = 1 \). Then we define a \( p \)-adic analytic function

\[ L_{p,f} : \text{Gal}(K(f)/K) \times \Gamma \to \hat{\mathcal{O}}_{L,\infty, y}^+[1/p], \quad L_{p,f}(\chi) := \mu_{\text{glob}}(f)(\chi^{-1}) \]

Here, \( \mu_{\text{glob}}(f) \in \Lambda(\text{Gal}(K(f)/K) \times \Gamma, \hat{\mathcal{O}}_{L,\infty, y}^+[1/p]) \) as in (46).

We recall Robert’s invariants. For more details in a concise and nice exposition, see [53, Chapter II.2.6].

**Definition 4.28** (Robert’s invariants). Let \( f \subset \mathcal{O}_K \) be an integral ideal with \( f \neq \mathcal{O}_K \). As before, let \( \mathbb{N}(f) \) denote the positive generator of the ideal \( \mathbb{N}_{\mathcal{O}_K / \mathcal{O}_f}(f) \). Let \( \mathcal{C}(f) \) denote the ray class group modulo \( f \), so that Artin reciprocity gives \( \mathcal{C}(f) \cong \text{Gal}(K(f)/K) \). Let \( b \subset \mathcal{O}_K \), \( (b,6f) = 1 \). For any \( \sigma \in \text{Gal}(K(f)/K) \), let

\[ \phi_f(\sigma) = \theta(1, f\zeta^{-1})^f, \quad \sigma = (c, K(f)/K), \quad c \subset \mathcal{O}_K. \]

Here, given a lattice \( L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C} \), with \( \tau = \omega_1 / \omega_2, \text{im}(\tau) > 0, \)

\[ \theta(z, L) = \Delta(L) \cdot e^{-6\eta(z, L)z} \cdot \sigma(z, L)^{12}, \]

where

\[ \sigma(z, L) = z \cdot \prod_{\omega \in L, \omega \neq 0} \left(1 - \frac{z}{\omega} \right) \exp \left( \frac{z}{\omega} + \frac{1}{2} \left( \frac{z}{\omega} \right)^2 \right), \quad \Delta(L) = (2\pi i / \omega_2)^{12} \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i \tau}, \]

and \( \eta \) is an \( \mathbb{R} \)-linear form on \( \mathbb{C} \) given by

\[ \eta(z, L) = \frac{\omega_1 \eta_1 - \omega_2 \eta_2}{2\pi i A(L)} + \frac{\omega_2 \eta_1 - \omega_1 \eta_2}{2\pi i A(L)} z, \quad A(L) = (2\pi i)^{-1}(\omega_1 \omega_2 - \omega_1 \omega_2) = \pi^{-1} \text{Area}(\mathbb{C}/L), \]

\[ \eta_1 = \omega_1 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}, n \neq 0} (m\omega_1 + n\omega_2)^{-2}, \quad \eta_2 = \omega_2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}, n \neq 0} (m\omega_1 + n\omega_2)^{-2}. \]
Recall the Galois group (see (32)), curves with CM by imaginary quadratic fields of class number 1. Let \( \mathfrak{g} = \mathfrak{p}^n \) where \( (\mathfrak{f}, \mathfrak{p}) = 1 \). Then we have
\[
\mathcal{L}_{p, \mathfrak{g}}(\chi) = -\frac{1}{12N(\mathfrak{g})w_0} \cdot G(\chi^{-1}) \left(1 - \frac{\chi^{-1}(\mathfrak{p})}{N(\mathfrak{p})}\right) \cdot \sum_{c \in \mathcal{C}(\mathfrak{g})} \chi(c) \cdot \log \phi_\mathfrak{g}(c).
\]
Here, \( G(\chi^{-1}) \) is defined in (49), and \( \phi_\mathfrak{g} \) is defined as in Definition 4.28.

**Proof.** Given our definition of \( \mu_{\text{glob}}(\mathfrak{g}) \) in (45), and the interpolation property (44), this argument is entirely analogous to the one given in [53, Proof of Theorem II.5.2].

### 4.6. Formulation of the Main Conjectures

In this section, we let \( L = K(\mathfrak{f}) \), where \( \mathfrak{f} \subset \mathcal{O}_K \), but do not impose any assumptions on \( \mathfrak{f} \) unless specified. We can let \( \mathfrak{f} = 1 \) (which will be the situation for our later applications to the Birch and Swinnerton-Dyer conjecture for elliptic curves with CM by imaginary quadratic fields of class number 1).

**Definition 4.30.** Recall the Galois group (see (32)) \( G_\infty = \text{Gal}(L_\infty/K) \). Let \( M_\infty \) denote the maximal abelian pro-\( p \)-extension of \( L_\infty \) unramified outside (primes above) \( p \). Let \( N_\infty \) denote the maximal abelian pro-\( p \)-extension of \( L_\infty \) unramified at all places of \( L_\infty \). Then we let
\[
\mathcal{X} := \text{Gal}(M_\infty/L_\infty), \quad \mathcal{Y} := \text{Gal}(N_\infty/L_\infty).
\]
Recalling that \( \text{Gal}(L_\infty/L) \xrightarrow{i_\mathfrak{p}} \text{Gal}(L_\infty/L_\mathfrak{p}) \xrightarrow{\Lambda} \mathcal{O}_K^\times \) via the reciprocity map, and that \( \Lambda_R = \Lambda(\text{Gal}(L_{\infty, \mathfrak{p}}/L_\mathfrak{p}), R) = R[\text{Gal}(L_{\infty, \mathfrak{p}}/L_{\mathfrak{p}})] \) for a \( \mathcal{O}_\mathfrak{p} \)-algebra \( R \), we can view \( \mathcal{X} \) and \( \mathcal{Y} \) as \( \Lambda_{\mathcal{O}_{\mathfrak{p}, \infty}} \)-modules. Hence, we can view \( \mathcal{X} \otimes_{\mathcal{O}_{\mathfrak{p}, \infty}} \mathcal{Y} \otimes_{\mathcal{O}_{\mathfrak{p}, \infty}} \) as \( \Lambda_{\mathcal{O}_{\mathfrak{p}, \infty}} \)-modules. It is well-known (see Remarks before Theorem 2) that \( \mathcal{X} \) has \( \Lambda_{\mathcal{O}_{\mathfrak{p}, \infty}} \)-rank equal to \( d = [L_\mathfrak{p} : K_\mathfrak{p}] = r_2(L) \) (the number of pairs of complex embeddings of \( L \)), while \( \mathcal{Y} \) is \( \Lambda_{\mathcal{O}_{\mathfrak{p}, \infty}} \)-torsion. We wish to formulate a main conjecture for \( \mathcal{X} \), which will be facilitated by considering an appropriate \( \Lambda_{\mathcal{O}_{\mathfrak{p}, \infty}} \)-torsion quotient of \( \mathcal{X} \otimes_{\mathcal{O}_{\mathfrak{p}, \infty}} \mathcal{O}_{\mathfrak{p}, \infty} \).

This quotient will be induced from the kernel \( \mathfrak{M}(\mathfrak{y}) \) of \( \mu_{\text{glob}}(\mathfrak{y}) \) or \( D\mu_{\text{glob}}(\mathfrak{y}) \) in the twisted case. See [51] for the precise formulation.

We will also wish to consider certain pure \( \mathcal{O}_{\mathfrak{p}, \infty} \)-extensions of \( L_\infty \), and hence isotypic components of all the relevant \( \Lambda_{\mathcal{O}_{\mathfrak{p}, \infty}} \)-modules, so that we can study the associated Iwasawa invariants of these isotypic components. In this article, we will study only the \( \mu \)-invariant in-depth, as from its vanishing we will deduce a more general version of the Main Conjecture for \( \mathcal{Y}_\infty \) proven in [45, Theorem 4.1]. Note that we have a natural isomorphism \( \text{Gal}(L_\infty/L) \xrightarrow{i_\mathfrak{p}} \text{Gal}(L_{\infty, \mathfrak{p}}/L_\mathfrak{p}) \xrightarrow{\Lambda} \mathcal{O}_K^\times \) induced by \( i_\mathfrak{p} \) from (3), since \( L_\infty/L \) is totally ramified. Let \( K \subset L' \subset L_\infty \) be such that \( \text{Gal}(L_\infty/L') \cong 1 + p^\epsilon \mathcal{O}_{K_\mathfrak{p}} \). Let
\[
\Gamma' := \text{Gal}(L_\infty/L') \cong 1 + p^\epsilon \mathcal{O}_{K_\mathfrak{p}}.
\]
Let
\[
\Delta := \text{Gal}(L'/K),
\]
where \( \epsilon \) is as in (29). Let \( K \subset K_\infty \subset L_\infty \) such that
\[
\Gamma' := \text{Gal}(K_\infty/K) \cong 1 + p^\epsilon \mathcal{O}_{K_\mathfrak{p}}.
\]
Let \( K \subset K_n \subset K_\infty \) be the unique finite subextension such that \( \text{Gal}(K_n/K) \cong (\mathbb{Z}/p^n)^{\oplus 2} \). If \( K(1) \cap K_\infty = K_1 \), then the image of \( \Gamma' \) in \( \Gamma' \) under restriction to \( K_\infty \) is \((\Gamma')^p\). Similarly, the image of \( \Gamma' \) in \( \Gamma' \) (given by the restriction from \( L_\infty \) to \( K_\infty \)) is \((\Gamma')^p\) where \( \Gamma'_- = \text{Gal}(K_\infty/K) \).
Fix a decomposition 

$$G_\infty = \Delta' \times \Gamma'$$

where $\Delta' = \text{Gal}(L_\infty/K_\infty)$ is a finite abelian group. Suppose that $\chi \in \hat{\Delta}'$. Let $L_p[\chi]$ be the finite extension of $L_p$ generated by the values of $\chi$, and let $\mathcal{O}_{L_p,\chi}$ be its valuation ring. Then given a $\Lambda(G_\infty, \mathcal{O}_{L_p,\infty})$-module $M$, we can define the $\chi$-isotypic component $M_\chi$ as follows. Consider $\Lambda(\Gamma', \mathcal{O}_{L_p,\chi})$ as a $\Lambda(G_\infty, \mathcal{O}_{L_p,\chi})$-module by letting $\Delta'$ act through $\chi$. Then let

$$M_\chi := M \otimes_{\Lambda(G_\infty, \mathcal{O}_{L_p,\infty})} \chi \Lambda(\Gamma', \mathcal{O}_{L_p,\chi}),$$

which is the largest quotient of $M \otimes_{\mathcal{O}_{L_p}} \mathcal{O}_{L_p,\chi}$ on which $\Delta'$ acts through $\chi$.

If $p \nmid \left[ L' : K \right]$ (recalling $L = K(\overline{f})$, $L' = K(\overline{f'})$, $(f, p) = 1$), then we have $\Delta = \Delta'$, $\Gamma = \Gamma'$, $L_p = L_p[\chi]$, and naturally we have isotypic decompositions

$$\Lambda(G_\infty, \mathcal{O}_{L_p,\infty}) = \bigoplus_{\chi \in \Delta} \Lambda(G_\infty, \mathcal{O}_{L_p,\infty}) \chi, \quad M = \bigoplus_{\chi \in \Delta} M_\chi.$$

However, for many applications the assumption $p \nmid \left[ L' : K \right]$ is cumbersome, which is why it is necessary to consider the equivariant main conjecture (i.e. a main conjecture involving $\Lambda(G_\infty, \mathcal{O}_{L_p,\infty})$-modules).

**Definition 4.31.** Recall $\check{\chi} : \hat{\Delta} = \text{Gal}(L'/K) \rightarrow \mathbb{Q}_p^\times$ is any nontrivial character. Let $\mathcal{M} := \ker \mu_{\text{glob}}(y) \subset U_{\check{\chi}}^1 \otimes \hat{L}_{p,\infty}$ and $\mathcal{D}\mathcal{M} := \ker D\mu_{\text{glob}}(y)$. Moreover, recalling our fixed identification $G_\infty = \text{Gal}(L_\infty/K) \cong \text{Gal}(L_\infty/K_\infty) \times \text{Gal}(K_\infty/K)$, we define for any $\Lambda(G_\infty, R)$-module,

$$M^- := M \otimes_{\Lambda(G_\infty, R)} \Lambda(\text{Gal}(L_\infty/K_\infty) \times \Gamma_-, R).$$

We define the following $\Lambda(\text{Gal}(K(f)/K) \times \Gamma_-, \mathcal{O}_{L_p,\infty})[1/p]$-modules

$$U_{\check{\chi}} := (U_{\check{\chi}}^1 \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty})/(\mathcal{M}, \mathcal{C}(f) \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty}), \quad \mathcal{X}'_{\check{\chi}} := (\mathcal{X}_{\check{\chi}}^\sim \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty})/\text{rec}(\mathcal{M}).$$

Here, rec : $U \rightarrow \mathcal{X}$ is the global reciprocity map, which we extend $\otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty}$-linearly to a map

rec : $U \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p,\infty} \rightarrow \mathcal{X} \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty}$. When we wish to emphasize the dependence on the auxiliary conductor $f$, we write $U' := U'(f), \mathcal{X}' = \mathcal{X}'(f)$, etc.

Similarly, let $\mathcal{D}\mathcal{M} := \ker D\mu_{\text{glob}}(y) \subset U_{\check{\chi}}^1 \otimes (\varphi^{-1}) \otimes \hat{L}_{p,\infty}$. We define

$$DU_{\check{\chi}} := (U_{\check{\chi}}^1 \otimes (\varphi^{-1}) \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty})/(\mathcal{D}\mathcal{M}, \mathcal{C}(f) \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty}), \quad D\mathcal{X}'_{\check{\chi}} := (\mathcal{X}_{\check{\chi}}^\sim \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty})/\text{rec}(\mathcal{D}\mathcal{M}).$$

**Theorem 4.32 (Fundamental Exact Sequence).** We have the following exact sequence of torsion $\Lambda(\Gamma_-, \mathcal{O}_{L_p,\infty})[1/p]$-modules:

$$0 \rightarrow (\mathcal{E}/\mathcal{C}(f))_{\check{\chi}} \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty} \rightarrow U'_{\check{\chi}} \rightarrow \mathcal{X}'_{\check{\chi}} \rightarrow \mathcal{Y}_{\check{\chi}} \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty} \rightarrow 0$$

where the middle arrow is given by the global Artin reciprocity map. Similarly, we have a twisted version of the above exact sequence

$$0 \rightarrow (\mathcal{E}/\mathcal{C}(f))_{\check{\chi}} \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty} \rightarrow DU'_{\check{\chi}} \rightarrow D\mathcal{X}'_{\check{\chi}} \rightarrow \mathcal{Y}_{\check{\chi}} \otimes_{\mathbb{Z}_p} \hat{L}_{p,\infty} \rightarrow 0$$

which is also an exact sequence of torsion $\Lambda(\Gamma_-, \mathcal{O}_{L_p,\infty})[1/p]$-modules.

**Proof.** This follows immediately from the definitions of $U'$ and $\mathcal{X}'$ and the exact sequence

$$0 \rightarrow \mathcal{E}/\mathcal{C}(f) \rightarrow \mathcal{E} \rightarrow \mathcal{Y} \rightarrow 0.$$

\[\square\]
Proposition 4.33 (cf. Lemma III.1.10 of [53]). We have
\begin{equation}
\text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}(\mathcal{U}_{\chi}^\prime) = \mu_{\text{glob}}(f)\Lambda(\text{Gal}(L/K) \times \Gamma_-,\mathcal{O}_{L_p,\infty})[1/p].
\end{equation}

Proof. This follows from (45) and the surjectivity of $D\mu_{\text{glob}}$ (Proposition 4.15).

The exact sequences (51) and (52) suggest the following “Rubin-type main conjectures”.

Conjecture 4.34 (Rubin-type Main Conjecture). For any nontrivial $\tilde{\chi} \in \hat{\Delta}$, we have the following equality of determinants of torsion $\Lambda(\text{Gal}(L/K) \times \Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]_{\tilde{\chi}}$-modules:
\begin{equation}
\text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}(\mathcal{U}_{\chi}^\prime) = \text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}(\mathcal{X}_{\chi}^\prime)
\end{equation}
\begin{equation}
\text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}(D\mathcal{U}_{\chi}^\prime) = \text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}(D\mathcal{X}_{\chi}^\prime).
\end{equation}

In particular, we have
\begin{equation}
D\mu_{\text{glob}}(f)\Lambda(\text{Gal}(L/K) \times \Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]_{\tilde{\chi}} \cong \text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}(D\mathcal{U}_{\chi}^\prime) = \text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{L_p,\infty})[1/p]}((D\mathcal{X}_{\chi}^\prime)_{\tilde{\chi}}).
\end{equation}

Our approach for proving the above conjecture will be through proving a suitable main conjecture for elliptic units. Unlike in Conjecture 4.34, we will be interested in an integral equivariant main conjecture [56], from which will follow an appropriate anticyclotomic specialization [57]. Let $\det$ be Mumford’s determinant functor, as defined as in [28, Section 1].

Conjecture 4.35 (Main Conjecture). We have the following equality of determinants of torsion $\Lambda(G_\infty,\mathbb{Z}_p)$-modules:
\begin{equation}
\det_{\Lambda(G_\infty,\mathbb{Z}_p)}((E/\mathcal{C}(f))) = \det_{\Lambda(G_\infty,\mathbb{Z}_p)}(\mathcal{Y}).
\end{equation}

Furthermore, for any nontrivial $\tilde{\chi} \in \hat{\Delta}$ (recall $\hat{\Delta} = \text{Gal}(L(A[p^\infty])/K)$), we have
\begin{equation}
\text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{K_p,[\chi]}[1/p])}(\mathcal{Y} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{\tilde{\chi}} = \text{char}_{\Lambda(\Gamma_-,\mathcal{O}_{K_p,[\chi]}[1/p])}(\mathcal{X} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_{\tilde{\chi}}.
\end{equation}

Proposition 4.36. (1) Suppose that (56) holds. Then (54) holds.
(2) Suppose that (57) holds. Then (55) holds.

Proof. The first assertion follows immediately from the exact sequence (51), (52) and the additivity of determinants in exact sequences. The second assertion follows from the first.

In fact, we will later show that (57) follows from (56). However, we will need to adopt the equivariant main conjecture viewpoint of [28].

4.7. The cyclotomic algebraic $\mu$-invariant. We continue the notation of the previous section.

Definition 4.37. Recall $\mathcal{U}$ as defined in (35), and recall we let $L = K(f)$ with $(f, p) = 1$, $\omega_f = 1$. Let $f$ be the smallest positive rational integer belonging to $f$ (so that $(f) = \text{Nm}(f)$), let $L^+ = K\mathbb{Q}(\mu_f)$, and let $L^+_n = L^+((\mu_f)^n)$ and $L^+_\infty = L^+((\mu_f)^\infty)$, and so $\text{Gal}(L^+_\infty/K) \cong \text{Gal}(L^+_\infty/L^+) \times \text{Gal}(L^+/K) \cong \mathbb{Z}_p \times \Delta_f$ for some finite abelian group $\Delta_f$. For $0 \leq n \leq \infty$, let $M_n^+$ be the maximal pro-$p$ abelian extension of $L^+_\infty$ which is unramified outside $p$, and let $N_n^+$ be the maximal pro-$p$ abelian extension of $L^+_\infty$ which is unramified everywhere. The we let
\begin{equation}
\mathcal{X}^+ := \text{Gal}(M^+_\infty/L^+_\infty) = \lim_{\leftarrow n} \text{Gal}(M_n^+/L_n^+).
\end{equation}
Let $\Gamma_+ \subset \text{Gal}(L_{\infty}^+/L)$ be the maximal subgroup isomorphic to $\mathbb{Z}_p$, so that

\begin{equation}
(\mathcal{X}^+)_{\Gamma_+^{p^n}} = \text{Gal}(M_n^{+}/L_{\infty}^+)
\end{equation}

where $s = \text{ord}_p(q)$, so that $s = 1$ if $p > 2$ and $s = 2$ if $p = 2$. By the non-vanishing of the $p$-adic regulator ([7]), one can show that $\text{Gal}(M_n^{+}/L_{\infty}^+)$ is finite. Note that $\mathcal{X}^+$ is a $\Lambda(\text{Gal}(L_{\infty}^+/K), \mathbb{Z}_p)$-module. For any $\chi \in \hat{\Delta}^+$, we can consider the $\chi$-isotypic component $\mathcal{X}^+$. Moreover, for each $0 \leq n \leq \infty$, let $K \subset K_n^+ \subset K(\mu_{p^n})$ be the unique subfield with $\text{Gal}(K_n^+/K) \cong \mathbb{Z}/p^n$ if $n < \infty$, and $\text{Gal}(K_n^+/K) \cong \mathbb{Z}_p$ if $n = \infty$. Note that (by standard theory of cyclotomic extensions), $K_n^+/K$ is totally ramified. Let

$$\Gamma'_n = \text{Gal}(K_n^+/K).$$

Note that if we let $t \in \mathbb{Z}_{\geq 0}$ such that $K_n^+ \cap K(1) = K_n^t$, then we have a natural identification $\Gamma_n = (\Gamma_n')^t$ given by restriction from $L_n^+$ to $K_n^+$. As $L_n^+/\mathbb{Q}$ is an abelian extension of $\mathbb{Q}$ which is a finite extension of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, by [23], we $\mathcal{X}^+$ is a torsion $\Lambda(\text{Gal}(L_{\infty}^+/K), \mathbb{Z}_p)$-module.

**Remark 4.38.** Note that $\mathcal{X}^+$ is not the cyclotomic specialization of $\mathcal{X}$, as the latter would have rank at least 1. In fact, there is a natural map from the latter into the former.

**4.8. Computation of algebraic Iwasawa invariants: a valuation calculation.** We follow the strategy of [12] for studying

$$\text{ord}_p(\#(\mathcal{X}^+)_{\Gamma_+^{p^n}}) = \text{ord}_p(\#\text{Gal}(M_n^{+}/L_{\infty}^+)).$$

We need some lemmas and propositions which are refinements of results from loc. cit. We follow [53] Chapter III.2 throughout this section.

**Definition 4.39** ($p$-adic regulator). Let $F/K$ be any abelian extension of degree $r$. Let $\sigma_1, \ldots, \sigma_r$ be the embeddings $F \hookrightarrow \mathbb{C}_p$. (Note that all embeddings induce the same embedding $K_p \hookrightarrow \mathbb{C}_p$, since there is only one prime above $p$.) Let $E$ be a subgroup of finite index in $\mathcal{O}_F^r$, and choose generators $e_1, \ldots, e_{r-1}$ for $E/E_{\text{tors}}$. Then the $p$-adic regulator of $E$ is defined to be

$$R_p(E) = \det(\log \sigma_i(e_j))_{1 \leq i, j \leq r-1}.$$ 

This is well-defined up to sign because $\log \text{Nm}_{F/K}(e) = 0$ for $e \in E$. As short-hand, let $R_p(F) = R_p(\mathcal{O}_F^r)$.

**Theorem 4.40 ([7]).** Assume $F/K$ is abelian (as is the case in Definition 4.39). Then $R_p(E) \neq 0$.

**Definition 4.41.** We follow the notation and presentation of [53] Chapter III.2.4. Fix an abelian extension $F/K$ of degree $r$ as in Definition 4.39. For each prime $\mathfrak{P}$ of $F$ above $p$ (the unique prime of $K$ above $p$), let $w_{\mathfrak{P}}$ denote the number of $p$-power roots of unity in $F_{\mathfrak{P}}$. Let $\Phi = F \otimes_K K_p = \prod_{\mathfrak{P} \mid p} F_{\mathfrak{P}}$, and let $U$ be the group of principal units in $\Phi$. Then the $p$-adic logarithm gives a homomorphism

$$\log : U \rightarrow \Phi$$

whose kernel has order $\prod_{\mathfrak{P} \mid p} w_{\mathfrak{P}}$, and whose image is an open subgroup of $\Phi$. Let $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. Let $E \subset \mathcal{O}_F^r$ be a subgroup of finite index, and let $D = D(E) = E \cdot \langle 1 + q \rangle$, and let $\overline{D}$ be its closure in $\Phi^r$, and $\langle \overline{D} \rangle$ the projection of $\overline{D}$ to $U$. We write $\Phi(F), U(F), E(F)$ when we refer to these objects for a specific $F$. Let $\Delta_p(F/K)$ denote the sum over $\mathfrak{P} \mid p$ of the relative discriminants of $F_{\mathfrak{P}}/K_p$. Given any number field $F$, let $h(F)$ denote its class number and let $h(F)[p^\infty]$ denote the $p$-part of the class number.
Definition 4.42. Suppose we are in the situation of Definition 4.37, i.e. with $F = L_n^+$. Then letting $U_n = U(L_n^+/L)$ and $E_n = E(L_n^+/L)$, Artin reciprocity gives
\[
U_n/\langle E_n \rangle \cong \text{Gal}(M_n^+/N_n^+).
\]
Recall that $N_n^+$ (resp. $M_n^+$) is the maximal pro-$p$ abelian extension unramified everywhere (resp. unramified outside $p$) of $L_n^+$. Note that $N_n^+ \cap L_\infty^+ = L_n^+$ and $L_\infty^+ \subset M_n$. Let $Y_n \subset U_n$ be the subgroup such that
\[
Y_n/\langle E_n \rangle \cong \text{Gal}(M_n^+/N_n^+L_\infty^+).
\]
Recall that $L_n^+ = K_n^+$, and $K_n^+ \subset K(\mu_{p\infty})$ is such that $[K_n^+ : K] = p^n$. In particular, $K_n^+/Q$ is an abelian extension. Consider the norm map $\text{Nm}_{L_n^+/Q} : U_n \to 1 + p\mathcal{O}_K$.

Lemma 4.43 (cf. Lemma III.2.6 of [53]). We have $Y_n = \text{Nm}_{L_n^+/Q}U_n$.

Proof. If $u \in Y_n$, then since $U_n/Y_n \cong \text{Gal}(N_n^+L_\infty^+/N_n^+) \cong \text{Gal}(L_\infty^+/L_n^+)$, we have $(u, L_\infty^+/L_n^+) = 1$ (as usual, denoting the Artin symbol of an abelian extension $E'/E$ by $(\cdot, E'/E)$). Then by the functoriality of the Artin symbol, we get $(u, L_\infty^+/L_n^+) = (\text{Nm}_{L_n^+/Q}(u), K_\infty^+/Q) = 1$. Hence the idèle $\text{Nm}_{L_n^+/Q}(u)$, which is 1 outside of $p$, is necessarily 1. The argument in reverse shows the converse.

Let $D_n = D(L_n^+)$, and similarly with $\overline{D}_n$.

Lemma 4.44. Assume that we are in the situation of Definition 4.37, i.e. with $F = L_n^+$. Let $p^\delta||L_n^+ : K_n^+|K_n^+|$, so that $p^{n+\delta}||L_n^+ : K_n^+|K_n^+|$. Then we have $[\log(U_n) : \log((\overline{D}_n))] < \infty$, and in fact
\[
\text{ord}_p ([\log(U_n) : \log((\overline{D}_n))]) = \text{ord}_p \left( \frac{q^{n+\delta}R_p(L_n^+)}{\sqrt{\Delta_p(L_n^+)/K_n^+}} \prod_{\mathfrak{p} | p} (w_{\mathfrak{p}|\mathfrak{p}}N(\mathfrak{p}))^{-1} \right),
\]
where $\mathfrak{p}$ runs over all primes of $L_n^+$ above $p$.

Proof. This is [12, Lemma 8], the argument of which goes through using $\epsilon_d = 1 + q$ in place of $\epsilon_d = 1 + p$.

Corollary 4.45. Retain the situation of Lemma 4.44. Let $w(E)$ be the number of roots of unity in $E$. Then
\[
\text{ord}_p ([U_n : (\overline{D}_n)]) = \text{ord}_p \left( \frac{q^{n+\delta}R_p(L_n^+)}{w(L_n^+) \sqrt{\Delta_p(L_n^+)/K_n^+}} \prod_{\mathfrak{p} | p} N(\mathfrak{p})^{-1} \right),
\]
where $\mathfrak{p}$ runs over all primes of $L_n^+$ above $p$.

Proof. This is an immediate consequence of Lemma 4.44 using the snake lemma, see [12, Lemma 9] for details.

Proposition 4.46 (cf. Proposition III.2.17 of [53]). Retain the notation of Lemma 4.45. Recall $p^\ell = [(K(1) \cap K_\infty^+) : K]$ and let $p^e = [(L \cap K_\infty^+) : K]$. We have
\[
\text{ord}_p ([M_n^+ : L_\infty^+]) = \text{ord}_p \left( \frac{q^{n+\delta+\epsilon}R_p(L_n^+)R_p(L_n^+)\prod_{\mathfrak{p} | p} (1 - N(\mathfrak{p})^{-1})}{w(L_n^+) \sqrt{\Delta_p(L_n^+)/K_n^+}} \right),
\]
where the product on the right-hand side runs over primes of $L_n^+$ above $p$. 34
Proof. Let $D_n = E_n \cdot (1 + q)$. Then since $\text{Nm}_{L_n^+/K}(E_n) = 1$, we have $\text{Nm}_{L_n^+/Q}(\langle \overline{D}_n \rangle) = 1 + q^2 p^{n+\delta-1}Z_p$. Now we have a diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \langle E_n \rangle \\
\downarrow & & \downarrow \\
0 & \rightarrow & \langle \overline{D}_n \rangle \xrightarrow{\text{Nm}_{L_n^+/Q}} 1 + q^2 p^{n+\delta-1}Z_p \\
\end{array}
\quad (61)
$$

where the horizontal rows are exact and the vertical arrows are injective, by the above discussion and Lemma 4.43. Then $[L_n^+ : K]$ is exactly divisible by $p^{n+\delta}$, and so $[L_n^+ : K]$ follows from Lemma 4.45. Then $[L_n^+ : L_{n,s}^+][N_n^+ : L_{n,s}^+] = [N_n^+ : L_n^+] = h(L_n^+)[p^\infty]$.

\begin{definition}
Let $\Delta'_+ = \text{Gal}(L_{n,s}^+/K_{n,s}^+)$ and recall $\Gamma'_+ = \text{Gal}(K_{n,s}^+/K)$. For $\chi \in \Delta'_+$, recall the torsion $\Lambda(\Gamma'_+, Z_p)$-module $(\chi^\pm)$. We let

$$
\hat{\mu}_\chi := \text{char}_{\Lambda(\Gamma'_+, O_{K_{p,s}})}((\chi^\pm)).
$$

Denote the $\mu$-invariant by $\mu_{\text{inv}}(f_\chi^\pm)$ and the $\lambda$-invariant by $\lambda_{\text{inv}}(f_\chi^\pm)$ and $\lambda_{\text{inv}}(g_\chi^\pm)$.

\begin{corollary}
Let $K_{n,s}^+ = K_{n,s}^+ \cap K(1)$, and recall that $s = 1$ if $p > 2$ and $s = 2$ if $p = 2$. We have for all $n \gg 0$,

$$
\sum_{\chi \in \Delta'_+} \mu_{\text{inv}}(f_\chi^+) \cdot p^{t+n-1}(p-1) + \sum_{\chi \in \Delta'_+} \lambda_{\text{inv}}(f_\chi^+) = 1 + \text{ord}_p \left( \frac{\text{h}R_p}{w \sqrt{\Delta_{n+s}}/K} \right). \quad (62)
$$

Proof. By (60), the right-hand side of (62) is ord_p([M_{n+s}^+ : L_{n,s}^+]/[M_{n+s-1}^+ : L_{n,s}^+]). Now (62) follows from (58) and Weierstrass preparation (using the standard definitions of $\mu_{\text{inv}}$ and $\lambda_{\text{inv}}$).

\end{corollary}

4.9. Computation of analytic Iwasawa invariants: application of analytic class number formula and $p$-adic Kronecker limit formula. We continue to follow [53, Chapter III.2].

\begin{lemma}
For any finite-order character $\nu \in \hat{\Gamma}'_+$, define $m_\nu = r$ if $\nu((\Gamma'_+)^r) = 1$ but $\nu((\Gamma'_+)^{p-1}) \neq 1$. Retaining the notation of Corollary 4.48, we have for all $n \gg 0$,

$$
\sum_{\chi \in \Delta'_+} \mu_{\text{inv}}(f_\chi^+) \cdot p^{t+n-1}(p-1) + \sum_{\chi \in \Delta'_+} \lambda_{\text{inv}}(f_\chi^+) = \text{ord}_p \left( \prod_{\nu, m_\nu = t+n} \nu(g_\chi^+) \right). \quad (63)
$$

Proof. This is just [53, Lemma III.2.9].

\end{lemma}

\begin{definition}
For any ramified character $\varepsilon \in \hat{\text{Gal}}(L_{n,s}^+/K)$, let $g = \text{cond}(|\varepsilon|)$, and let $g$ denote the smallest positive integer in $g$. Then in the same notation as for [50], define

$$
S_p(\varepsilon) := -\frac{1}{12gw} \sum_{c \in \mathbb{C}^+(g)} \varepsilon^{-1}(c) \log \phi_g(c).
$$

Fix $(\zeta_{f^p})_n$ a compatible system of $f^p$th roots of unity, and for a character $\varepsilon \in \hat{\text{Gal}}(L_{n,s}^+/K)$ of conductor dividing $f^p$, define the Gauss sum by

$$
g(\varepsilon) := \frac{1}{f^p} \sum_{a \in \text{Gal}(L_{n,s}^+/K)} \varepsilon(a) \zeta_{f^p}^{-a}.
$$
and let
\[ S_n := \{ \varepsilon \in \hat{\Gamma}_+ : p^\alpha || \text{cond}(\varepsilon) \}, \]
where the conductor is computed viewing \( \varepsilon : \text{Gal}(K(fp^\infty)/K) \to \mathbb{Q}_p^{	imes} \) (so that \( \Gamma_+ \) is the image of \( \Gamma \subset \text{Gal}(K(fp^\infty)/K) \) under the restriction from \( K(fp^\infty) \) to \( L_\infty^+ = K(\mu_{fp^\infty}), \) and \( \Gamma_+ = (\Gamma_+')^{p^\infty} \).

**Proposition 4.51.** In the notation of (4.50), for all \( n \gg 0 \) we have
\[ \text{ord} \left( \prod_{\varepsilon \in S_n} g(\varepsilon)S_p(\varepsilon) \right) = \text{ord}_p \left( \frac{hR_p}{w \sqrt{\Delta_p}} (L_n^+)/w \sqrt{\Delta_p} (L_{n-1}) \right). \]

**Proof.** This follows from the same argument as in [56, Chapter III.2.11], which is a standard calculation using the analytic class number formula. \( \square \)

4.10. **Relating to Kubota-Leopoldt \( p \)-adic \( L \)-functions.** We will show that the total \( \mu \)-invariant of \( \chi^+ \) is 0 by showing it is equal to the total \( \mu \)-invariant of a product of Kubota-Leopoldt \( p \)-adic \( L \)-functions, which has \( \mu \)-invariant 0 by the fundamental result of Ferrero-Washington (see also [56]).

The following Proposition relates the quantities \( S_p(\varepsilon) \) to a special value of a product of Kubota-Leopoldt \( p \)-adic \( L \)-functions. This is reminiscent of a theorem of Gross [24] on the factorization of the Katz \( p \)-adic \( L \)-function specialized to the cyclotomic line.

**Definition 4.52.** Recall \( \varepsilon \in \hat{\text{Gal}}(L_\infty^+/K) \). As \( L_\infty^+/Q \) is an abelian extension, \( \varepsilon \) has two extensions \( \varepsilon_+, \varepsilon_+ \varepsilon_K \in \hat{\text{Gal}}(L_\infty^+/Q) \) restricting to \( \varepsilon \), where \( \varepsilon_K \) denotes the quadratic character attached to \( K/Q \). Without loss of generality, let \( \varepsilon_+ \) denote the even extension, i.e. with \( \varepsilon_+(-1) = 1 \) and let \( \varepsilon_- = \varepsilon_+ \varepsilon_K \).

**Proposition 4.53.** We have
\[ g(\varepsilon)S_p(\varepsilon) = \frac{L_p(\varepsilon_+^{-1}\omega,0)}{2} \frac{L_p(\varepsilon_+,1)}{2}, \]
where \( L_p(\chi,s) \) denotes the Kubota-Leopoldt \( p \)-adic \( L \)-function (see [64, Chapter 5]).

**Proof.** We note that for \( n \gg 0 \), \( w_0 = 1 \). By argument of [24, Proof of Theorem 3.1], we have
\[ g(\varepsilon)S_p(\varepsilon) = -fp^nB_{1,\varepsilon_+} g(\varepsilon_+) \sum_{a \in (\mathbb{Z}/(fp^n))^{\times}} \varepsilon_+^{-1}(a) \log_p(C^+(a)), \]
where \( C^+(a) = (1 - \zeta_{fp^n}^a)(1 - \zeta_{fp^n}^{-a}) \) is the cyclotomic unit in loc. cit. The right-hand side of the above equation is, by the special value formulas recalled in loc. cit., equal to \( \frac{L_p(\varepsilon_+^{-1}\omega,0)}{2} \frac{L_p(\varepsilon_+,1)}{2} \). \( \square \)

4.11. **Vanishing of the \( \mu \)-invariant.**

**Corollary 4.54.** Recall the notation of Definition 4.47. Then for any \( \chi \in \hat{\Delta}_+ \), we have
\[ \mu_{inv}(f^+_\chi) = 0. \]

**Proof.** This follows from Proposition 4.53 and the vanishing of the \( \mu \)-invariant of (half of) any branch of half the Kubota-Leopoldt \( p \)-adic \( L \)-function \( \frac{1}{2}L_p \), established first by Ferrero-Washington, see for example [56]. \( \square \)
4.12. Proof of the Main Conjecture from vanishing of the cyclotomic algebraic $\mu$-invariant. In this section, we prove the Main Conjecture (Conjecture 4.35) from the result of [28], using as input the vanishing of the cyclotomic algebraic $\mu$-invariant.

**Theorem 4.55.** We have that (56) is true, and (57) is true for any non-trivial $\chi \in \hat{\Delta}$.

**Proof.** This will follow from the results of [28], and our results on the vanishing of the algebraic total $\mu$-invariant of $\mathcal{X}$. By Corollary 4.54, we have that $\mathcal{X}$ is a finitely generated $\Lambda$-module. Recall that $N_+^+$ is the maximal unramified $p$-abelian extension of $L_+^+$, and $\mathcal{Y}^+ = \lim_{\rightarrow} \text{Gal}(N_+^+ / L_+^+)$.

The natural surjection $\mathcal{X} \rightarrow \mathcal{Y}^+$ implies that $\mathcal{Y}^+$ is also a finitely generated $\mathbb{Z}_p$-module. Now let $\mathcal{H} := \ker(\text{Gal}(L_+^+ / K) \rightarrow \text{Gal}(L_+^+ / K)) \cong \mathbb{Z}_p \times \Delta_0^\prime$ for some finite abelian group $\Delta_0^\prime$. Since $\mathbb{Z}_p/\Delta_0^\prime$ is a finitely generated $\mathbb{Z}_p$-module, $\mathcal{Y}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta_0^\prime]$ is also a finitely generated $\mathbb{Z}_p$-module. Moreover, it is clear that as $L_+^+$ is the compositum of $\mathbb{Q}(\mu_{p^\infty})$ and a finite abelian extension, all primes are finitely decomposed in $L_+^+ = L^+(\mu_{p^\infty})$ and all primes of $L^+ = \mathbb{Q}(\mu_f)K$ above $p$ are totally ramified in $L_+^+ / L^+$.

Hence in the notation of loc. cit., $H^2(\mathcal{O}_K[1/p], \Lambda_{\mathcal{O}_K}(\Delta_0^\prime))((\text{Gal}(L_+^+ / K))(1))$ is a finitely generated $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module, and hence as in Corollary 5.10 of loc. cit., $H^2(\mathcal{O}_K[1/p], \Lambda_{\mathcal{O}_K}(1))$ is a finitely generated $\Lambda_{\mathcal{O}_K}(\mathcal{H})$-module by an application of Nakayama’s lemma for compact modules over completed group rings. Now the argument of Lemma 5.11 of loc. cit. goes through replacing the $\mathcal{H}$ of loc. cit. with the above $\mathcal{H}$ to show that Conjecture 5.5 of loc. cit. holds. By [28] Theorem 5.7, Section 5.4, this gives (56).

For (57), we first show that the anticyclotomic specialization of (56) holds, i.e.

$$\det_{\Lambda(\Delta \times \Gamma, \mathbb{Z}_p)}((E/\overline{E}(f))^-) = \det_{\Lambda(\Delta \times \Gamma, \mathbb{Z}_p)}((\mathcal{Y}^-)),$$

However, this follows from [18] Proposition 1.6.5 (3) and [28] Theorem 5.7. Now the discussion of Section 5.4 of loc. cit. implies that after inverting $p$ and taking isotypic components, we get (57).

**Corollary 4.56.** Se have that (54) is true for any $\chi \in \hat{\Delta}$.

**Proof.** This follows immediately from Theorem 4.55 and Proposition 4.36.

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5. Descent, Rank 0 converse and results on BSD

Let $K$ be an imaginary quadratic field of class number 1, and $E/K$ any elliptic curve with CM by $\mathcal{O}_K$. Denote its associated algebraic Hecke character of type $(1, 0)$ by $\lambda$ and denote its conductor by $\mathfrak{g}$. We now apply the results of the previous section to $\tilde{L} = K(1) = K$ (see Proposition 4.8), $L_n = K(p^n)$, so that $G_\infty = \text{Gal}(L_\infty / K) \cong \Gamma \times \Delta$, where $\Gamma \cong \text{Gal}(L_\infty / L^\prime)$ and $\Delta = \text{Gal}(L^\prime / K)$. Note that $K(p^\infty) = K(E[p^\infty])$ (where the last equality follows from the theory of complex multiplication).

Let $\lambda : \text{Gal}(L_\infty / K) \rightarrow \mathcal{O}^\times_{K_p}$ be the local reciprocity character, and let $\chi_E : \text{Gal}(L^\prime / K) \rightarrow \mathcal{O}^\times_{K_p}$ be its restriction to $\Delta$.

Descent in the rank 0 case essentially entails replacing the map “$\delta$” in [45] Section 11.4 with the map $(\lambda/\chi_E)^{\mu_1} : (U^1)_{\chi_E} \rightarrow \Lambda(\text{Gal}(L_{p,\infty} / K_p), \mathcal{O}_{L_p})_{\chi_E}$, and following the same arguments. In essence, $(\lambda/\chi_E)^{\mu_1}$ is a $\Lambda(\text{Gal}(L_{p,\infty} / K_p), \mathcal{O}_{L_p})_{\chi_E}$-adic lifting of $\delta$, which is itself simply the “first moment” of the explicit local reciprocity law.

5.1. The Selmer group via local class field theory.

**Definition 5.1.** Let $T(\lambda) := \mathcal{O}_{L_p} (\lambda)$, $V(\lambda) := T(\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $W(\lambda) := V(\lambda)/T(\lambda) = T(\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p$.

We define

$$S_n(\lambda) = \ker \left( \prod_v \text{loc}_v : H^1(L(E[p^n]), W(\lambda)) \rightarrow \prod_v H^1(K(E[p^n]), E) \right).$$
where the product runs over all places of \( K(E[p^n]) \). Let \( S_\infty(\lambda) = \lim_{n \to \infty} S_n(\lambda) \). In particular, we see that \( S_\infty(\lambda) \) is a \( \Lambda(G_\infty, \mathcal{O}_{L_p, \infty}) \)-module (note that \( \mathcal{O}_{L_p, \lambda} = \mathcal{O}_{L_p} \)). Following the notation of [45], we define a relaxed Selmer group

\[
S'_n(\lambda) = \ker \left( \prod_{v \nmid p} \text{loc}_v : H^1(K(E[p^n]), W(\lambda)) \to \prod_{v \nmid p} H^1(K(E[p^n])_v, E) \right),
\]

where the product runs over all places of \( K(E[p^n]) \) not dividing \( p \).

**Definition 5.2.** Given a group \( H \) and an \( H \)-module \( M \), let \( M^H \) denote the invariants by \( H \).

**Proposition 5.3** (Theorem 11.2 of [45]). For all \( n \geq 2 \), the restriction map \( H^1(K_n, W(\lambda)) \to H^1(K_\infty, W(\lambda)) \) induces isomorphisms

\[
S'_n(\lambda) \cong \text{Hom}(\mathcal{X}_n, W(\lambda))^{\text{Gal}(K(E[p^n])_\infty)/K(E[p^n])} \cong \text{Hom}(\mathcal{X}_\infty, W(\lambda))^{\text{Gal}(K(E[p^n])_\infty)/K(E[p^n])},
\]

**Proof.** This is well-known. See [45, Proof of Theorem 11.2] for an outline with detailed references.

\[ \square \]

**Lemma 5.4** (cf. Lemma 11.6 of [45]). We have an isomorphism

\[
H^1(K_p(E[p^n]), W(\lambda)) \cong \text{Hom}(\mathcal{U}_1^1, W(\lambda))
\]

as \( \text{Gal}(\overline{K_p(E[p^n])}/K_p(E[p^n])) \)-modules.

**Proof.** Since \( \text{Gal}(\overline{K_p}/K_p) \) acts on \( W(\lambda) = E[p^n] \) through \( \lambda \), we have that \( \text{Gal}(\overline{K_p(E[p^n])}/K_p(E[p^n])) \) acts trivially on \( W(\lambda) \), and so

\[
H^1(K_p(E[p^n]), W(\lambda)) = \text{Hom}(\text{Gal}(\overline{K_p(E[p^n])}/K_p(E[p^n])), W(\lambda))
\]

By local class field theory, since \( K = K(1) \), we have an isomorphism

\[
\mathcal{U}_1 \times \mu(\mathcal{O}_K) \times \hat{\mathbb{Z}} = \mathcal{U} \times \hat{\mathbb{Z}} \cong \text{Gal}(K_p(E[p^n])^{ab}/K_p(E[p^n])).
\]

By Lemma 11.5, we have that \( \chi_E \) is non-trivial on the decomposition subgroup of \( \Delta \) above \( p \), and so \( \hat{\mathbb{Z}}_{\chi_E} = 1 \). Clearly, \( \text{Gal}(K_p(E[p^n])_\infty)/K_p(E[p^n]) \) acts trivially on \( \mu(\mathcal{O}_K) \), and so \( \mu(\mathcal{O}_K)_{\chi_E} = 1 \). Hence (67) gives an isomorphism

\[
(\mathcal{U}_1^1)_{\chi_E} \cong \text{Gal}(K_p(E[p^n])^{ab}/L_{p, \infty})_{\chi_E}.
\]

This, along with (66), gives (65).

Let \( M_n/K(E[p^n]) \) denote the maximal pro-\( p \) abelian extension of \( K(E[p^n]) \) unramified outside of places above \( p \). Let \( M_{p, \infty} = \bigcup_{n \in \mathbb{Z}_{>0}} M_{n, p} \) and \( \chi_p = \text{Gal}(M_{p, \infty}/K_p(E[p^n])) \). The \( \chi_E \)-part \( (\mathcal{U}_1^1)_{\chi_E} \stackrel{\text{rec}}{\to} (\chi_p)_{\chi_E} \) of the local reciprocity map \( \mathcal{U}_1^1 \stackrel{\text{rec}}{\to} \chi_p \) induces a map

\[
\text{Hom}((\chi_p)_{\chi_E}, W(\lambda)) \to \text{Hom}((\mathcal{U}_1^1)_{\chi_E}, W(\lambda)).
\]
We denote this map by \( f \to f|_{(U^1)_{\chi E}} \). We have the following commutative diagram (cf. [45, p. 62]):

\[
\begin{array}{ccc}
\text{Hom}((\mathcal{X}_p)_{\chi E}, W(\lambda))^\Delta \\
\downarrow \phi \\
\text{Hom}((U^1)_{\chi E}, W(\lambda))^\Delta
\end{array}
\]

\[ 0 \to \left( E(K_p(E[p^\infty])) \otimes_{O_{K_p}} K_p/O_{K_p} \right)^\Delta \to H^1(K_p(E[p^\infty]), W(\lambda))^\Delta \to (H^1(K_p(E[p^\infty]), E[p^\infty])^\Delta \to 0 \]

where the middle row is exact by the local descent exact sequence. Given \( f \in \text{Hom}((\mathcal{X})_{\chi E}, W(\lambda)) \), denote by \( f_p \) the image under the natural map \( \text{Hom}((\mathcal{X})_{\chi E}, W(\lambda)) \to \text{Hom}((\mathcal{X}_p)_{\chi E}, W(\lambda)) \) given by the natural inclusion \( \mathcal{X}_p \subset \mathcal{X} \) (induced by \( i_p \) from [3]). By Proposition 5.3 and (69), we have

\[
\text{S}_\infty(\lambda)^\Delta = \{ f \in \text{Hom}((\mathcal{X})_{\chi E}, W(\lambda))^\Delta : f_p|_{(U^1)_{\chi E}} \in \text{im}(\phi) \}.
\]

We have a map \((\mu^1)_{\chi E} := (\mu^1_{\text{glob}})_{\chi E} : (U^1)_{\chi E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \Lambda(\text{Gal}(K_p(E[p^\infty])/K_p), O_{K_p})_{\chi E}[1/p] \) induced by taking the \( \chi_E \)-component of \([35]\) (with \( f = 1 \), so that \( U^1 = U^1 \)).

**Definition 5.5.** Given a measure \( \mu \in \Lambda(G, R) \) and a continuous function \( f \) on \( G \), we define the twist of \( \mu \) by \( f \) by

\[ f^* \mu(g) = \mu(fg) \]

for any continuous function \( g \) on \( G \).

**Definition 5.6.** Define

\[
\delta := (\lambda/\chi E)^*(\mu^1)_{\chi E} : (U^1)_{\chi E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \Lambda(\text{Gal}(K_p(E[p^\infty])/K_p), O_{K_p})_{\chi E}[1/p].
\]

Here, recall our notation for pullback: for any function \( f \) on \( \text{Gal}(L_{p,\infty}/K_p) \), and any \( u \in (U^1)_{\chi E} \),

\[ (\lambda/\chi E)^*(\mu^1)_{\chi E}(u)(f) = (\mu^1)_{\chi E}(u)((\lambda/\chi E)f). \]

5.2. **Interlude: technical lemmas relating \( U^1 \) to good reduction twists.** Recall the fixed CM elliptic curve \( A/L \) from Definition 4.1 so that \( w_1 = 1, (f, p) = 1, L = K(f) \), and \( F = A \) is a relative Lubin-Tate group for the unramified extension \( L_{p}/K_p \). Henceforth make the following choice.

**Choice 5.7.** Recalling that \( g \) is the conductor of \( \lambda \), let \( g_0 \) denote the prime-to-\( p \) part of \( g \). Choose \( f \) as above with \( g_0/f \).

This ensures that \( K(E[p^n]) \subset K(fp^n) = L(A[p^n]) \) (where the first equality follows because \( K(E[p^n])/K \) is ramified at primes dividing \( gp \), and the last equality follows from [53 Proposition II.1.6]). Denote the associated tower of local units by \( \mathcal{U}' \), and the associated tower of principal local units by \( (U')^1 \). Recalling that \( U^1 \) denotes the tower of local units attached to \( E \), by the previous sentence, we have \( U^1 \subset (U')^1 \). Moreover, the norm from \( L(A[p^n]) \to K(E[p^n]) \) induces a norm map \( \text{Nm} : (U')^1 \to U^1 \). Denote the type \((1, 0)\) Hecke character associated with \( A \) by \( \lambda_A \), and viewed as a Galois character on \( \text{Gal}(L(A[p^\infty])/K) \) let \( \chi_A = \lambda_A|_{\text{Gal}(L(A[p^\infty])/K_{\infty})} \), where \( K_{\infty}/K \) denotes the unique \( \mathbb{Z}_p^{\infty} \)-extension of \( K \).

**Proposition 5.8.** We have

\[
\lambda/\chi_E = \lambda_A/\chi_A
\]

as characters \( \text{Gal}(K_{\infty}/K) \to O_{K_p}^\times \).
Proof. By the theory of complex multiplication ([46, Theorem 5.11]), we see that $\lambda/\chi_E$ and $\lambda_A/\chi_A$ both map $\text{Gal}(K/\lambda)$ into $1 + p\mathcal{O}_{K_p}$ with finite cokernel. Moreover, since $\lambda$ and $\lambda_A$ differ by a finite-order character (both are of infinite type $(1,0)$), we have that $\lambda/\chi_E$ and $\lambda_A/\chi_A$ differ by a finite-order character. Since the images of both $\lambda/\chi_E$ and $\lambda_A/\chi_A$ lie in the torsion-free group $1 + p\mathcal{O}_{K_p}$, this finite-order character must be trivial, and so we have $\lambda/\chi_E = \lambda_A/\chi_A$ on all of $\text{Gal}(K/\lambda)$. 

\[ \text{Proposition 5.9.} \]

Let $\delta': \mathcal{U} \to \text{Hom}(E(K_p,\lambda)\otimes\mathcal{O}_{K_p}, K_p/\mathcal{O}_{K_p}, W(\lambda)), \delta_A': (\mathcal{U})^1 \to \text{Hom}(A(L_p,\lambda)\otimes\mathcal{O}_{K_p}, K_p/\mathcal{O}_{K_p}, W(\lambda))$ be the natural Kummer maps precomposed with the local Artin maps (cf. [53, Chapter I.4]). Note that since $A(L_p,\lambda) = E(L_p,\lambda)$, there is a trace map $\text{Tr}_{L_p,\lambda}/K_p: A(L_p,\lambda) = E(L_p,\lambda) = E(K_p,\lambda)$.

There is commutative diagram of $\Lambda(\text{Gal}(K_p,\lambda)\otimes\mathcal{O}_{K_p}[1/p])$-equivariant maps (cf. (36))

\[
\begin{array}{ccc}
\mathcal{U}^1 & \xrightarrow{\delta'} & \text{Hom}(E(K_p,\lambda)\otimes\mathcal{O}_{K_p}, K_p/\mathcal{O}_{K_p}, W(\lambda)) \\
\downarrow \text{Nm} & & \downarrow \text{Tr}_{L_p,\lambda}/K_p, \\
(\mathcal{U})^1 & \xrightarrow{\delta_A'^{1}} & \text{Hom}(A(L_p,\lambda)\otimes\mathcal{O}_{K_p}, K_p/\mathcal{O}_{K_p}, W(\lambda)) \\
\downarrow \text{Nm} & & \downarrow \text{incl} \\
\mathcal{U}^1 & \xrightarrow{\delta'} & \text{Hom}(E(K_p,\lambda)\otimes\mathcal{O}_{K_p}, K_p/\mathcal{O}_{K_p}, W(\lambda)),
\end{array}
\]

where the first right vertical arrow is pullback by $\text{Tr}_{L_p,\lambda}/K_p: A(L_p,\lambda) \to E(K_p,\lambda)$, the second right vertical arrow is pullback by the inclusion $\text{incl}: E(K_p,\lambda) \subset E(L_p,\lambda) = A(L_p,\lambda)$, and the left vertical arrows and right vertical arrows are isomorphisms of $\Lambda(\text{Gal}(K_p,\lambda)\otimes[1/p])$-modules.

Proof. This follows immediately from the restriction/corestriction functorialities of the local reciprocity map. 

\[ \text{5.3. Description of the Selmer group via Wiles’s explicit reciprocity law.} \]

\[ \text{Proposition 5.10 (cf. Proposition 11.10 of [45]).} \]

Let

\[ \delta W(\lambda) = \delta E[p^\infty] := \{ \text{maps } u \mapsto \delta(u)v : v \in W(\lambda) = E[p^\infty] \}. \]

We have

\[ \text{im}(\phi)^\infty \supset \delta W(\lambda)^\infty = \text{Hom}(\delta(\mathcal{U})^1, W(\lambda))^{\infty} \]

with the left-hand side containing the right-hand side with index a finite power of $p$.

Proof. The equality in (44) follows immediately from the definitions. When $E$ has good reduction, the containment is a consequence of the explicit local reciprocity law of Wiles as stated in [53, Theorem I.4.2]. Indeed, first assume that $E/\mathcal{O}_{K_p}$ has good reduction. We have a natural map $\delta': (\mathcal{U})^1_{\chi_E} \to \text{Hom}(E(L_p,\lambda)\otimes\mathcal{O}_{K_p}, K_p/\mathcal{O}_{K_p}, W(\lambda))_{\chi_E}$ given by the Kummer pairing. Taking $G_{\infty}$-invariants, by loc. cit. we have

\[ \delta'(\mathcal{U})^1_{\chi_E} = \left( x \otimes \pi^{-k} \mapsto \left[ \frac{d \log(\beta_0)}{\log E(x)} \right] \right). \]

Using the fact that $\theta_{\alpha_0}^E Q d^{-1} = \Omega_p \cdot d \log(\lambda)^/\lambda x$, by (42) we have that $\ker(\delta^\infty) \supset \ker((\delta')^\infty)$. Now (44) and the finite index assertion follow after noting that $\log E: E[p^\infty\mathcal{O}_{K_p}] \rightarrow p^\infty\mathcal{O}_{K_p}$ for all $r \gg 0$.

Now if $E/\mathcal{O}_{K_p}$ has bad reduction, using (73) and the above argument for a good reduction twist $A/\mathcal{O}_{K_p}$, we again see that $\ker(\delta^\infty) \supset \ker((\delta')^\infty)$, which gives (44) in this case.
A key consequence of (74) is the following.

**Theorem 5.11.** There is a natural containment

\[(75) \quad S_0(\lambda)^{G_\infty} \supset \text{Hom}((\mathcal{X}')_{\chi_E}, W(\lambda))^{G_\infty}\]

where the left-hand side contains the right-hand side with finite $p$-power index, and where $\mathcal{X}'$ is defined as in Definition 4.47. Hence,

\[(76) \quad \#S_0(\lambda)^{G_\infty} \sim \#\text{Hom}((\mathcal{X}')_{\chi_E}, W(\lambda))^{G_\infty}\]

where “~” denotes equality up to a finite power of $p$. (Here, we follow the usual convention that one side is infinite if an only if the other side is infinite.)

**Proof.** This follows immediately from (74) and (70). □

**Proposition 5.12.** We have

\[(77) \quad \#(\text{Hom}((\mathcal{X}')_{\chi_E}, W(\lambda))^{\text{Gal}(L_\infty/K)}) \sim \#(\text{Hom}((\mathcal{U}')_{\chi_E}/C(\chi_E), W(\lambda))^{\text{Gal}(L_\infty/K)}).\]

**Proof.** This follows from the same arguments as in [45, Proof of Theorem 11.16], using the Rubin-type Main Conjecture (55) for $\chi = \chi_E$ and the fact that $\mathcal{X}$ has no nonzero pseudo-null submodules by [23, Theorem 2]. □

Theorem 5.11 has the following consequence for rank 0 BSD for elliptic curves $E/\mathbb{Q}$ with CM by $\mathcal{O}_K$.

**Theorem 5.13.** In the setting of Theorem 5.11, write

\[\mathcal{L} = \frac{L(\chi, 1)}{\Omega_\infty},\]

where $\Omega_\infty$ is the usual Néron period associated with $E$, and in particular $\mathcal{L} \in \mathbb{Q}$. Then we have

\[(78) \quad \#S_0(\lambda) \sim \#(\mathcal{O}_{K(p)}/\mathcal{L} \mathcal{O}_{K(p)}),\]

where “~” denotes equality up to a finite power of $p$. In particular,

\[L(E/\mathbb{Q}, 1) = L(\chi, 1) \neq 0 \iff \#S_0(\lambda) < \infty.\]

**Proof.** We have

\[\#S_0(\lambda) \overset{(76)}{=} \#(\text{Hom}((\mathcal{X}')_{\chi_E}, W(\lambda))^{G_\infty})\]

\[\overset{(77)}{=} \#(\text{Hom}((\mathcal{U}')_{\chi_E}, W(\lambda))^{G_\infty}).\]

Now we have

\[\#(\text{Hom}((\mathcal{U}')_{\chi_E}/C(\chi_E), W(\lambda))^{G_\infty}) = \#\{\phi \in \text{Hom}((\mathcal{U}')_{\chi_E}, W(\lambda))^{G_\infty} : \phi(C(\chi_E)) = 0\}\]

\[= \{v \in W(\lambda) : (\lambda/\chi_E)^* \mu_{\text{glob}}(\chi_E)v = 0\}\]

\[\overset{(14)}{=} \{v \in W(\lambda) : \left(1 - \frac{\chi(p)}{N(p)}\right) \mathcal{L} v = 0\}\]

\[= \#(\mathcal{O}_{K(p)}/\left(1 - \frac{\chi(p)}{N(p)}\right) \mathcal{L} \mathcal{O}_{K(p)}).\]

Now (78) immediately follows. □
Remark 5.14. Note that $S_0(\lambda) = \text{Sel}_p(E/Q)$. Hence Theorem 5.13 implies that if $\#\text{Sel}_p(E/Q) < \infty$, i.e. corank_{\Z_p}(\text{Sel}_p(E/Q)) = 0, then $L(E/Q, 1) = L(\lambda, 1) \neq 0$, i.e. $\text{ord}_{s=1}L(E/Q, s) = 0$. As mentioned before, it is possible to make the constant $c(E)$ explicit, and hence with more work establish the $p$-part of the BSD formula for $E/Q$ in this case.

Together with the $2\infty$-Selmer distribution results of Smith [57], we have the following result establishing that 50% of quadratic twists of certain elliptic curves with CM by $K = \Q(i)$ and with full rational 2-torsion have analytic rank 0. Given an elliptic curve $E : y^2 = x^3 + ax + b$ defined over $\Q$, and an integer $d$, recall the quadratic twist $E^d : dy^2 = x^3 + ax + b$.

Corollary 5.15. Suppose $E/Q$ has full rational 2-torsion (i.e. $E[2] = E[2](\Q)$), that $E$ admits no cyclic rational 4-isogeny (i.e. $E(\Q)$ does not contain a cyclic group of order 4), and that $E$ has CM by $\mathcal{O}_K$ for $K = \Q(i)$. We have that

$$\lim_{X \to \infty} \frac{\# \{ 0 < |d| < X : d \text{ squarefree}, \text{ord}_{s=1}L(E^d/Q, s) = 0 \}}{\# \{ 0 < |d| < X : d \text{ squarefree} \}} = \frac{1}{2}.$$

Proof. By the Selmer distribution results of [57], we have that corank_{\Z_2}(\text{Sel}_2(E^d/Q)) = 0 for 50% of fundamental discriminants $d$ (with respect to the ordering by increasing $|d|$). Now the assertion follows from Theorem 5.13.

In particular, the above theorem has the following consequence for the congruent number family $E^d : y^2 = x^3 - d^2 x$.

Corollary 5.16. Consider $E^d : y^2 = x^3 - d^2 x$. Then $\text{ord}_{s=1}L(E^d/Q, s) = 0$ for 100% of squarefree $d \equiv 1, 2, 3 \pmod{8}$, ordering by increasing $|d|$.

Proof. This follows from the result of [57] that corank_{\Z_2}(\text{Sel}_2(E^d/Q)) = 0 for 100% of squarefree $d \equiv 1, 2, 3 \pmod{8}$.

Remark 5.17. This verifies one half of Goldfeld’s conjecture (Conjecture 7.6) for the congruent number family, i.e. that “50% of quadratic twists of $E : y^2 = x^3 - x$ have analytic rank 0”. We will later show (Corollary 7.7) that the other half of Goldfeld’s conjecture is true for $E$, i.e. that “50% of quadratic twists of $E$ have analytic rank 1”. This is the first instance of the verification of Goldfeld’s conjecture for any quadratic twist family for an elliptic curve over $\Q$.

6. Factorizations of $p$-adic $L$-functions and Selmer groups

In this section, we show a factorization along the anticyclotomic line of the $p$-adic $L$-function of a certain CM form into two $p$-adic Hecke $L$-functions, which are special cases of the construction from the previous sections. This factorization will be crucial for relating the Iwasawa theory of elliptic units to the Iwasawa theory of Heegner points, and hence for obtaining rank 1 converse theorems.

Henceforth, let $E/Q$ be an elliptic curve with CM by an imaginary quadratic field $K$, so that $K$ has class number 1. Let $\lambda$ be the Hecke character over $K$ associated with $E/K$, of infinity type $(1,0)$, and $\theta_\lambda$ its associated theta series. Let $\chi$ be any Hecke character over $K$, and let $\chi^*(x) = \chi(x)$ (and as before $\bar{x}$ denotes the complex conjugate of $x$). Now let $g = \theta_\psi$, and let $F = \Q(g, \chi)$ denote the finite extension of $\Q$ generated by the Hecke eigenvalues of $g$ and by the values of $\chi$. Note that we have the following compatibility of the central character $w_g$ of $g$ and $\chi$:

$$w_g \cdot \chi|_{K^\times_Q} = 1.$$

Note that we have the factorization of $L$-series

$$L(g \times \chi, s) = L(\lambda, s)L(\psi^* \chi, s)$$

(80)
where on the left-hand side, we are considering the base change of \( g \) to \( K \).

We also have a corresponding factorization of Galois representations.

**Definition 6.1.** Let \( V_g \) denote the \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-representation associated with \( g \), so that we have

\[
V_g|_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{K})} = F_p(\psi) \oplus F_p(\psi^*) \quad \text{and} \quad V_{g,\chi} := V_g|_{\text{Gal}(\overline{\mathbb{K}}/\mathbb{K})} \otimes_{O_{\mathbb{F}_p}} \chi = F_p(\lambda) \oplus F_p(\psi^*) \chi.
\]

Let \( T_g \subset V_g|_{\text{Gal}(\overline{\mathbb{K}}/\mathbb{K})} \) and \( T_{g,\chi} \subset V_{g,\chi} \) be \( \text{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \)-invariant lattices such that

\[
T_g \cong O_{\mathbb{F}_p}(\psi) \oplus O_{\mathbb{F}_p}(\psi^*), \quad T_{g,\chi} \cong O_{\mathbb{F}_p}(\lambda) \oplus O_{\mathbb{F}_p}(\psi^*) \chi
\]

and let \( W_g = V_g/T_g \) and \( W_{g,\chi} = V_{g,\chi}/T_{g,\chi} \).

### 6.1. Heegner points

**Assumption 6.2.** Again, let \( E/\mathbb{Q} \) be an elliptic curve with CM by \( O_K \) where \( K \) is an imaginary quadratic field of class number 1, and let \( \lambda \) be the type \((1,0)\) Hecke character associated with \( E/K \). Assume that the root number of \( L(\lambda, s) = L(E/\mathbb{Q}, s) \) is \(-1\).

**6.2. A choice of twist of \( \lambda \).** We now consider a certain twist of \( \lambda \) by a Hecke character \( \chi \) over \( K \), such that (in particular) the associated theta-series \( \theta_{\lambda} \) will have level prime to \( p \) when viewed as a modular form over \( K \). Recall that given a Hecke character \( \chi \) of conductor \( \mathfrak{f} \subset O_K \), then we can view \( \chi \) as a character on ideals \( \mathfrak{a} \) prime to \( \mathfrak{f} \) via

\[
\chi(\mathfrak{a}) = \prod_{v\mid\mathfrak{f}} \chi_v(\mathfrak{a}_v)
\]

where \( \mathfrak{a}_v \in O_{K_v} \) is any uniformizer of \( \mathfrak{a}O_{K_v} \).

**Proposition 6.3.** There exist infinitely many finite order Hecke characters \( \chi : \mathbb{A}_K^\times \to \mathbb{C}^\times \) such that, letting \( \varphi = \lambda/\chi \),

1. \( \chi \) is valued in \( K^\times \),
2. \( \varphi \) is unramified at \( p \), i.e. \( \chi|_{O_{K_p}^\times} = \lambda|_{O_{K_p}^\times} \),
3. \( L(\varphi^*\chi, 1) \neq 0 \).

**Proof.** The first condition is clearly satisfied for infinitely many \( \chi \); for example, see [46, Corollary 5.22]. By the theory of complex multiplication, \( \lambda|_{K_p} \subset K^\times \), so one can moreover choose \( \chi \) to satisfy (2) by restricting to the (infinitely) many Dirichlet characters \( \chi : (O_K/\mathfrak{f})^\times \to O_{K_p}^\times \).

We claim that condition (3) holds for all but finitely many \( \chi \) satisfying (1) and (2) above. Indeed, by [44] we have that \( L(\lambda^*\chi_0, 1) \neq 0 \) for all but finitely many anticyclotomic characters \( \chi_0 \). By [26, Lemma 5.31], we have that any anticyclotomic \( \chi_0 \) can be written as \( \chi/\chi^* \) for some Hecke character \( \chi \). Clearly, as \( \chi \) varies through Dirichlet characters \( \mathbb{A}_K^\times \to O_{K_p}^\times \) satisfying (1) and (2), \( \chi/\chi^* \) varies through infinitely many anticyclotomic characters. Hence \( L(\varphi^*\chi, 1) \neq 0 \) for all but finitely many \( \chi \).

**Definition 6.4.** Note that the associated theta series \( g = \theta_\varphi \) has central character \( \chi^{-1}|_{\mathbb{A}_Q^\times} \) when viewed as a modular form on \( \text{GL}_2(\mathbb{A}_Q) \). Recall that we have an associated abelian variety \( A_g/\mathbb{Q} \) of \( \text{GL}_2 \)-type. Let \( F_g \) denote the finite extension of \( K \) obtained by adjoining the Hecke eigenvalues of \( g \), so that naturally \( \text{End}(A_g) = O_{F_g} \). Note that if \( \chi \) is chosen as in Proposition 6.3 then \( F_g = K \).

In any case, \( F_g \subset F \) (recalling that \( F \) is the finite extension of \( K \) obtained by adjoining the Hecke eigenvalues of \( g \) and values of \( \chi \)), and \( O_{F_g} \cong \text{End}(A_g) \). We consider the self-dual Rankin-Selberg pair \((g, \chi)\). Then we have the Serre tensor product

\[
B := A_g \otimes_{O_{F_g}} O_F(\chi)
\]
associated with \((g, \chi)\) where we consider \(g\) over \(K\), or more precisely its base change from \(\mathbb{Q}\) to \(K\). Here, \(\mathcal{O}_F(\chi)\) denotes the free \(\mathcal{O}_F\)-module space \(\mathcal{O}_F\) with the action of \(\text{Gal}(K^{ab}/K)\) given by multiplication through \(\chi\).

**Convention 6.5.** To lessen notation, henceforth we use the same notation to denote an algebraic Hecke character, its \(p\)-adic avatar and its associated \(p\)-adic Galois character. The particular avatar of the character in use should be clear from context.

### 6.3. Heegner points

In this section, for \(\chi\) satisfying Proposition 6.3, we define Heegner points on \(B\) as follows.

**Definition 6.6.** Let \(D/\mathbb{Q}\) be an indefinite quaternion algebra ramified at \(\ell \nmid \infty\) such that the local root numbers \(\epsilon_\ell(D)\) satisfy
\[
\epsilon_\ell(D) = \epsilon_\ell(g \times \chi)(\chi|_{\mathcal{O}_D^\times})_p \epsilon_\ell(\eta_\ell(-1)),
\]
where \(\eta\) denotes the quadratic character associated with \(\chi\). In this setting, we have a family of Shimura curves \(X = \varprojlim_U X_U\), running over open compact subgroups \(U \subset D^\times(\mathbb{A}_\mathbb{Q}, f)\). Let \(J_U\) denote the Albanese of \(X_U\), so that using the Hodge class we have an Abel-Jacobi map
\[
X \to J := \varprojlim_U J_U
\]
defined over \(\mathbb{Q}\).

**Proposition 6.7.** In addition to the properties listed in Proposition 6.3, we can choose \(\chi\) to have
\[
\epsilon_p(D) = \epsilon_p(g, \chi)((\chi|_{\mathcal{O}_D^\times})_p \epsilon_p (-1) = \epsilon_p(g, \chi) = +1,
\]
i.e. \(D \otimes_{\mathbb{Q}} \mathbb{Q}_p = M_2(\mathbb{Z}_p)\).

**Proof.** Let \(\pi\) be the automorphic representation on \(GL_2(\mathbb{A}_\mathbb{Q})\) associated with \(g = \theta_\omega\). Recall that the central character \(w_g\) of \(g\) is \(\chi|_{\mathcal{O}_D^\times}\), and so by property (1) of Proposition 6.3, we have \(w_g(p)(-1) = (\chi|_{\mathcal{O}_D^\times})_p^{-1}(-1) = \eta_p(-1)\), and so \(\chi|_{\mathcal{O}_D^\times})_p \epsilon_p(-1) = +1\), and we are reduced to showing \(\epsilon_p(g, \chi) = +1\). Note that we have \(\epsilon_p(g, \chi) = \epsilon_p(\lambda)\epsilon_p(\varphi^*\chi)\) (see the properties of epsilon factors given in [62]), where \(\epsilon_p(\chi')\) denotes the local root number at \(p\) of a Hecke character \(\chi'\) over \(K\). By [60] Formula for \(\rho_0(c), p. 2.19\), we have \(\epsilon_p(\varphi^*\chi) = \epsilon_p(\chi)\) (since \(\varphi\) is unramified at \(p\)) and \(\epsilon_p(\chi) = \epsilon_p(\lambda)\) (by property (2) of \(\chi\) given in Proposition 6.3), and so we have \(\epsilon_p(g, \chi) = \epsilon_p(\lambda)^2 = \epsilon_p(E/\mathbb{Q})^2 = +1\). \(\square\)

**Definition 6.8.** Henceforth, fix an embedding \(K_p \subset M_2(\mathbb{Q})\) such that \(K_p \cap M_2(\mathbb{Z}_p) = \mathcal{O}_{K_p}\). Recall that \(g = \theta_{\varphi}\), and that the central character of \(g\) is \(\omega_g = \chi|_{\mathcal{O}_D^\times}\). In particular, \(\omega_g\) is nontrivial. By our choice of \(\chi\) given in Proposition 6.3, we have that \(\alpha := \varphi(\mathfrak{p})\) is a well-defined value since \(\varphi\) is unramified at \(p\), and hence there exists a nonzero vector
\[
\phi \in \pi_g := \text{Hom}^0(X, A_g) := \text{Hom}(X, A_g) \otimes_{\mathbb{Z}} \mathbb{Q},
\]
where \(A_g\) is the quotient of the \(J_U\) cut out by \(g\) via the Eichler-Shimura construction, such that
\[
U_p\phi = \alpha \phi.
\]

By Proposition 6.7, the quaternion algebra \(D\) on which \(g\) lives is split at \(p\), i.e. we can identify \(D \otimes_{\mathbb{Q}} \mathbb{Q}_p = M_2(\mathbb{Z}_p)\). For any \(m \in \mathbb{Z}_{\geq 0}\), define the Iwahori subgroup
\[
U_{p, m} = \left\{ g \in GL_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^m} \right\} \subset GL_2(\mathbb{Z}_p).
\]

Henceforth, fix a point \(P \in X_{K^\times}\); using the moduli-theoretic interpretation of \(X\), we can view \(P\) as a tuple consisting of a CM abelian surface (in fact a “false elliptic curve”) \(A\) together with
a Rosatti idempotent $\epsilon$, a trivialization $\alpha: \mathbb{Z}_p^{\geq 2} \sim \epsilon T_p A$ of its Tate module, tame level structure, polarization, and other data, see [6] for an exposition. Recall that we let $K_n^-/K$ denote the degree $p^n$ subextension of the anticyclotomic $\mathbb{Z}_p$-extension $K_{\infty}/K$, i.e. $\text{Gal}(K_n^-/K) \cong \mathbb{Z}/p^n$. For our purposes, it will be enough to suppress all tame data and only consider the data $(A, \alpha)$ when considering $P$. Since $\epsilon_p(D) = +1$, i.e. $D$ is split at $p$, the supersingular locus of the $p$-adic fiber of $X$ is has an étale cover by copies of LT. We can furthermore fix $P$ so that it satisfies [51] Choice 8.6 (i.e. an appropriate trivialization $\mathcal{O}_{K_p} \cong \mathbb{Z}_p^{\geq 2}$ as in loc. cit.). Note that $P$ induces a point $P_U$ on $X_U$ for every compact open subgroup $U \subset B(\mathbb{A}_f)^{\times}$ (where here $\mathbb{A}_f \subset \mathbb{A}_Q$ denotes the finite rational adèles); for any $P \in X_U$, where $U \subset U'$ as open compact sugroups of $B^{\times} = (B \otimes \mathbb{A}_f)^{\times}$, let $\phi(P)_U = \phi(P_U)$. We now define a family of Heegner points

\begin{equation}
(P_{x,n}(\phi) := \int_{\text{Gal}(\overline{K}/K_n^-)} \phi(P^\sigma)_{U_{p,n}} \otimes \chi(\sigma)d\sigma) \in \Delta(K_n^-) := B(K_n^-) \otimes \mathbb{Z} \mathbb{Q}
\end{equation}

where the Haar measure $d\sigma$ on $\text{Gal}(\overline{K}/K)$ is chosen to have total volume 1.

**Definition 6.9.** Let $A_{\text{univ}} \rightarrow X$ denote the universal false elliptic curve. Then $A_{\text{univ}}$ is isogenous to a product of elliptic curves $A_1 \times A_2$, one of which is picked out by the Rosatti idempotent $\epsilon$. On a covering Lubin-Tate space $LT \rightarrow X^\text{ss}$ of the supersingular locus, we therefore have a natural identification $A_1|_{LT} \cong F_{\text{univ}}$, and we fix such isomorphisms henceforth.

**Definition 6.10.** Note that by Faltings’ isogeny theorem, the factorization of $L$-functions \([80]\) implies that $E$ appears as an isogeny factor of $B$ (even the isotypic component in the isogeny category corresponding to $g \times \chi$). Hence we may project $P_{x,n}$ in the isogeny category to $E$. Denote the image of this projection by $P_{x,n}^1$.

6.4. A Rankin-Selberg $p$-adic $L$-function. In this section, we give a construction of a Rankin-Selberg $p$-adic $L$-function using our coordinate $Q - 1$ to construct a suitable $GL_2$ half-Coleman map.

Recall our universal deformation $\rho : F_{\text{univ}} \rightarrow LT$ with normalized differential

$$\omega_0^{\text{univ}} = \log'_F \langle X^{\text{univ}} \rangle dX^{\text{univ}} \in \rho_\ast \Omega^{1}_{F_{\text{univ}}/LT}.$$

We have the universal logarithm $\log_{F_{\text{univ}}} \in \mathcal{O}_{LT, \eta}(LT, \eta)$.

**Definition 6.11.** We have a power series

$$\log_{F_{\text{univ}}}(X^{\text{univ}}(q_{\text{DR}} - 1)) \in \mathcal{O}_{F_{\text{univ}}}(F_{\text{univ}} \times_{LT, \eta} LT_{\infty})[q_{\text{DR}} - 1].$$

Define

$$\tilde{\log}_{F_{\text{univ}}}(X^{\text{univ}}) = \log_{F_{\text{univ}}}(X^{\text{univ}}) - \sum_{p \in F^{\text{univ}}[x] = (f \alpha^{\text{univ}}_{1,1})} \log_{F_{\text{univ}}}(X^{\text{univ}}[+f x])$$

$$= \log_{F_{\text{univ}}}(X^{\text{univ}}) - \sum_{j=0}^{p-1} \log_{F_{\text{univ}}}(X^{\text{univ}}[+f j \alpha_{1,1}^{\text{univ}}]).$$

Here, the “$\sim$” Since $(X^{\text{univ}}[+f j \alpha_{1,1}])^{(q_{\text{DR}} - 1)} = X^{\text{univ}}((\zeta_p^{\text{univ}})^{q_{\text{DR}} - 1})$, we then have

$$\tilde{\log}_{F_{\text{univ}}}(X^{\text{univ}}(q_{\text{DR}} - 1)) = \log_{F_{\text{univ}}}(X^{\text{univ}}(q_{\text{DR}} - 1)) - \sum_{j=0}^{p-1} \log_{\omega_0^{\text{univ}}}(X^{\text{univ}}[+f j \alpha_{1,1}^{\text{univ}}])(q_{\text{DR}} - 1)$$

$$= \tilde{\log}_{\omega_0^{\text{univ}}}(q_{\text{DR}} - 1)$$

where $\tilde{\log}_{\omega_0^{\text{univ}}}(q_{\text{DR}} - 1)$ is as in Definition 3.34.
Our next Proposition shows that the above power series has integral coefficients.

**Proposition 6.12.** We have
\[
\log_{\text{univ}}(X_{\text{univ}}) \in \hat{O}_{L_{\infty},y}^+[X_{\text{univ}}^y][1/p].
\]

**Proof.** As before, let \( A_{\text{univ}} \to X \) denote the universal false elliptic curve, isogenous to \( A_1 \times A_2 \), so that pulling back along a uniformization \( L_T \to X^{ss} \), we get identifications \( A_{\text{univ}}^{\text{univ}} \cong F_{\text{univ}} \) and hence \( \log_{F_{\text{univ}}} = \log_{A_{\text{univ}}^{\text{univ}}} \). Letting \( U_p \) and \( V_p \) denote the usual Atkin operators on \( X \), one can check that 
\( (1 - U_p V_p)^* \log_{A_{\text{univ}}^{\text{univ}}} \) is a global rigid analytic function on \( X \), where \( T \) is the Serre-Tate coordinate on any ordinary residue disc (cf. [6, Chapter V]). Moreover by the moduli-theoretic interpretation we see that
\[
(1 - U_p V_p)^* \log_{A_{\text{univ}}^{\text{univ}}} = \log_{F_{\text{univ}}}.
\]

In particular we have
\[
\log_{F_{\text{univ}}} \in O_{F_{\text{univ}}}^+[1/p] \subset O_{F_{\text{univ}}^+,y}^+[1/p] = \hat{O}_{L_{\infty},y}^+[X_{\text{univ}}^y][1/p].
\]

**Corollary 6.13.** We have
\[
\log_{\omega,0}^+(q_{\text{DR}} - 1) \in \hat{O}_{L_{\infty},y}^+[q_{\text{DR}} - 1][1/p].
\]

**Proof.** Apply (21) to the previous Proposition.

Let
\[
\hat{O}_{L_{\infty},y}^+[q_{\text{DR}} - 1][1/p] = \{ f(q_{\text{DR}} - 1) \in \hat{O}_{L_{\infty},y}^+[q_{\text{DR}} - 1][1/p] : f(q_{\text{DR}} - 1) = 0 \}.
\]

**Corollary 6.14.** We have
\[
\log_{\omega,0}^+(q_{\text{DR}} - 1) \in \hat{O}_{L_{\infty},y}^+[q_{\text{DR}} - 1][1/p].
\]

**Proof.** This follows from the previous Corollary and Definition 6.11.

We have the natural identification using the Amice transform
\[
\hat{O}_{L_{\infty},y}^+[q_{\text{DR}} - 1][1/p] = \Lambda(Z_{p,x}^{\times} \hat{O}_{L_{\infty},y}^+[q_{\text{DR}} - 1][1/p]).
\]

We continue to use \( i \) to view \( \Gamma_- \cong 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times \) (see (30)), which gives a natural projection \( \mathbb{Z}_p^\times \to \Gamma_- \), and hence induces via pushforward \( \Lambda(Z_{p,x}^{\times}, R) \to \Lambda(\Gamma_-, R) \). We let \( \mu_F \) denote the image of \( \log_{\omega,0}^+(q_{\text{DR}} - 1) \) under the composition of the two above maps (for \( R = \hat{O}_{L_{\infty},y}^+ \)).

We have the following \( p \)-adic Waldspurger formula (cf. [34, Chapter 9] for a similar \( p \)-adic Waldspurger formula for another \( p \)-adic \( L \)-function).

As before, we let \( \mu_F(y) \) denote the image of \( \mu_F \) under the specialization map \( \mathcal{O}_{\Delta, L_{\infty}, y} \to \hat{O}_{\Delta, L_{\infty}, y}(y) \).

**Proposition 6.15.** (cf. Theorem 9.10 of [34]) For any \( \rho \in \hat{\Gamma}_- \) of finite order, supposing that \( \rho \in \hat{\Gamma}_{-n}^\times \) we have
\[
\mu_F(y)(\rho) = \frac{\tau(\rho)}{q p^{n-1}} \sum_{\tau \in \hat{\Gamma}_{-n}^\times} \rho^{-1}(\tau) \log_{\omega_0}((P_{n,\tau})^\tau).
\]
Proof. This follows from the identity
\[(j \alpha_{1,n} + \alpha_{2,n})^* q_{dR} = \zeta_{p^n}^{\text{univ}},\]
and so
\[\sim \log_{\omega_{0}}((\zeta_{p^n}^{\text{univ}})^{i} - 1)(y) = \sim \log_{F}(j \alpha_{1,1} + \alpha_{1,2}) = \log_{F}(P_{n}^{\sigma_{i}}).\]
\[\square\]

Definition 6.16. For any continuous function \(h\) on a topological abelian group \(G\) and measure \(\mu\) on \(G\), we define the pullback
\[h^* \mu(F) = \mu(hF)\]
for any continuous function \(F\) on \(G\).

Proposition 6.17.
\[(\varphi/\varphi^*)^* \mu_{F} = \left(\frac{q_{dR}d}{dq_{dR}}\right) \sim \log_{\omega_{0}}(q_{dR} - 1)|_{q_{dR}=1}^{\text{univ}}.\]

Proof. Let \(\nu : \mathbb{Z}_p^e \to \mathbb{Z}_p^e\) denote the “moment character” \(z \mapsto z\). As the projection \(O_{K_v} \to O_{K_v}^e/\mathcal{O}_p^e\) is given by \(x \mapsto x/\bar{x}\) (with \(\bar{x}\) the image of \(x\) under the nontrivial element of \(\text{Gal}(K_p/Q_p)\)), under our fixed choice of \(i\) \((30)\), we have that \(\varphi/\varphi^*|_{\Gamma} = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^e\). Now by definition of \(\mu_{F}\), we have
\[(\varphi/\varphi^*)^* \mu_{F} = \sim \log_{\omega_{0}}(q_{dR} - 1)(\nu)\]
where we view \(\sim \log_{\omega_{0}}(q_{dR} - 1) \in \Lambda(\mathbb{Z}_p^e, \hat{O}_{LT_{\infty},y}[q_{dR} - 1])[1/p]\) using \((83)\). By “Kummer’s logarithmic derivative” formula for the first moment of a measure on \(\mathbb{Z}_p\), we have
\[\sim \log_{\omega_{0}}(q_{dR} - 1)(\nu) = \left(\frac{q_{dR}d}{dq_{dR}}\right) \sim \log_{\omega_{0}}(q_{dR} - 1).\]
\[\square\]

Proposition 6.18. Recall we let \(dz_{dR} = \nabla(z_{dR})\). We have
\[\frac{q_{dR}d}{dq_{dR}} = \frac{d}{dz_{dR}}.\]

Proof. This follows from the formula
\[e^* q_{dR} = \exp((z_{dR} - \bar{z}_{dR}) \log e^* Q),\]
recalling that \(\nabla(e^* Q) = 0\). For this latter equality, note that \(e^* Q \in \hat{O}_{LT_{\infty}}(LT_{\infty})\), the latter ring being \(p\)-adically complete. Now
\[\nabla(e^* Q) = d(e^* Q) = d(((e^* Q)^{1/p^n})^{p^n}) = p^n(e^* Q)^{(p^n-1)/p^n} d(((e^* Q)^{1/p^n}) \to 0\]
as \(n \to \infty\) (recall \((e^* Q)^{1/p^n}\) is unit), and so \(\nabla(e^* Q) = 0\) in this \(p\)-adically complete ring. \[\square\]
6.5. A $p$-adic Maass-Shimura differential operator on measures. Let

$$d_k = \frac{d}{dz} \left[ \frac{1}{z - \tilde{z}} \right] \left[ q_{dR} - 1 \right] \rightarrow \hat{O}_L^{+} \left[ \frac{1}{z - \tilde{z}} \right] \left[ q_{dR} - 1 \right].$$

which is the $p$-adic Maass-Shimura operator $\delta_{\Delta_k}$ of \cite{34}, and $\tilde{z} \in \hat{O}_{LT}(LT_{\infty})$ is the Hodge-Tate period recalled in loc. cit. (first defined in \cite{49}). We can hence view

$$d_k : \Lambda(\mathbb{Z}_p, \hat{O}_L^{+} \left[ \frac{1}{z - \tilde{z}} \right]) \rightarrow \Lambda(\mathbb{Z}_p, \hat{O}_L^{+} \left[ \frac{1}{z - \tilde{z}} \right]).$$

In particular, we have $\Lambda(\mathbb{Z}_p, \hat{O}_L^{+}, y) \subset \Lambda(\mathbb{Z}_p, \hat{O}_L^{+} \left[ \frac{1}{z - \tilde{z}} \right])$, so we can apply $d_k$ to $\mu_{\text{glob}}$. By Proposition 6.18 we have

$$d_k = \frac{q_{dR}d}{dz} + \frac{k}{z - \tilde{z}},$$

and so by Proposition 6.17

$$d_k \mu_{\text{glob}} = (\varphi/\varphi^*)^* \mu_{\text{glob}} + \frac{k}{z - \tilde{z}} \mu_{\text{glob}}.$$

Recall that $\chi$ is the finite order character such that $\varphi = \lambda/\chi$ is unramified at $p$. Let $f$ be the conductor of $\varphi$. Recall that given a continuous function $h$ on an abelian group $G$ and a measure $\mu$ on $G$, $h^* \mu$ denotes the twist by $h$ (Definition 6.16). In practice, we will take $h = \varphi/\varphi^*, \chi^* \varphi/\varphi^*$, or $\chi^* \chi$. Recall that $\mu(h)$ denotes the evaluation of a measure at $h$, and given a measure $\mu \in \Lambda(G, \hat{O}_{LT_{\infty},y}^+ \{1/p\})$, we let $\mu(y) \in \Lambda(G, \mathcal{O}_{I_{p,\infty}}^+ \{1/p\})$ denote the image under the specialization of coefficients $\hat{O}_{LT_{\infty},y}^+ \rightarrow \hat{O}_{LT_{\infty},y}^+(y) = \mathcal{O}_{I_{p,\infty}}$.

**Proposition 6.19.** We have, for any $y \in \hat{\Gamma}_-$,

$$d_1(\chi^*)^* \mu_{\text{glob}}(f)(y)(\rho) = G(\varphi \rho)(y) \mathcal{L}_{q_{\text{alg}}}^+(\varphi^{-2} \varphi^*(\chi^* \rho)^{-1}, 0)$$

and if $L(\varphi^* \chi, 1) \neq 0$, then

$$d_1(\chi^*)^* \mu_{\text{glob}}(f)(y)(\varphi^*/\varphi) \neq 0.$$

Here, $\mathcal{L}_{q_{\text{alg}}}$ is defined as in \cite{1}, Proof of Theorem 3.17, last line of (3-17), and the subscript “$p$” denotes removing the Euler factor at $p$.

**Proof.** The first assertion concerning the algebraic $L$-value follows by the comparison of $d_k$ and the complex Maass-Shimura operator (\cite{34} Theorem 5.48) and the interpolation properties of (Proposition 8.11 of loc. cit.). By \cite{48}, we have

$$d_1(\chi^*)^* \mu_{\text{glob}}(f) = (\chi^* \varphi/\varphi^*)^* \mu_{\text{glob}}(f) + \frac{1}{z - \tilde{z}} (\chi^*)^* \mu_{\text{glob}}(f).$$

Hence, letting $1$ denote the trivial character,

$$d_1(\chi^*)^* \mu_{\text{glob}}(f)(\varphi^*/\varphi) = (\chi^*)^* \mu_{\text{glob}}(f)(1) + \frac{1}{z - \tilde{z}} (\chi^*)^* \mu_{\text{glob}}(f)(\varphi^*/\varphi).$$

We want to show that the evaluation at the fiber $(y)$ is not zero given $L(\varphi^* \chi, 1) \neq 0$. However, from \cite{44} and \cite{45}, we have for some nonzero $C' \in \mathbb{C}_p^\times$,

$$d_1(\chi^*)^* \mu_{\text{glob}}(f)(1)(y) = C' \cdot L(\varphi^*(\chi^*)^{-1}, 0) = C' L(\varphi^* \chi, 1) \neq 0.$$
where \( C'' \in \mathbb{C}_p \) and \( C'L(\varphi^* \chi, 1) \neq 0 \) by assumption. The second assertion follows.

Finally, we have another Proposition on the image of \( \mu_F \) under \( \mathfrak{d}_0 = \frac{d\mathfrak{d}R}{dq_{\mathfrak{d}R}} \). Recall that for an element \( f \in \hat{\mathcal{O}}_{LT, \infty, y} \), we let \( f(y) \) denote the image in the residue field \( \hat{\mathcal{O}}_{LT, \infty, y}(y) \).

**Proposition 6.20.** We have for any \( \rho \in \hat{\Gamma}_{n,-} \),

\[
((\varphi/\varphi^*)^* \mu_F(y)(\rho))^2 = \Omega(y)^2 (G(\varphi \rho))^2 (y) L^\text{alg}_p(g \times \chi \rho^{-1}, 0),
\]

for some

\[
\Omega \in \hat{\mathcal{O}}_{LT, \infty, y} \left[ \frac{1}{z_{\mathfrak{d}R} - \bar{z}} \right]
\]

with \( \Omega(y) \neq 0 \), where \( L^\text{alg} \) is the algebraic normalization of the Rankin-Selberg \( L \)-value in \([34, \text{Discussion after Definition 8.5}]\), and the subscript “\( \mathfrak{p} \)” denotes removing the Euler factor at \( \mathfrak{p} \).

**Proof.** As in Proposition 6.18 we have that pulling back by \( (\varphi/\varphi^*) \) corresponds to differentiating by

\[
\left( \frac{d\mathfrak{d}R d}{dq_{\mathfrak{d}R}} \right) = \mathfrak{d}_0.
\]

Now the assertion follows from the same calculation involving Waldspurger’s period sum formula for Rankin-Selberg \( L \)-values as in \([34, \text{Chapter 8.1}]\), after noting that

\[
\mathfrak{d}_0 \log_\text{univ} = \frac{d}{dz_{\mathfrak{d}R}} \log_\text{univ}(y) = \frac{\omega_0}{d\mathfrak{d}R}(y) = \Omega(y)^2 F,
\]

where \( F \) is the weight 2 newform corresponding to \( \omega_0 = \omega_{A_p} \) and \( \Omega(y) \) is the period coming from comparing \( e^* dX^\text{univ} \) and \( d_{\mathfrak{d}R} \). The non-vanishing follows because each of these latter differentials is a generator of the proétale sheaf \( \Omega_1^1(\Gamma_{LT, \infty}) \) at \( y \). \( \square \)

**Remark 6.21.** We note that \( \mu_F \) satisfies an interpolation property for finite order anticyclotomic characters, whereas the \( p \)-adic \( L \)-function constructed in \([34, \text{Chapter 8}]\) only allows tame ramification at \( p \). On the other hand, the \( p \)-adic \( L \)-function of loc. cit. is a continuous function of anticyclotomic weight characters (of the form \( z \mapsto (z/\bar{z})^j \) at \( \mathfrak{p} \)).

**6.6. Factorization of anticyclotomic \( p \)-adic \( L \)-functions.**

**Theorem 6.22.** We have the following identity in \( \Lambda(\Gamma_-, \mathcal{O}_{F_p, \infty})[1/p] \):

\[
D(\chi^*)^* \mu_{\text{glob}}(f)(y) \cdot \mathfrak{d}_1(\chi^*)^* D \mu_{\text{glob}}(f)(y) = C(y) ((\varphi/\varphi^*)^* \mu_F(y))^2
\]

for some nonzero constant \( C(y) \in \hat{\mathcal{O}}_{LT, \infty, y} \left[ \frac{1}{z_{\mathfrak{d}R} - \bar{z}} \right] \).

**Proof.** This follows from \([44, 45]\), Propositions 6.15 and 6.19 and the Artin formalism

\[
L^\text{alg}_p(\lambda^{-1} \rho^{-1}, 0) L^\text{alg}_p(\varphi^{-2} \varphi^*(\chi^*)^{-1}, 0) = L^\text{alg}_p(g \times \chi \rho^{-1}, 0).
\]

**Remark 6.23.** This factorization hints at an underlying Heegner point (or Perrin-Riou)-type main conjecture as in \([9]\) involving a certain “\( + \)” Heegner class whose “big logarithm” gives rise to \( \mu_F \), where as the \( p \)-adic \( L \)-functions on the left-hand side give the structure of a “\( + \)”-codiscrete \( \Lambda \)-adic Selmer group which is a \( \Lambda \)-adic interpolation of \( \text{III} \). In fact, such a Heegner point main conjecture is formulated and proven in forthcoming work.
7. Descent, Rank 1 \(p\)-converse, Sylvester’s Conjecture and Goldfeld’s conjecture for the Congruent Number Family

In this section, we give the rank-1 converse theorem implied by factorization of \(p\)-adic \(L\)-functions from the previous section.

### 7.1. \(p\)-converse theorem.

#### Definition 7.1.
Fix a topological generator \(\gamma^- \in \Gamma^-\), and let \(I^- := (\gamma^- - 1)\Lambda_{\mathcal{O}_K}\) be the associated augmentation ideal.

#### Lemma 7.2.
Suppose \(\text{corank}_{\mathbb{Z}_p} \text{Sel}_{\mathbb{Q}}(E/\mathbb{Q}) = 1\). Then
\[
D(\chi^*)\mu_{\text{glob}}(f)(y)/((\varphi/\varphi^*)(\gamma^- - 1) \neq 0.
\]

**Proof.** By Proposition 5.3 and Mazur’s anticyclotomic control theorem, we have a surjection
\[
\mathcal{X}(\lambda^{-1})_{\chi}\mathcal{E}/(\gamma^- - 1) \xrightarrow{\sim} \text{Hom}(\text{Sel}_{\mathbb{Q}}(E/\mathbb{Q}), \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_K
\]
with finite kernel. Hence the corank 1 assumption implies that \(\mathcal{X}(\lambda^{-1})_{\chi}\mathcal{E}/(\gamma^- - 1)\) has \(\mathcal{O}_K\)-rank 1. Since the quotient map \(\mathcal{X}(\lambda^{-1})_{\chi}\mathcal{E} \to \mathcal{X}(\lambda^{-1})_{\chi}\mathcal{E}\) has kernel of rank 1, we thus see that \(\mathcal{X}'(\lambda^{-1})_{\chi}\mathcal{E}/(\gamma^- - 1)\) is \(\mathcal{O}_K\)-torsion. Hence by (4.34), we see that
\[
D(\chi^*)\mu_{\text{glob}}(f)(y)/((\varphi/\varphi^*)(\gamma^- - 1) \neq 0.
\]

\(\square\)

#### Theorem 7.3.
Suppose that \(K\) is an imaginary quadratic field with class number 1 in which \(p = 2, 3\) is ramified, and that \(E/\mathbb{Q}\) is an elliptic curve with CM by \(\mathcal{O}_K\). Then
\[
\text{corank}_{\mathbb{Z}_p} \text{Sel}_{\mathbb{Q}}(E/\mathbb{Q}) = 1 \implies 0 \neq P_{\chi,0}(\phi) \in B(K) \otimes_{\mathbb{Z}} \mathbb{Q}, \text{ord}_s L(E/\mathbb{Q}, s) = \text{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1.
\]

**Proof.** Let \(f\) be the conductor of \(\varphi\), and recall the factorization in \(\Lambda(\Gamma_-, \mathcal{O}_{\mathcal{L}_{\mathbb{Q}}})[1/p]
\[
D(\chi^*)\mu_{\text{glob}}(f)(y)/((\varphi/\varphi^*)(\gamma^- - 1) \neq 0.
\]

By Lemma 7.2
\[
D(\chi^*)\mu_{\text{glob}}(f)(y)/((\varphi/\varphi^*)(\gamma^- - 1) \neq 0.
\]

By our choice of \(\chi\), we have \(L(\varphi^* \chi, 1) \neq 0\), and so
\[
\mu_1 D_{\mu_{\text{glob}}}(f)(y)/((\varphi/\varphi^*)(\gamma^- - 1) \neq 0
\]
by Proposition 6.19. Hence the above factorization implies
\[
(\mu_F(y))^2/(\gamma^- - 1) \neq 0.
\]

Now by Theorem 6.15, we have
\[
P_{\chi,0}^1 \neq 0
\]
which implies
\[
P_{\chi,0} \neq 0
\]
and so by Yuan-Zhang-Zhang’s [66] generalized Gross-Zagier formula:
\[
1 = \text{ord}_s L(g \times \chi, s) \overset{[60]}{=} \text{ord}_s L(\lambda, s) + \text{ord}_s L(\varphi^* \chi, s),
\]
and so by Proposition 6.3 (which ensures \(L(\varphi^* \chi, 1) \neq 0\), we have
\[
\text{ord}_s L(E/\mathbb{Q}, s) = \text{ord}_s L(\lambda, s) = 1.
\]
Now the remaining assertions of the Theorem follow from the Gross-Zagier formula [25] (or Yuan-Zhang-Zhang [66]) and Kolyvagin [31].

\(\square\)
7.2. Sylvester’s conjecture. Recall the cubic twist family of elliptic curves
\[ E_d : x^3 + y^3 = d. \]
Sylvester conjectured in 1879 (\cite{58}) that for \( d = p \) prime, if \( d \equiv 4, 7, 8 \pmod{9} \) then \( E_d \) has a rational solution.

**Corollary 7.4.** Sylvester’s conjecture is true. That is, for any prime \( p \), if \( p \equiv 4, 7, 8 \pmod{9} \) then there exist \( x, y \in \mathbb{Q} \) such that \( x^3 + y^3 = p \).

**Remark 7.5.** Previously, the case \( p \equiv 4, 7 \pmod{9} \) was announced by Elkies \cite{17}, though the full proof remains unpublished. See also the article of Dasgupta-Voight \cite{15}, which gives another proof of Elkies’s result under additional assumptions.

**Proof of Corollary 7.4.** By standard 3-descent (see \cite{15}), we have \( \text{corank}_{\mathbb{Z}/3}(\text{Sel}_{\infty}(E_d)) = 1 \) for \( d \equiv 4, 7, 8 \pmod{9} \). Now the Corollary follows immediately from Theorem 7.3 with \( p = 3 \) and \( K = \mathbb{Q}(\sqrt{-3}) \).

7.3. Goldfeld’s conjecture for the congruent number family. For a general elliptic curve \( E : y^2 = x^3 + ax + b \), let \( r_{an}(E/\mathbb{Q}) = \text{ord}_{s=1} L(E/\mathbb{Q}, s) \) denote the analytic rank. Let \( E^d : y^2 = x^3 + ad^2x + bd^3 \) be the quadratic twist by \( \mathbb{Q}(\sqrt{d}) \). The celebrated conjecture of Goldfeld states that:

**Conjecture 7.6 (Goldfeld’s conjecture \cite{22}).** For \( r = 0, 1, \)
\[ \lim_{X \to \infty} \frac{\# \{ 0 < |d| < X : d \text{ squarefree}, r_{an}(E^d/\mathbb{Q}) = r \}}{\# \{ 0 < |d| < X : d \text{ squarefree} \}} = \frac{1}{2}. \]

The best known unconditional results towards Goldfeld’s conjecture in general are \cite{32}, \cite{35} and \cite{36}. For the congruent number family \( E^d : y^2 = x^3 - d^2x \), the previously best known result follows from the main result of \cite{57}. Using the results of loc. cit., we can establish Goldfeld’s conjecture for certain elliptic curves among those considered in loc. cit. including the congruent number family.

**Corollary 7.7.** Suppose \( E/\mathbb{Q} \) is an elliptic curve with \( E(\mathbb{Q})[2] \cong (\mathbb{Z}/2)^{\oplus 2} \) and no cyclic 4-isogeny defined over \( \mathbb{Q} \). Suppose further that \( E \) has CM by \( K \) (so that \( K \) necessarily has class number 1), and that 2 is ramified in \( K \). Then Goldfeld’s conjecture (Conjecture 7.6) is true for \( E \).

In particular, 100% of squarefree \( d \equiv 1, 2, 3 \pmod{8} \) are not congruent numbers and 100% of squarefree \( d \equiv 5, 6, 7 \pmod{8} \) are congruent numbers, and Goldfeld’s conjecture is true for the congruent number family \( E^d : y^2 = x^3 - d^2x \).

**Proof.** This follows from the Selmer distribution results of Smith \cite{57}, Corollary 5.16 and Theorem 7.3 for \( p = 2 \) ramified in \( K \) of class number 1 (so that, in particular, \( K \) satisfies the assumptions of Theorem 7.3).

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**References**
