SUPERSINGULAR MAIN CONJECTURES, SYLVESTER’S CONJECTURE AND GOLDFELD’S CONJECTURE

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Abstract. We formulate and prove a Rubin-type main conjecture for imaginary quadratic fields $K$ in which a prime $p$ is inert or ramified, relating the characteristic ideal of a certain $\Lambda$-torsion quotient of the maximal abelian pro-$p$-extension of the $\mathbb{Z}_p^{\geq 2}$-extension of $K$ unramified outside $p$, to a supersingular Katz-type $p$-adic $L$-function introduced by the author in previous work. A key component of this work is the development of a new interplay between Iwasawa theory and $p$-adic Hodge theory, in order to extend the method of Coleman power series and construct measures from norm-compatible systems of local units in the height 2 setting. We also formulate a Perrin-Riou-type Heegner point main conjecture in the supersingular CM setting, and in certain cases relate it to the above Rubin-type main conjecture. As an application, we prove rank 0 and 1 converse theorems for supersingular CM elliptic curves $E/K$ when $K$ has class number 1: “for $r = 0, 1$ if $\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^{\infty}} (E/\mathbb{Q}) = r$ then $\text{ord}_{s=1} L(E/\mathbb{Q}, s) = r$. We hence get results on two classical problems of arithmetic. First, we prove the 1879 conjecture of Sylvester on primes expressible as the sum of rational cubes: “If $p$ is a prime with $p \equiv 4, 7, 8 \pmod{9}$, then there exist $x, y \in \mathbb{Q}$ such that $x^3 + y^3 = p$”. Moreover, we show that the rank part of BSD holds for these elliptic curves. Second, by combining our $p$-converse theorem with the Selmer distribution results of Smith, we prove that 100% of squarefree integers congruent to 5, 6, 7 (mod 8) are congruent numbers, thus settling the congruent number problem in 100% of cases, and verifying the rank part of BSD and Goldfeld’s celebrated conjecture on quadratic twists for the congruent number family.

1. Introduction

Certain special cases of the Birch and Swinnerton-Dyer conjecture have long been known to have connections with classical problems in number theory. For example, the rank part of the conjecture for the cubic twists family $x^3 + y^3 = p$, $p$ a prime which is $4, 7, 8 \pmod{9}$, is Sylvester’s conjecture on primes expressible as the sum of two rational cubes. These questions often consider “exceptional cases” of elliptic curves with CM, which are supersingular or of bad reduction at primes for which it is natural to study descent, thus making these curves less amenable to classical methods of Iwasawa theory.

In this article, we prove converse theorems to the well-known theorem of Gross-Zagier and Kolyvagin in such exceptional cases, through establishing a suitable version of a $\Lambda$-adic version of the Birch and Swinnerton-Dyer conjecture, often known as a “Heegner point Main Conjecture” (see Section 7). Converse theorems in this vein were first developed by Skinner [47], though the hypotheses of loc. cit. rule out the CM case. A special case of our supersingular CM converse theorem implies Sylvester’s conjecture (Theorem 8.6), while another special case, combined with work of Smith [49], shows that 100% of squarefree integers congruent to 5, 6, 7 (mod 8) are congruent numbers and establishes Goldfeld’s conjecture on quadratic twists [19] for the congruent number family. The key step for establishing this Heegner point Main Conjecture (a “supersingular $GL_2$ main conjecture”) is to reduce it to a supersingular analogue of a Rubin-type Main Conjecture for imaginary quadratic fields $K$ involving elliptic units (a “supersingular $GL_1$ main conjecture”), and an essential for this reduction is the $p$-adic Waldspurger formula established by the author in [30, Chapter 9]. Proving the latter Main Conjecture in this supersingular setting (i.e. when $p$ is inert or ramified in $K$) involves overcoming several fundamental obstacles, among which include identifying
the correct torsion $\Lambda$-modules for which to formulate the main conjecture, and constructing the appropriate “Katz-type” $p$-adic $L$-function.

Substantial progress toward the latter step was already made by the author in [30] and [29], in which supersingular Rankin-Selberg $p$-adic $L$-functions in the relevant framework were constructed. Among the various results toward establishing the Main Conjecture in Section 4.15 we show that the Katz-type $p$-adic $L$-function constructed in loc. cit. is an element of the relevant Iwasawa algebra, and in fact is the restriction of a two-variable Katz-type $p$-adic measure. In order to construct the two-variable Katz-type $p$-adic measure, we adapt the method of Coleman power series in the height 2 setting, combining it with $p$-adic Hodge theory in a fundamentally new way in order to find the “correct” power series ring for constructing appropriate measures. Perhaps the largest conceptual leap made in this step is the shift of focus from the traditional framework of Lubin-Tate groups to “transcendental Lubin-Tate groups”, which are extensions of scalars of Lubin-Tate groups via the Serre tensor product construction. Another innovation is the use of $p$-adic Hodge theory to make explicit computations on these larger formal groups in order to construct our measures; the natural coordinate we use for constructing our measures comes from exponentiating a de Rham period arising from the $p$-adic de Rham comparison theorem, as in [30] and [29]. The Rankin-Selberg type $p$-adic $L$-function arising in the Heegner point Main Conjecture can be shown to be equal to the $p$-adic $L$-function constructed in loc. cit., via an explicit reciprocity law involving appropriate $+\text{-Selmer}$ groups and big logarithm maps. In particular, we establish that the $p$-adic $L$-functions constructed in [30] and [29] belong to appropriate anticyclotomic Iwasawa algebras in the case of CM forms.

Finally, we point out that actually proving the Rubin-type Main Conjecture involves studying the usual fundamental exact sequence from class field theory, and reducing it to the Elliptic units Main Conjecture, which is the assertion of a certain equality of characteristic ideals involving elliptic units and the maximal unramified pro-$p$ abelian extension of the $\mathbb{Z}_p^{\oplus 2}$-extension of $K$. Rubin’s strategy for proving the Elliptic units Main Conjecture involves the Euler system of elliptic units, which however entails introducing certain assumptions on the prime $p$ inconvenient for our applications. In order to remove these assumptions, we instead show that the $\mu$-invariant of the restriction of our Katz-type $p$-adic $L$-function to the cyclotomic line is 0 (Section 4.13), which by an additional argument and work of Johnson-Leung-Kings [25] on equivariant main conjectures, gives the Elliptic units Main Conjecture.

1.1. Outline of the paper, main results and proofs. We give a brief overview of the paper.

The first half of the paper concerns the $GL_1/K$ setting, where $K$ is an imaginary quadratic field. Let $L = K(\mathfrak{f})$ for some ideal $(\mathfrak{f},p) = 1$, and let $L_{\infty}/L$ denote the compositum of $L$ and the $\mathbb{Z}_p^{\oplus 2}$-extension $K_{\infty}$ of $K$. Fix an embedding $\mathbb{Q} \subset \mathbb{Q}_p$, and for any algebraic extension $F$ of $\mathbb{Q}$, let $F_p$ denote the induced $p$-adic completion of $F$. Let $\Lambda = \mathbb{Z}_p[\mathrm{Gal}(L_{\infty}/K)]$, and for any $\mathbb{Z}_p$-algebra, let $\Lambda_R = \Lambda \hat{\otimes}_{\mathbb{Z}_p} R$. From class field theory, one gets an exact sequence of the form

$$0 \to \mathcal{E} \to \mathcal{U} \xrightarrow{\text{rec}} \mathcal{X} \to \mathcal{Y} \to 0,$$

where $\mathcal{E}$ is the usual tower of global units associated with $L_{\infty}/L$, $\mathcal{U}$ is the tower of principal semilocal units associated with $L_{p,\infty}/L_p$, $\mathcal{X}$ is the Galois group of the maximal pro-$p$ abelian extension of $L_{\infty}$ unramified outside $p$, and $\mathcal{Y}$ is the Galois group of the maximal pro-$p$ abelian extension of $L_{\infty}$ unramified everywhere. Letting $\mathcal{C} \subset \mathcal{E}$ denote a rank-1 free suitable module of elliptic units, we can descend the above exact sequence to an exact sequence

(1) $$0 \to \mathcal{E}/\mathcal{C} \to \mathcal{U}/\mathcal{C} \xrightarrow{\text{rec}} \mathcal{X} \to \mathcal{Y} \to 0.$$ 

In the “ordinary” case, i.e. when $p$ splits $K$, all modules in the above exact sequence are $\Lambda$-torsion, and one can naturally formulate Iwasawa main conjectures. Namely, Coleman (see [15]
Chapter I-II] produces a \( \Lambda \)-linear map \( U \hookrightarrow \Lambda W(\mathbb{F}_p) \) with pseudonull cokernel sending \( C \) to a Katz \( p \)-adic \( L \)-function \( \mathcal{L}_p \). Let \( \Delta = \text{Gal}(L_\infty/K_\infty) \), and suppose that \( p \nmid \# \Delta \). Then for each nontrivial character \( \chi \) on \( \Delta \), Rubin \( [10] \) proved that the isotypic component \( \mathcal{L}_{p,\chi} \) generates the characteristic ideal of the isotypic component \( (\mathcal{X} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_p))_\chi \) by using the Euler system of elliptic units to show \( \text{char}_{\Lambda}(\mathcal{E}/\mathcal{C})_\chi = \text{char}_{\Lambda}(\mathcal{Y})_\chi \) under certain assumptions on \( p \). This Rubin-type main conjecture has immediate applications to the arithmetic of certain CM elliptic curves, see \([30] \) Section 11).

In the “supersingular” case when \( p \) is inert or ramified in \( K \), several complications arise. First, the middle two terms of the above exact sequence are not \( \Lambda \)-torsion, meaning there is no immediate formulation of a main conjecture. One of the key innovations of this paper is to use \( p \)-adic Hodge theory, namely the de Rham comparison theorem for Lubin-Tate groups and the Cartier duality formulation of a main conjecture. One of the key innovations of this paper is to use

\[
\begin{align*}
\text{Theorem } 5.13, \quad 
&\text{if } E/\mathbb{Q} \text{ has CM by } K, \ corank_{\mathbb{Z}_p, \text{Sel}_p}(E/\mathbb{Q}) = 0 \implies \text{ord}_{s=1} L(E/\mathbb{Q}, s) = 0. \\
&\text{This in particular applies to rank 0 members of the congruent number family, and invoking Selmer distribution results of Smith, we show that } 100\% \text{ of the curves } y^2 = x^3 - d^2 x \text{ with squarefree } d \equiv 1, 2, 3 \pmod{8} \text{ have analytic rank 0.}
\end{align*}
\]

The second half of the paper concerns the \( GL_2/\mathbb{Q} \) setting in the case of CM elliptic curves, and is motivated primarily by the desire to establish a rank 1 \( p \)-converse theorems, and apply it to the classical sums of cubes problem (Sylvester’s conjecture) and the congruent number problem. We follow the general strategy of formulating a Perrin-Riou type Heegner point main conjecture, and showing (in the CM setting) that it is equivalent to the Rubin-type main conjecture proven in the first half of the paper. This philosophy of reducing “difficult” main conjectures to more amenable “Greenberg-type” main conjectures was first realized by Skinner in his pioneering paper on \( p \)-converse theorems \([17] \). The supersingular Heegner point main conjecture we formulate here (Conjecture 7.70) follows the framework considered by Castella-Wan \([8] \) in introducing “+”
Selmer groups and “+” Heegner point classes which satisfy norm-compatibilities. The ideas for these “+”-type constructions have their origins in the works of Pollack [35] and Kobayashi [26], but have hitherto been developed only for height 1 Lubin-Tate formal groups. As we must deal with height 2 Lubin-Tate formal groups in the supersingular CM setting, we develop the “+” theory over the imaginary quadratic field $K$ in the case where $p$ is ramified in $K$ and $K$ has class number 1. Using Kummer theory and the + construction, we produce a norm-compatible system of cohomology classes from the usual family of Heegner points. The core of our argument for relating the supersingular CM Heegner point main conjecture to the supersingular Rubin-type main conjecture is the explicit reciprocity law proven in Theorem 7.50, relating the image of the + Heegner class under a big logarithm map to the superinsingular Rankin-Selberg $p$-adic $L$-function introduced in [30] and [29]. The proof of the explicit reciprocity law relies on the $p$-adic Waldspurger formula of [30, Chapter 9], Wiles’s explicit reciprocity law and several new constructions: a $+$-big logarithm map, a $+$-Coleman map, and our measure map $\mu : U \otimes \mathbb{Z}_p \mathcal{O}_L \rightarrow \Lambda_{\mathcal{O}_L}$ from the first half of the paper. Via careful calculations using Kummer theory and global duality, we then relate our + Selmer groups to the torsion Selmer groups in the supersingular Rubin-type main conjecture, and hence prove the Heegner point main conjecture in this supersingular CM setting (Theorem 7.72).

Finally, in Section 8, we prove our rank 1 $p$-converse theorem (Theorem 8.5): “if $E/\mathbb{Q}$ has CM by $K$, corank $\mathbb{Z}_p \operatorname{Sel}_{p,\infty}(E/\mathbb{Q}) = 1$ $\implies$ ord$_{s=1}$ $L(E/\mathbb{Q}, s) = 1$”. We can then apply this theorem to prove Sylvester’s conjecture on sums of cubes (Corollary 8.6) and Goldfeld’s conjecture for the congruent number family (Corollary 8.9).

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2. Review of the Coleman map

2.1. Notation. For the rest of this paper, $K/\mathbb{Q}$ will denote an imaginary quadratic field. Let $D \in \mathbb{Z}_{<0}$ denote the fundamental discriminant of $K$. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and henceforth view all number fields as embedded in $\overline{\mathbb{Q}}$.

Definition 2.1. Henceforth, fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, let $\mathbb{C}_p$ denote its $p$-adic completion, and fix embeddings

$$i : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}, \quad i_p : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p.$$  

Let $\mathfrak{p}$ be the prime ideal of $\mathcal{O}_K$ fixed by (3). Given any field $L \subset \overline{\mathbb{Q}}_p$, let $L_p$ denote its $p$-adic completion in $\overline{\mathbb{Q}}_p$. Given a local field $L_p$, let $k(L_p)$ denote its residue field.

2.2. Relative Lubin-Tate groups. We recall some notions concerning relative Lubin-Tate groups, following [40, Chapter I.1.1], to which we refer the reader for more details. Let $L_p/K_p$ be an unramified extension of local fields, and let $k(L_p)$ denote the residue field associated with $L_p$. Let $K_p^{ur}$ denote the $p$-adic completion of the maximal unramified extension of $K_p$. Let $\phi : K_p^{ur} \rightarrow \tilde{K}_p^{ur}$ denote the Frobenius automorphism corresponding to $K_p/\mathbb{Q}_p$. Let $q$ denote the order of the residue field $k(K_p)$ of $K_p$, so that $\phi$ lifts the $q$-power Frobenius. Let $v : \tilde{K}_p^{ur} \rightarrow \mathbb{Z}$ denote the normalized valuation on $\tilde{Q}_p^{ur}$, i.e. with $v(p) = 1$. Henceforth, let $F_j$ denote a relative Lubin-Tate formal group law (more precisely, the Lubin-Tate formal $\mathcal{O}_{K_p}$-module) of height $h := [K_p : \mathbb{Q}_p]$ with respect to
the unramified extension $L_p/K_p$, and with special $\phi$-linear endomorphism $f$ lifting the Frobenius $\phi$ on $k(K_p)$. In particular, there is an isomorphism
\[ [\cdot]_f : \mathcal{O}_{K_p} \cong \text{End}_{\mathcal{O}_{L_p}}(F_f). \]

We recall some properties of the special $\phi$-linear endomorphism $f$. Fix an element $\xi \in K_p^\times$ with $v(\xi) = d := [L_p : K_p]$, and fix $\pi' \in L_p$ with $\text{Nm}_{L_p/K_p}(\pi') = \xi$. Then $f \in \text{Hom}(F_f, F_f^p)$, is a homomorphism of formal groups over $\mathcal{O}_{L_p}$, and so in particular an element of $\mathcal{O}_{L_p}[X]$ where $X$ is the formal parameter on $F_f$. Here for $n \in \mathbb{Z}_{\geq 0}$, $F_f^{\phi^n}$ denotes the formal group law obtained by applying $\phi^n$ to the coefficients of $F_f$. Moreover, $f$ satisfies
\[ f(X) \equiv \pi' X \pmod{X^2}, \quad f(X) \equiv X^q \pmod{\pi' \mathcal{O}_{L_p}}. \]

Let $\phi^n f$ denote the $\phi$-linear endomorphism of $F_f^{\phi^n}$ obtained by applying $\phi^n$ to the coefficients of the $\phi$-linear endomorphism $f$ of $F_f$, and note that $F_f^{\phi^n} = F_{\phi^n f}$.

Let $F_f[f^n] \subset F_f(\mathcal{O}_{C_p})$ denote the subgroup of $f^n$-torsion points on $F_f$, and let $L_{p,n} = L_p(F_f[f^n])$. Let $\text{Nm}_n : L_{p,n} \to L_{p,n-1}$ denote the norm map. Henceforth, let $p_n$ denote the maximal ideal of $\mathcal{O}_{L_{p,n}}$, and let $p_\infty = \bigcup_n p_n$. Since $L_p/K_p$ is inert we have $p_0 = p \mathcal{O}_{L_p}$. Following the notation of [45 Chapter I.1.1], let
\[ f^n = \phi^{n-1} f \circ \phi^{n-2} f \circ \cdots \circ \phi f \circ f. \]

Let $F_f[f^n]$ denote the kernel of the $n$-fold composition $f^n$, and note that $F_f[f^\infty]$ is the $p$-divisible group associated with the formal group $F_f$. Let
\[ T_f F_f := \varprojlim_{\phi^n f} F_{\phi^{-n} f}[(\phi^{-n} f)^n]. \]

Let $p$ denote the prime ideal of $\mathcal{O}_{K_p}$ above $p$. By Lubin-Tate theory (see [45 Proposition I.1.7]), there exist isomorphisms $\alpha_n : \mathcal{O}_{K_p}/p^n \cong F_{\phi^{-n} f}[(\phi^{-n} f)^n]$ for all $n \in \mathbb{Z}_{\geq 0}$ such that we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{O}_{K_p}/p^{n+1} & \xrightarrow{\alpha_{n+1}} & F_{\phi^{-(n+1)} f}[(\phi^{-(n+1)} f)^{n+1}] \\
\downarrow & & \downarrow \phi^{-(n+1)} f \\
\mathcal{O}_{K_p}/p^n & \xrightarrow{\alpha_n} & F_{\phi^{-n} f}[(\phi^{-n} f)^n]
\end{array}
\]

This implies that abstractly, there exists an isomorphism
\[ \alpha_\infty = \varprojlim_n \alpha_n : \mathcal{O}_{K_p} \cong T_f F_f. \]

We henceforth refer to such an isomorphism as a $(p^\infty)$-level structure.

Let $\mathcal{G} := \text{Gal}(L_{p,\infty}/L_p)$. By [45 Proposition I.1.8], we have an isomorphism
\[ \kappa : \mathcal{G} \cong \mathcal{O}_{K_p}^\times, \quad \forall x \in F_f[f^n], \sigma(x) = [\kappa(\sigma)]_f(x). \]

By the remark after equation (9) in Chapter I.3.3 of loc. cit., composing $\kappa$ by the local Artin symbol $\mathcal{O}_{K_p}^\times \to \mathcal{G}$, $\sigma \mapsto \sigma^{-1}$. 

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By [51 Section 2.2, Proof of Proposition 1], identifying the category of connected formal groups
over complete local Noetherian rings with the category of connected $p$-divisible groups over complete
local rings, we have a non-canonical isomorphism
\[ \mathcal{F}_f \cong \text{Spf}(\mathcal{O}_{L_p}[X]). \]
When $\mathcal{F}_f$ is a Lubin-Tate group, its $p$-divisible group $\mathcal{F}_f[f^\infty]$ is connected. Hence we can and will
often freely identify the formal $\mathcal{O}_{K_p}$-module $\mathcal{F}_f$ with its $p$-divisible group $\mathcal{F}_f[f^\infty]$.

Henceforth, denote
\[ \mathcal{U} := \lim_{\xleftarrow{\text{Nim}_n}} \mathcal{O}_{L_p,n}^\times. \]

### 2.3. The Coleman map and Tsuji’s reformulation.

**Theorem 2.2 (Coleman).** Let $\beta = (\beta_n) \in \mathcal{U}$. Fix a level structure
\[ \alpha_\infty = \lim_{\xleftarrow{n}} \alpha_n : \mathcal{O}_{K_p} \xrightarrow{\sim} T_f \mathcal{F}_f. \]
There exists a unique $g_\beta \in \mathcal{O}_{L_p}[X]$ such that
\[ \phi^{-n} \circ g_\beta(\alpha_n) = \beta_n \]
for all $n \geq 0$. This defines a map
\[ \text{Col}_{(F_f, \alpha_\infty)} : \mathcal{U} \to \mathcal{O}_{L_p}[X]^{\times, N_f=\phi} \subset \mathcal{O}_{L_p}[X]^{\times} \]
where $N_f$ is Coleman’s norm operator attached to $f$ (see [45 Chapter I.2.1]). Moreover, for any
$\sigma \in \text{Gal}(L_{p,\infty}/L_p)$, we have
\[ \text{Col}_{(F_f, \alpha_\infty)}(\sigma \beta) = \text{Col}_{(F_f, \alpha_\infty)}(\beta) \circ [\kappa(\sigma)]_f. \]

Tsuji has the following “coordinate-free” formulation of Coleman’s theorem.

**Theorem 2.3 (Tsuji’s formulation of Coleman’s theorem).** Fix a level structure
\[ \alpha_\infty : \mathcal{O}_{K_p} \xrightarrow{\sim} T_f \mathcal{F}_f. \]
Then there exists a natural $\text{Gal}(\overline{L}_p/L_p)$-equivariant isomorphism
\[ \text{Col}_{(F_f, \alpha_\infty)} : \mathcal{U} \xrightarrow{\sim} \mathcal{O}_{F_f[f^\infty]}^{\times}(F_f[f^\infty])^{N_f=\phi} = \mathcal{O}_{F_f[f^\infty]}^{\times}(F_f[f^\infty]) \]
and so applying (5) to (9), we recover (7).

### 3. Construction of measure on the local Galois group

**Definition 3.1.** Henceforth, given a profinite group $G$, let $\mathbb{Z}_p[G]$ denote its completed group
algebra, and for any $\mathbb{Z}_p$-algebra $R$, let $\Lambda(G, R) := \mathbb{Z}_p[G] \hat{\otimes}_{\mathbb{Z}_p} R = R[G]$.

In this section, we prove the following.

**Theorem 3.2.** Assume that $\mathcal{F}_f$ has height 2 and is self-dual (see Definition 3.19), and fix a $p^\infty$-level
structure $\alpha : \mathcal{O}_{K_p} \xrightarrow{\sim} T_f \mathcal{F}_f$. Then there is a $\text{Gal}(\overline{L}_p/L_p)$-equivariant map
\[ \mu_{\mathcal{O}_{K_p}^\times} : \mathcal{U} \to \Lambda(G, \mathcal{O}_{L_p}). \]

The proof of this theorem will actually take up much of this section, with the final construction of (10) appearing in (68).
Definition 3.3. Let
\[ \tilde{F}_f[f^n] := \text{Hom}(F_f[f^n], \mu_{p^n}) \]
denote the Cartier dual of the finite group scheme \( F_f[f^n] \), so that we have a bilinear pairing
\[ \tilde{F}_f[f^n] \times F_f[f^n] \to \mu_{p^n}. \]
Taking the limit, we get a bilinear pairing
\[ (\cdot, \cdot) : T_f \tilde{F}_f \times F_f[f^\infty] \to \mu_{p^\infty}. \]
By construction, one sees that \( \tilde{F}_f[f^\infty] := \lim_{\to} \tilde{F}_f[f^n] \) is a \( p \)-divisible group. In the height 2 case, it gives rise to a connected formal group \( \tilde{F}_f \) equipped with \( \mathcal{O}_{K_p} \)-module structure \([ \cdot ]_f : \mathcal{O}_{K_p} \xrightarrow{\sim} \text{End}(\tilde{F}_f[f^\infty]/\mathcal{O}_{L_p}) \) defined by the relation
\[ \langle \lambda \rangle_f(x) = \langle x, [\lambda]_f(y) \rangle, \]
where \( \lambda \mapsto \bar{\lambda} \) is the non-trivial element of \( \text{Gal}(K_p/\mathbb{Q}_p) \). Note that from (12), we see that \([ \cdot ]_f \) is the only \( \mathcal{O}_{K_p} \)-module structure on \( F_f \) which makes (11) \( \text{Gal}(\overline{K_p}/K_p) \)-equivariant with the natural \( \text{Gal}(\overline{K_p}/K_p) \)-actions.

3.1. **Height 1 (ordinary) case.** Suppose first that \( F_f \) has height \( h = 1 \), so that \( \mathcal{O}_{K_p} = \mathbb{Z}_p \). Then as \( F_f[f^n] \) is connected and of height 1, \( \tilde{F}_f[f^n] \) is étale, and so any section of \( T_f \tilde{F}_f \) is defined over \( W = W(\mathbb{F}_p) \). From (11) we get an identification
\[ \theta : T_f \tilde{F}_f \xrightarrow{\sim} \text{Hom}(F_f[f^\infty], \mu_{p^\infty}). \]
For any section \( \alpha \in T_f \tilde{F}_f \) (which is thus defined over \( W \)), we hence get a map of \( p \)-divisible groups
\[ \theta_{\alpha} : F_f[f^\infty] \xrightarrow{\sim} \mu_{p^\infty}, \quad \theta_{\alpha}(\alpha) = \langle \alpha, \alpha \rangle \]
defined over \( W \). (It is an isomorphism by the perfectness of the Cartier duality pairing.) In fact, identifying \( F_f[f^\infty] \) and \( \mu_{p^\infty} \) with their corresponding formal groups \( F_f \) and \( \hat{G}_m \), \( \theta_{\alpha} \) is exactly the isomorphism on associated \( p \)-divisible groups of the isomorphism
\[ \theta_{\text{ds}} : F_f \xrightarrow{\sim} \hat{G}_m \]
over \( W \) constructed in [15, Chapter I.3]. The map (14) is key for the construction of the map \( i : \mathcal{U} \to \Lambda(\mathcal{G}, W) \) in Chapter I.3.4 of loc. cit. (where it is denoted simply by “\( \theta \)”; we have changed the notation to avoid potential confusion later, as we will use \( \theta \) in defining (19) below), which is the height 1 analogue of our desired map (10).

3.2. **Height 2 (supersingular) case.** Now suppose that \( F_f \) has height \( 2 \). In this case, \( \tilde{F}_f[f^n] \) is still connected and so it has no obvious quotient with which to produce maps as in (13). Moreover, we wish to produce a measure on \( \mathcal{G} \) valued in a finite extension of \( \mathcal{O}_{K_p} \). We hence seek to find a natural linear subspace of a certain scalar extension of \( T_f \tilde{F}_f \) defined over \( \mathcal{O}_{L_p} \). This subspace arises from the Hodge filtration and the \( p \)-adic de Rham comparison theorem, as follows.

3.2.1. **Brief review of the de Rham comparison theorem from \( p \)-adic Hodge theory.** Recall Fontaine’s map \( \vartheta : B_{dR}^+ \to \mathbb{C}_p \), and that the natural filtration on \( B_{dR}^+ \) is given by \( \text{Fil}^i B_{dR}^+ = (\ker \vartheta)^i B_{dR}^+ \). It is well-known that \( \ker \vartheta \) is principal, and is in fact generated by an element \( t \) (often known as a “Fontaine 2\pi i t”) which is a period for the cyclotomic character: \( \sigma(t) = \chi_{\text{cyc}}(\sigma) t \). By the \( p \)-adic de Rham comparison theorem for the \( p \)-divisible group \( F_f[f^\infty] \), we have a canonical isomorphism
\[ H^i_{dR}(F_f[f^\infty]/\mathcal{O}_{L_p}) \xrightarrow{\text{dR}} (T_f \tilde{F}_f \otimes \mathbb{Z}_p B_{dR})^{\text{Gal}(\overline{\mathcal{T}_p}/\mathbb{L}_p)} \]
Tensoring (11) by Definition 3.5. choice of ω since any other choice of (17) ι we have /p algebras not containing 1 O Definition 3.4. Ω the natural filtrations on H where "(" G S gives rise to a functor S' × Ff[ω]) ⊗Zp B, and an inclusion
\[ \Omega^1_{Ff[ω]}/O_{L_p} \subset H^1_{dr}(Ff[ω]/O_{L_p}) \subset H^1_{dr}(Ff[ω]/O_{L_p}) \otimes O_{L_p} B_{dr}^+ \] (16) \[ \iota_{dr} \subset H^1_{et}(Ff[ω] \times O_{L_p} \bmod{Z_p} \otimes_Z B_{dr}^+ \text{ Weil pairing } T_f \hat{F}_f(-1) \otimes Z_p B_{dr}^+ \cong T_f \hat{F}_f \otimes Z_p B_{dr}^+ t^{-1}, \]
where "(i)" denotes the i-th Tate twist. The comparison inclusion \( \iota_{dr} \) in (16) is compatible with the natural filtrations on \( H^1_{dr}(Ff[ω]/O_{L_p}) \otimes O_{L_p} B_{dr}^+ \) (given by the convolution of the Hodge filtration \( \text{Fil}^1 H^1_{dr}(Ff[ω]/O_{L_p}) \) with the natural filtration on \( B_{dr}^+ \)) and on \( T_f \hat{F}_f(-1) \otimes Z_p B_{dr}^+ \cong T_f \hat{F}_f \otimes Z_p B_{dr}^+ t^{-1} \) (given by the convolution of the trivial filtration on \( T_f \hat{F}_f \) and the natural filtration on \( B_{dr}^+ \)). Hence the above composition (16) factors through
\[ \Omega^1_{Ff[ω]}/O_{L_p} =: \text{Fil}^1 H^1_{dr}(Ff[ω]/O_{L_p}) \subset H^1_{dr}(Ff[ω]/O_{L_p}) \iota_{dr} T_f \hat{F}_f \otimes Z_p B_{dr}^+ \subset T_f \hat{F}_f \otimes Z_p B_{dr}^+ t^{-1}. \]

Since \( Ff[ω] \) is 1-dimensional, \( \Omega^1_{Ff[ω]} \) is a rank-1 \( O_{L_p} \)-module, where the natural action of \( O_{K_p} \cong \text{End}_{O_{L_p}}(Ff[ω]) \) is given by multiplication through the natural inclusion \( O_{K_p} \subset O_{L_p} \).

As we will work with \( p \)-divisible groups, we only want to tensor our \( p \)-divisible groups with \( \mathbb{Z}_p \)-algebras not containing 1/p. (Tensoring \( p \)-divisible groups over \( \mathbb{Z}_p \) with rings containing 1/p will result in 0 by \( p \)-divisibility.)

**Definition 3.4.** For any \( O_{L_p} \)-module generator \( ω \in \Omega^1_{Ff[ω]}/O_{L_p} \), \( ω \) has filtration degree 1 and so we have \( \iota_{dr}(ω) = \sum_i α_i \otimes_\mathbb{L} \in \text{Fil}^1(T_f \hat{F}_f \otimes Z_p B_{dr}^+ t^{-1}) = T_f \hat{F}_f \otimes Z_p B_{dr}^+ \). Letting \( B \subset B_{dr}^+ \) be the sub-\( O_{L_p} \)-algebra generated by \( \{α_i\} \), we have \( 1/p \notin B \) and \( \iota_{dr}(\Omega^1_{Ff[ω]}/O_{L_p}) \subset T_f \hat{F}_f \otimes Z_p B \).

Since any other choice of \( ω \) will differ by a scalar in \( \mathbb{O}_{L_p}^\times \), the definition of \( B \) is independent of the choice of \( ω \).

**Definition 3.5.** Tensoring (11) by \( \otimes \mathbb{Z}_p B \) we get a \( \text{Gal}(\mathbb{L}/L_p) \)-equivariant pairing
\[ \langle ·, · \rangle : (T_f \hat{F}_f × Ff[ω]) \otimes \mathbb{Z}_p B → \mu_ψ \otimes \mathbb{Z}_p B, \]
we get a map
\[ \theta : T_f \hat{F}_f \otimes \mathbb{Z}_p B → \text{Hom}(Ff[ω] \otimes \mathbb{Z}_p B, \mu_ψ \otimes \mathbb{Z}_p B). \]

**3.2.2. The map \( θ_α \).** We begin by recalling a seemingly formal yet very useful construction of formal group laws with prescribed endomorphism ring, which is a special case of a more general construction due to Serre.

**Definition 3.6** (Serre tensor product). Let \( G \) be a formal group (or connected \( p \)-divisible group) defined over a complete local ring \( S \), and let \( R \subset \text{End}(G/S) \) be any subring of the ring of endomorphisms of \( G \). Note that we can view \( G \) as a functor from \( S \)-algebras to abelian groups in the usual way (sending an \( S \)-algebra \( S' \) to the group of \( S' \)-valued points \( G(S') \)). For any \( R \)-module \( T \), the tensor product
\[ G \otimes_R T \]
gives rise to a functor \( S' → G(S') \otimes_R T \) and hence defines a formal group law over \( S \). We refer to \( G \otimes_R T \) as the Serre tensor product (with \( \otimes_R T \)) of \( G \). Note that if \( T \) is an \( R \)-algebra, \( \text{End}((G \otimes_R T)/S) = \text{End}(G/S) \otimes_R T \), where \( T \) acts on \( G \otimes_R T \) by multiplication on the right, and that \( G \otimes_R T \) has dimension \( \text{dim}(G) \cdot \text{rank}_RT \). Note that for any endomorphism \( λ \in R \subset \text{End}(G/S) \otimes_R T \), and any point \( x \otimes t \in G \otimes_R T \), we have
\[ λ(x \otimes t) = x \otimes tλ = λ(x) \otimes t. \]
We can also extend the construction to tensor products $G \otimes_R M$ for $R$-modules $M$. In this case, for any $R$-module $T$ acting on $M$ compatibly with the $R$-module structure, we have $\text{End}(G/S) \otimes_R T \subset \text{End}(G \otimes_R M/S)$.

We shall primarily be interested in the case where $G = \mu_{p^\infty}$, $S = \mathcal{O}_{L_p}$, $R = \mathbb{Z}_p$ and $T = B$.

Consider the inclusion induced by the $p$-adic de Rham comparison inclusion
\[
\Omega^1_{F_j[f^\infty]/\mathcal{O}_{L_p}} \xrightarrow{\iota_{\text{dR}}} T_f \hat{F}_f \otimes_{\mathbb{Z}_p} B.
\]

Since $\iota_{\text{dR}}$ in (16) is $\text{Gal}(L_p/L_p)$-equivariant, from (15) and we have that $\text{Gal}(L_p/L_p)$ acts trivially on $\iota_{\text{dR}}(\Omega^1_{F_j[f^\infty]/\mathcal{O}_{L_p}})$.

Thus by the Galois equivariance of (18) (which we recall is induced by the Weil pairing), given any $\omega_0 \in \Omega^1_{F_j[f^\infty]/\mathcal{O}_{L_p}}$ so that $\iota_{\text{dR}}(\omega_0) \in T_f \hat{F}_f \otimes_{\mathbb{Z}_p} B$ by (17), we have that the homomorphism associated to $\iota_{\text{dR}}(\omega_0)$ by (19)
\[
\theta_{\iota_{\text{dR}}(\omega_0)} := \theta(\iota_{\text{dR}}(\omega)) \in \text{Hom}(F_j[f^\infty] \otimes_{\mathbb{Z}_p} B, \mu_{p^\infty} \otimes_{\mathbb{Z}_p} B)^{\text{Gal}(L_p/L_p)}
\]
is a homomorphism of $p$-divisible groups (viewing $\mu_{p^\infty} \otimes_{\mathbb{Z}_p} B$ as a Serre tensor product) which is invariant under $\text{Gal}(L_p/L_p)$. Hence $\theta_{\iota_{\text{dR}}(\omega_0)}$ is a map of $p$-divisible groups over $\mathcal{O}_{L_p}$.

**Definition 3.7.** Let $\log_{F_j}$ be the logarithm of the formal group $F_j$ normalized so that $\log_{F_j}(0) = 1$, so that $\log_{F_j}(X) \in \mathcal{O}_{L_p}[X]^\infty$ (see [45, Chapter I.1.1]). Henceforth, fix the $\mathcal{O}_{L_p}$-module generator
\[\omega_0 = d \log_{F_j} \in \Omega^1_{F_j[f^\infty]/\mathcal{O}_{L_p}},\]
and let
\[\tilde{\alpha}_0 := \iota_{\text{dR}}(\omega_0) \overset{(17)}{\in} T_f \hat{F}_f \otimes_{\mathbb{Z}_p} B.\]

**Definition 3.8.** From $\tilde{\alpha}_0$ and the construction (20), we get a map
\[\theta_{\tilde{\alpha}_0} : F_j[f^\infty] \otimes_{\mathbb{Z}_p} B \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} B\]
of $p$-divisible groups over $\mathcal{O}_{L_p}$. By restriction to $F_j[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \subset F_j[f^\infty] \otimes_{\mathbb{Z}_p} B$, we hence get a map
\[\theta_{\tilde{\alpha}_0} : F_j[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} B,\]
which is still a map of $p$-divisible groups over $\mathcal{O}_{L_p}$. Further restricting to $F_j[f^\infty] \subset F_j[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}$, we get a map
\[\theta_{\tilde{\alpha}_0} : F_j[f^\infty] \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} B.\]

3.3. **The injectivity and image of $\theta_{\tilde{\alpha}_0}$**. Recall the element $\xi = \text{Nm}_{L_p/K_p}(\pi') \in \mathcal{O}_{K_p}$ with $v(\xi) = d = [L_p : K_p]$, where $\pi' = f'(0)$, from Section 2.1. Note that
\[f^d = [\xi]_f \in \text{End}(F_j/\mathcal{O}_{L_p})\]
since $(f^d)'(0) = \xi$, and homomorphisms between two fixed formal group laws over rings of characteristic 0 are uniquely determined by the value of their first derivatives at $X = 0$ (see [45, Chapter I.1.1]).

The main result of this section is the following Theorem.

**Theorem 3.9.** (1) \[\text{is injective.}\]
(2) Moreover, fixing a level structure $\alpha_\infty : \mathcal{O}_{K_p} \to T_f F_f$, (22) induces an isomorphism
\begin{equation}
\theta_{\alpha_0} : F_f[f^\infty] \to t_\kappa : \mathcal{O}_{K_p} \subseteq \mu_{p^\infty} \otimes_{\mathbb{Z}_p} B,
\end{equation}
where
\begin{equation}
t_\kappa = t_\kappa(\alpha_0) := \lim_{n \to \infty} \xi^n \theta_{\alpha_0}(\alpha_{nd}) \in \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} B
\end{equation}
(writing $t_\kappa = t_\kappa(\alpha_0)$ when we wish to emphasize the dependence on $\alpha_0$, or equivalently on $\omega_0$).

(3) Moreover, $t_\kappa$ is a period for $\kappa$, in the sense that
\begin{equation}
\sigma^* t_\kappa = \kappa(\sigma) t_\kappa \quad \forall \sigma \in \text{Gal}(\overline{K}_p / K_p).
\end{equation}

Remark 3.10. One should think of $t_\kappa$ as a “Fontaine period associated to $\kappa$”, and it can similarly be equivalently defined, using Colmez’s description (see [12]) of the p-adic comparison map $\iota_{\text{dR}}$ via an integration pairing, as the integration of a differential over a generator of $T_f F_f$:
\begin{equation}
t_\kappa = t_\kappa(\alpha_0) := \lim_{n \to \infty} \xi^n \theta_{\alpha_0}(\alpha_{nd}) = \lim_{n \to \infty} \xi^n (\alpha_{0}, \alpha_{nd}) = \lim_{n \to \infty} \xi^n \int_0^{\alpha_{nd}} \omega_0.
\end{equation}

To prove Theorem 3.9 we first study (21).

Definition 3.11. Note that the module $F_f[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}$ has an $\mathcal{O}_{K_p}$-action induced by the $\mathcal{O}_{K_p}$-module structure $[\cdot]_f$ on $F_f[f^\infty]$ via
\begin{equation}
[\lambda]_f(x \otimes_{\mathbb{Z}_p} g) = [\lambda]_f(x) \otimes_{\mathbb{Z}_p} g.
\end{equation}
Note that $g \in \mathcal{O}_{K_p}$ also acts on $x \in F_f[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}$ by multiplication on the right, and we denote this multiplication by $xg$. We define $\mathcal{O}_{K_p}$-submodules of $F_f[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}$
\begin{equation}
H_{1,0} := \{ x \in F_f[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} : [\lambda]_f(x) = x \lambda \forall \lambda \in \mathcal{O}_{K_p} \},
\end{equation}
\begin{equation}
H_{0,1} := \{ x \in F_f[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} : [\lambda]_f(x) = x \lambda \forall \lambda \in \mathcal{O}_{K_p} \}.
\end{equation}
By standard linear algebra, we then get a decomposition
\begin{equation}
F_f[f^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} = H_{1,0} \oplus H_{0,1}.
\end{equation}

Remark 3.12. Recall that $\text{End}_{\mathcal{O}_{L_p}}(F_f[f^\infty])$ acts on $\iota_{\text{dR}}(\Omega^1_{F_f[f^\infty] / \mathcal{O}_{K_p}}) \subseteq T_f \tilde{F}_f \otimes_{\mathbb{Z}_p} B$, via pullback $[\cdot]_f$, as simply by multiplication by $\lambda$:
\begin{equation}
[\lambda]_f^* \omega = \omega \lambda, \quad \forall \omega \in \Omega^1_{F_f[f^\infty] / \mathcal{O}_{L_p}}, \forall \lambda \in \mathcal{O}_{K_p} \cong \text{End}_{\mathcal{O}_{L_p}}(F_f[f^\infty]).
\end{equation}
In particular, we have
\begin{equation}
[\lambda]_f(\alpha_0) = [\lambda]_f(\iota_{\text{dR}}(\omega_0)) = \iota_{\text{dR}}([\lambda]_f^* \omega_0) = \iota_{\text{dR}}(\omega_0) \lambda = \alpha_0 \lambda,
\end{equation}
where the second equality follows from standard properties of $\iota_{\text{dR}}$ (see, for example, [12], where $\iota_{\text{dR}}$ is described via an integration pairing, and the equality then follows from the properties of the integration pairing in loc. cit.).

Lemma 3.13. $\ker((21)) = H_{0,1}$, and so (21) factors through an injection $H_{1,0} \hookrightarrow \mu_{p^\infty} \otimes_{\mathbb{Z}_p} B$.

Proof. Let $\lambda \in \mathcal{O}_{K_p}$. Again by definition of the $\mathcal{O}_{K_p}$-action on $\tilde{F}_f$, we have
\begin{equation}
\theta_{\alpha_0}([\lambda]_f(y)) = \langle \tilde{\alpha}_0, [\lambda]_f(y) \rangle \quad (\overline{[\lambda]_f(\alpha_0), y}) \quad (\tilde{\alpha}_0 \lambda, y) = \langle \tilde{\alpha}_0, y \rangle \lambda = \theta_{\alpha_0}(y) \lambda.
\end{equation}
Suppose now that $y \in H_{0,1}$, so that $[\lambda]_f(y) = y \lambda$. Then we have
\begin{equation}
\theta_{\alpha_0}(y) = \theta_{\alpha_0}(y) \lambda = \theta_{\alpha_0}(\overline{[\lambda]_f(y)}) \quad \theta_{\alpha_0}(y) \lambda,
\end{equation}
and so taking any \( \lambda \in \mathcal{O}_{K_p} \setminus \mathbb{Z}_p \), so that \( \overline{\lambda} \neq \lambda \), we see that \( \theta_{\delta_0}(y) = 0 \). Hence \( H_{0,1} \subset \ker ((21)). \) 
Since (16) is an inclusion, we then get from (26) the equality \( H_{0,1} = \ker ((21)). \) □

**Definition 3.14.** Note that there is a natural \( \mathcal{O}_{K_p} \)-linear map
\[
m : F_f[f^{\infty}] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \to F_f[f^{\infty}], \quad x \otimes_{\mathbb{Z}_p} \lambda \mapsto [\lambda]_f(x).
\]

Note that the inclusion \( F_f[f^{\infty}] \subset F_f[f^{\infty}] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \) composed with \( m \) is simply the identity on \( F_f[f^{\infty}] \).

**Proposition 3.15.** We have that \( \ker (m) = H_{0,1} \), and hence induces an \( \mathcal{O}_{K_p} \)-linear isomorphism \( m : H_{1,0} \sim \to F_f[f^{\infty}] \).

*Proof.* By definition, we have
\[
m([\lambda]_f(x)) = [\lambda]_f(m(x)).
\]
If \( x \in H_{0,1} \), we have
\[
m([\lambda]_f(x)) = m(x \overline{\lambda}) = [\overline{\lambda}]_f(m(x)).
\]
Hence, taking \( \lambda \in \mathcal{O}_{K_p} \setminus \mathbb{Z}_p \), we see from the above two displayed equations that \( m(x) = 0 \). So \( H_{0,1} \subset \ker (m) \). Now since \( m \) is clearly surjective, from (26) we see that \( H_{0,1} = \ker (m) \). □

**Corollary 3.16.** The composition
\[
F_f[f^{\infty}] \subset F_f[f^{\infty}] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \overset{(26)}{\longrightarrow} H_{1,0} \oplus H_{0,1} \to H_{1,0}
\]
is an isomorphism.

*Proof.* By definition, the composition of the inclusion \( F_f[f^{\infty}] \subset F_f[f^{\infty}] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \) with \( m \) is the identity on \( F_f[f^{\infty}] \). From Proposition 3.15 we see that \( m \) factors through (29). This gives the Corollary. □

*Proof of Theorem 3.4.* (1): This follows immediately from Lemma 3.13 and Corollary 3.16.

(2): From the \( p^{\infty} \)-level structure \( \alpha_{\infty} : \mathcal{O}_{K_p} \sim \to T_f F_f \), we get a sequence of elements \( \alpha_{nd} \in F_f[\xi^n] \) for \( n \in \mathbb{Z}_{\geq 0} \). Note that by (1), \( \theta_{\delta_0} \) induces isomorphisms
\[
\theta_{\delta_0} : F_f[\xi^n] \sim \sim \mu_{p^{\infty}} \otimes_{\mathbb{Z}_p} (\xi^{-n}B/B).
\]
Define
\[
t_{n,n} := \theta_{\delta_0}(\alpha_{nd}) \in \mu_{p^{\infty}} \otimes_{\mathbb{Z}_p} (\xi^{-n}B/B), \quad t_{n} := \lim_{n \to \infty} \xi^n \theta_{\delta_0}(\alpha_{nd}) \in \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} B.
\]
As \( \alpha_{nd} : (\xi^{-n}\mathcal{O}_{K_p}/\mathcal{O}_{K_p}) \sim \to F_f[\xi^n] \), then
\[
\theta_{\delta_0}(F_f[\xi^n]) = \theta_{\delta_0}((\mathcal{O}_{K_p}, f)(\alpha_{nd})) \overset{(1)}{=} \theta_{\delta_0}(\mu_{p^{-1}}((\mathcal{O}_{K_p}, f)(\alpha_{nd}))) \overset{\text{Proposition 3.13}}{=} \theta_{\delta_0}(\alpha_{nd}\mathcal{O}_{K_p}) = t_{n,n} \cdot (\mu_{p^{\infty}} \otimes_{\mathbb{Z}_p} (\xi^{-n}\mathcal{O}_{K_p}/\mathcal{O}_{K_p})).
\]
Now the fact that \( \theta_{\delta_0} : F_f[\xi^n] \sim \to t_{n,n}\mathcal{O}_{K_p}/(\xi^n\mathcal{O}_{K_p}) \) follows immediately from \( \mathcal{O}_{K_p} \)-linearity of \( \theta_{\delta_0} \). Putting these isomorphisms together, we get (23).

(3): This follows immediately from the construction of \( t_{n} \) given in (2). □
Remark 3.22. Note that for any \( \lambda \in \mathbb{Z}_p \) there is a “Fontaine \( 2\pi i \)”: 
\[
t := \log([1, \zeta_p, \zeta_p^2, \ldots]) \in B^+_{\text{dR}},
\]
on which \( \text{Gal}(L_{p,\infty}/L_p) \) acts via \( \chi_{\text{cyc}}|_{\text{Gal}(\mathbb{K}_p/K_p)} \). In particular, note that for any fixed orientation (or equivalently, fixed \( t \)), there is a canonical isomorphism of \( \mathbb{Z}_p \)-modules \( \mathbb{Z}_p(1) \cong \mathbb{Z}_p \cdot t \).

Convention 3.18. Henceforth, fix an orientation of \( \mathbb{C}_p \) (or equivalently, a Fontaine \( 2\pi i t \in B^+_{\text{dR}} \)). As mentioned above, this fixes an isomorphism \( \mathbb{Z}_p(1) \cong \mathbb{Z}_p \cdot t \).

Definition 3.19. If there exists an isomorphism \( \rho : F_f \xrightarrow{\sim} \hat{F}_f \) (or \( F_f[f^\infty] \xrightarrow{\sim} \hat{F}_f[f^\infty] \)) of formal \( \mathcal{O}_{K_p} \)-modules over \( \mathcal{O}_{L_p} \), so that in particular \( \hat{F}_f \) is a Lubin-Tate formal \( \mathcal{O}_{K_p} \)-module (i.e. has a special endomorphism \( \hat{f} : \hat{F}_f \to \hat{F}_f^\phi \)), we call \( F_f \) (or \( F_f[f^\infty] \)) self-dual. In this case, the Cartier duality pairing precomposed with \( \rho \) gives a nondegenerate pairing 
\[
F_f[p^n] \times F_f[p^n] \cong \hat{F}_f[p^n] \times F_f[p^n] \to \mu_{p^n}
\]
which shows that \( \mu_{p^n} \subset L_{p,\infty} \). This also shows that 
\[
\kappa \hat{R} = \chi_{\text{cyc}}|_{\text{Gal}(\mathbb{K}_p/K_p)} : \text{Gal}(\mathbb{K}_p/K_p) \to \mathbb{Z}_p^\times
\]
where \( \chi_{\text{cyc}} \) is the cyclotomic character. In particular, in the case where \( F_f \) arises from the formal group \( \hat{A} \) of an elliptic curve \( A/\mathcal{O}_L \) with complex multiplication by \( \mathcal{O}_K \) and of good reduction at the place of \( \mathcal{O}_L \) above \( p \) fixed by \([3]\), then such a \( \rho \) is induced by the canonical principal polarization on \( A \).

Remark 3.20. Suppose \( F_f \) is self-dual. By the above, we have \( \mu_{p^n} \subset L_{p,\infty} \). Then recalling our notation \( \xi = \text{Nm}_{L_p/K_p}(f'(0)) \), the norm-restriction functoriality of local class field theory implies that \( \text{Nm}_{L_p/K_p}(\xi) = q^d \), where \( d = [L_p : K_p] \) and \( q = \#k(K_p) \) (so that \( q = p \) or \( p^2 \), according to whether \( p \) is ramified or inert in \( K \)).

Definition 3.21. Henceforth, let 
\[
G_{\hat{\alpha}_0} := \theta_{\hat{\alpha}_0}(F_f[f^\infty]) \subset \mu_{p^n} \otimes_{\mathbb{Z}_p} B,
\]
so that by Theorem \([3.9]\) we have an \( \mathcal{O}_{K_p} \)-linear isomorphism
\[(30)\]
\[
\theta_{\hat{\alpha}_0} : F_f[f^\infty] \xrightarrow{\sim} G_{\hat{\alpha}_0}.
\]
When \( \hat{\alpha}_0 \) is clear from context, we sometimes write \( G = G_{\hat{\alpha}_0} \). We have 
\[
t \in \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} B \cong \mathbb{Z}_p t \otimes_{\mathbb{Z}_p} B = t \cdot B \implies t^{-1} := t_\pi := t \in B_{\text{dR}}.
\]
We can write 
\[
G_{\hat{\alpha}_0} \overset{\text{Theorem}\,[3.9]}{=} \frac{t_k}{t} \frac{K_p}{\mathcal{O}_K} = \frac{t(t_k/t)K_p}{\mathcal{O}_K} = \mu_{p^n} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}(t_k/t) = \mu_{p^n} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} t_k^{-1} \subset \mu_{p^n} \otimes_{\mathbb{Z}_p} B.
\]

Remark 3.22. Note that for any \( \lambda \in \mathbb{O}_{K_p}^\times \), by part (4) of Theorem \([3.9]\) we have
\[
G_{\hat{\alpha}_0,\lambda} := t_k(\hat{\alpha}_0,\lambda) K_p/\mathcal{O}_K = t_k(\hat{\alpha}_0,\lambda) K_p/\mathcal{O}_K = t_k K_p/\mathcal{O}_K = G_{\hat{\alpha}_0}.
\]
3.3.1. The canonical coordinate $Q$.

Assumption 3.23. Henceforth, we assume that $F_f$ is self-dual in the sense of Definition 3.19.

Definition 3.24. Choose and fix an isomorphism

\begin{equation}
O_{F_f[f^\infty]}(F_f[f^\infty]) \cong O_{L_p}[X].
\end{equation}

In fact, such an isomorphism is unique up to a change of variables $X \mapsto [u]_f(X)$ for some $u \in O_{K_p}^\times$.

Definition 3.25. Recall the notation $G_{\tilde{\alpha}_0} = \theta_{\tilde{\alpha}_0}(F_f[f^\infty])$ from Definition 3.21. The isomorphism $\theta_{\tilde{\alpha}_0} : F_f[f^\infty] \cong G_{\tilde{\alpha}_0}$ induces an isomorphism

\begin{equation}
\theta_{\tilde{\alpha}_0}^* : O_{G_{\tilde{\alpha}_0}}(G_{\tilde{\alpha}_0}) \cong O_{F_f[f^\infty]}(F_f[f^\infty]),
\end{equation}

with inverse $(\theta_{\tilde{\alpha}_0}^*)^{-1} = (\theta_{\tilde{\alpha}_0}^{-1})^*$. Let

\begin{equation}
Q_{\tilde{\alpha}_0} := 1 + (\theta_{\tilde{\alpha}_0}^{-1})^*(X)
\end{equation}

so that we get an identification

\begin{equation}
O_{G_{\tilde{\alpha}_0}}(G_{\tilde{\alpha}_0}) = O_{L_p}[Q_{\tilde{\alpha}_0} - 1].
\end{equation}

Similarly, for any $\lambda \in K_p$, we define

\begin{equation}
Q_{\tilde{\alpha}_0\lambda} := 1 + (\theta_{\tilde{\alpha}_0\lambda}^{-1})^*(X) \in O_{G_{\tilde{\alpha}_0\lambda}}(G_{\tilde{\alpha}_0\lambda}).
\end{equation}

Note that if $\lambda \in O_{K_p}$, then we can view

\begin{equation}
G_{\tilde{\alpha}_0} \cong t_\kappa(\tilde{\alpha}_0)K_p/O_{K_p} \quad \text{open sub-$p$-divisible group}
\end{equation}

as an open sub-$p$-divisible group, and in particular we can view $Q_{\tilde{\alpha}_0\lambda^{-1}}$, via restriction, as an element $Q_{\tilde{\alpha}_0\lambda^{-1}} \in O_{G_{\tilde{\alpha}_0}}(G_{\tilde{\alpha}_0})$. Note that since $G_{\tilde{\alpha}_0} \subset \mu_{p^\infty} \otimes \mathbb{Z}_p B$ is a sub-$p$-divisible group, and so the group law on $G_{\tilde{\alpha}_0}$ is induced by the one on $\mu_{p^\infty} \otimes \mathbb{Z}_p B$, then

\begin{equation}
d\log Q_{\tilde{\alpha}} = \frac{dQ_{\tilde{\alpha}}}{Q_{\tilde{\alpha}}} \in \Omega^1_{G_{\tilde{\alpha}}/O_{L_p}}
\end{equation}

is an invariant differential.

Definition 3.26. Henceforth, let

\begin{equation}
\theta_{\tilde{\alpha}_0}(X) := \theta_{\tilde{\alpha}_0}^*(Q_{\tilde{\alpha}_0} - 1) \in O_{F_f[f^\infty]}(F_f[f^\infty]) \cong O_{L_p}[X],
\end{equation}

\begin{equation}
\theta_{\tilde{\alpha}_0}^{-1}(Q_{\tilde{\alpha}_0} - 1) := (\theta_{\tilde{\alpha}_0}^{-1})^*(X) \in O_{G_{\tilde{\alpha}_0}}(G_{\tilde{\alpha}_0}) \cong O_{L_p}[Q_{\tilde{\alpha}_0} - 1].
\end{equation}

Definition 3.27. Let

\begin{equation}
\Omega_p := \frac{d}{dQ_{\tilde{\alpha}_0}}(\theta_{\tilde{\alpha}_0}^{-1})(Q_{\tilde{\alpha}_0} - 1)|_{Q_{\tilde{\alpha}_0} = 1} = (\theta_{\tilde{\alpha}_0}^{-1})'(0) \in O_{L_p}^\times.
\end{equation}

Here, the last inclusion follows because $\theta_{\tilde{\alpha}_0}^{-1}$ is an isomorphism.

Proposition 3.28. We have

\begin{equation}
\theta_{\tilde{\alpha}_0}^* \frac{dQ_{\tilde{\alpha}_0}}{Q_{\tilde{\alpha}_0}} = \Omega_p^{-1} \cdot \omega_0.
\end{equation}
Letting $\theta$ denote logarithms of formal groups over $\mathcal{O}_{L_p}$, the calculation

$$\frac{d}{dX}(\log(1 + \theta(X)))|_{X=0} = \theta'(0) = \Omega_p^{-1} = \Omega_p^{-1} \cdot \log_{F_f}(0)$$

implies

$$\log(1 + \theta(X)) = \Omega_p^{-1} \cdot \log_{F_f}(X).$$

from which (34) follows.

**Proof.** Since logarithms of formal groups over $\mathcal{O}_{L_p}$ are unique up to a multiple in $\mathcal{O}_{L_p}$, the calculation

$$\frac{d}{dX}(\log(1 + \theta(X)))|_{X=0} = \theta'(0) = \Omega_p^{-1} = \Omega_p^{-1} \cdot \log_{F_f}(0)$$

implies

$$\log(1 + \theta(X)) = \Omega_p^{-1} \cdot \log_{F_f}(X).$$

**Proposition 3.29.** We have $t_{\sigma} = \Omega_p$, and

$$\theta^* \frac{dQ_{\tilde{\alpha}_0}}{Q_{\tilde{\alpha}_0}} = t_{\sigma}^{-1} \cdot \omega_0 = \Omega_p^{-1} \cdot \omega_0 \in \Omega_{F_f[f^\infty]}^1/\mathcal{O}_{L_p}.$$

**Proof.** Letting $T$ be the standard coordinate on $\mu_{p^\infty}$, we have $Q_{\tilde{\alpha}_0} = T \otimes t_{\sigma}^{-1}$. Hence,

$$\frac{dQ_{\tilde{\alpha}_0}}{Q_{\tilde{\alpha}_0}} = t_{\sigma}^{-1} \frac{dT}{T}$$

in $\Omega_{\mu_{p^\infty} \otimes \mathbb{Z}_p B/\mathcal{O}_{L_p}}$. Now we have

$$\tilde{\alpha}_0 \frac{dT}{T} = t_{dR}(\omega_0)^* d\log T = \omega_0.$$

Here the last equality is a standard fact following from the “pullback property” of Colmez’s $p$-adic integration pairing description of $t_{dR}$ (see, for example, [30] Chapter 3, Theorem 4.9 and Section 5), where the identity $d\log \circ t_{dR} = \mathrm{id}$ is proved; note that $d\log$ is denoted as the “Hodge-Tate map” $\mathrm{HT}$ in loc. cit.). Putting the above equalities together, we arrive at the first equality (35). Now the rest of the Proposition follows from (34).

**Proposition 3.30.** For any $\lambda \in \mathcal{O}_{K_p}$, we have

$$[\lambda]_{G_{\tilde{\alpha}_0}}(Q_{\tilde{\alpha}_0} - 1) = Q_{\tilde{\alpha}_0}^{\text{Nm}_{K_p/\mathbb{Q}_p}(\lambda)} - 1.$$

**Proof.** Since $Q_{\tilde{\alpha}_0} = \langle \tilde{\alpha}_0, X \rangle$, we have

$$[\lambda]_{G_{\tilde{\alpha}_0}}(Q_{\tilde{\alpha}_0} - 1) = \langle \tilde{\alpha}_0, [\lambda]_f(X) \rangle = \langle [\lambda]_f[\lambda^{-1}]_f(\tilde{\alpha}_0), X \rangle \overset{(12)}{=} \langle [\lambda]_f(\tilde{\alpha}_0), [\lambda]_f(X) \rangle$$

$$= \langle [\lambda]_f(\tilde{\alpha}_0), X \rangle^{\text{Nm}_{K_p/\mathbb{Q}_p}(\lambda)} \overset{(22)}{=} \langle \tilde{\alpha}_0 \lambda^{-1}, X \rangle^{\text{Nm}_{K_p/\mathbb{Q}_p}(\lambda)} = Q_{\tilde{\alpha}_0}^{\text{Nm}_{K_p/\mathbb{Q}_p}(\lambda)}.$$

**Definition 3.31.** From the lift of the Frobenius $f : F_f[f^\infty] \rightarrow F_f^\phi[(\phi f)^\infty]$ the isomorphism $\theta_{\tilde{\alpha}_0} : F_f[f^\infty] \overset{\sim}{\rightarrow} G_{\tilde{\alpha}_0}$, we get a homomorphism $\tilde{f} : G_{\tilde{\alpha}_0} \rightarrow G_{\phi_{\tilde{\alpha}_0}}^\phi$ by writing $f$ as a power series in $Q_{\tilde{\alpha}_0} - 1$, lifting the Frobenius $\phi$ modulo $p\mathcal{O}_{L_p}$. That is,

$$f(Q_{\tilde{\alpha}_0} - 1) \equiv (Q_{\tilde{\alpha}_0} - 1)^{\#F_f[f]} \pmod{p\mathcal{O}_{L_p}[Q_{\tilde{\alpha}_0} - 1]}.$$

In the next proposition, we show that the modulo $p\mathcal{O}_{L_p}$ congruence (37) in fact holds modulo $p\mathcal{O}_{L_p}[Q_{\tilde{\alpha}_0} - 1]$.

**Proposition 3.32.** We have that

$$f(Q_{\tilde{\alpha}_0} - 1) \equiv (Q_{\tilde{\alpha}_0} - 1)^{\#F_f[f]} \pmod{p\mathcal{O}_{L_p}[Q_{\tilde{\alpha}_0} - 1]}.$$
Proof. As before, let \( \pi \) be a uniformizer of \( p\mathcal{O}_{K_p} \). Since \( f \in \text{Hom}(G_{\hat{\alpha}_0}, G_{\hat{\alpha}_0}) \), we have
\[
(f(Q_{\hat{\alpha}_0} - 1) + 1)^{Nm_{K_p/q_p}(\pi)} - 1 \overset{(36)}{=} c_{\hat{\alpha}_0}^\pi \equiv f(Q_{\hat{\alpha}_0} - 1) = f(Q_{\hat{\alpha}_0} - 1) \circ [\pi]G_\pi \overset{(36)}{=} G_{\hat{\alpha}_0} f((Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)}).
\]
The binomial theorem shows that
\[
(f(Q_{\hat{\alpha}_0} - 1) + 1)^{Nm_{K_p/q_p}(\pi)} - 1 \equiv f(Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} \pmod{p\mathcal{O}_{L_p}[Q_{\hat{\alpha}_0} - 1]}.
\]
Hence (39) implies
\[
f(Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} \equiv f((Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} \pmod{p\mathcal{O}_{L_p}[Q_{\hat{\alpha}_0} - 1]}.
\]
Now from (37), we get
\[
f(Q_{\hat{\alpha}_0} - 1) - (Q_{\hat{\alpha}_0} - 1)^{\#F_j[f]} \equiv 0 \pmod{p\mathcal{O}_{L_p}[Q_{\hat{\alpha}_0} - 1]},
\]
which, raising to the \( Nm_{K_p/q_p}(\pi) \)th power and noting that \( p\mathcal{O}_{L_p}[\#_{Nm_{K_p/q_p}(\pi)}] \), gives
\[
f(Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} - (Q_{\hat{\alpha}_0} - 1)^{\#F_j[f]Nm_{K_p/q_p}(\pi)} \equiv 0 \pmod{p\mathcal{O}_{L_p}[Q_{\hat{\alpha}_0} - 1]}.
\]
By (40), (41) is equivalent to
\[
f((Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} \equiv ((Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} \#F_j[f] \pmod{p\mathcal{O}_{L_p}[Q_{\hat{\alpha}_0} - 1]}.
\]
Replacing \( (Q_{\hat{\alpha}_0} - 1)^{Nm_{K_p/q_p}(\pi)} \) by \( (Q_{\hat{\alpha}_0} - 1) \) in the above identity immediately gives (38). \qed

**Convention 3.33.** For the remainder of this section, we will only talk about \( G_{\hat{\alpha}_0} \) for the \( \hat{\alpha}_0 \in \Omega_{F_j[f\infty]/\mathcal{O}_{L_p}} \) previously fixed in Definition 3.26, and hence we will simply write \( G = G_{\hat{\alpha}_0} \). Similarly, we will only talk about \( f_{G_{\hat{\alpha}_0}} \) and \( Q_{\hat{\alpha}_0} \) for this fixed \( \hat{\alpha}_0 \), and so we will write \( f = f_{G_{\hat{\alpha}_0}} \) and \( Q = Q_{\hat{\alpha}_0} \).

**Convention 3.34.** Henceforth, we use (37) and (38) to make an identification
\[
\mathcal{O}_{L_p}[X] = F_j[f\infty] = G = \mathcal{O}_{L_p}[Q - 1], \quad Q - 1 = \theta_{\hat{\alpha}_0}^{-1}(Q - 1), \quad X = \theta_{\hat{\alpha}_0}(X), \quad X = \theta_{\hat{\alpha}_0}^{-1}(Q - 1), \quad Q - 1 = \theta_{\hat{\alpha}_0}(X)
\]
where the latter two power series are defined in Definition 3.26. Given \( g \in \mathcal{O}_{F_j[f\infty]}(F_j[f\infty]) = \mathcal{O}_G(G) \), write \( g = g(X) \) when viewing it as in \( \mathcal{O}_{L_p}[X] \), and as \( g = g(Q - 1) \) when viewing it as in \( \mathcal{O}_{L_p}[Q - 1] \).

**Lemma 3.35.** Let \( H \subset G \) be any finite group scheme of order \( p \). For any \( n \in \mathbb{Z}_{\geq 0} \), we have
\[
\sum_{\varpi \in F_j[f]} (\theta_{\hat{\alpha}_0}(X[+f\varpi]) + 1)^n = \frac{\#F_j[f]}{p} \sum_{\gamma \in H} (((Q - 1)[+f\gamma]) + 1)^n = \frac{\#F_j[f]}{p} \sum_{j=0}^{p-1} (\varpi_j^2 Q)^n.
\]
Proof. Note that, by definition, we have
\[
Q = \theta_{\alpha_0}(X) + 1 = \langle \hat{\alpha}_0, X \rangle.
\]
Hence,
\[
\theta_{\hat{\alpha}_0}(X[+f\varpi]) + 1 = \langle \hat{\alpha}_0, X[+f\varpi] \rangle = \langle \hat{\alpha}_0, X \rangle \langle \hat{\alpha}_0, \varpi \rangle = (\theta_{\alpha}(X) + 1)\langle \hat{\alpha}_0, \varpi \rangle = Q\langle \hat{\alpha}_0, \varpi \rangle.
\]
We have
\[
\sum_{\varpi \in F_f[f]} (\theta_0(X[+]_f \varpi) + 1)^n = Q^n \sum_{\varpi \in F_f[f]} (\tilde{\alpha}_0, \varpi)^n = Q^n \sum_{\varpi \in F_f[f]} (\tilde{\alpha}_0, [n]_f \varpi) = Q^n \left\{ \begin{array}{cc} \#F_f[f] & p|n \\ 0 & p \nmid n \end{array} \right.
\]
where the last equality follows because \([n]_f(\varpi) = 0\) if \(p|n\) and \([n]_f\) acts simply transitively on \(F_f[f]\) if \(p \nmid n\). Now the Lemma follows after noting that by the same reasoning as above,
\[
\sum_{\gamma \in H} ((Q - 1)[+G\gamma] + 1)^n = Q^n \left\{ \begin{array}{cc} p & p|n \\ 0 & p \nmid n \end{array} \right.
\]
\[
\tag{46}
\]

\[\hat{h} := h - \frac{1}{\#F_f[f]} \log(h^\phi \circ f) = h(X) - \frac{1}{\#F_f[f]} \sum_{\varpi \in F_f[f]} h(X[+]_f \varpi)
\]
\[
\tag{45}
\]
\[h(Q - 1) - \frac{1}{p} \sum_{\gamma \in H} h((\gamma + 1)Q - 1).
\]

Using the second equality of (45) gives us the formula
\[
\hat{h} = h(Q - 1) - \frac{1}{p} \sum_{j=0}^{p-1} h(\zeta_p^j Q - 1)
\]
which will be essential for calculations later.

**Remark 3.37.** Note that this is analogous to the definition of \(\sim\) in [15, Chapter 1.3 (7')] which uses the \(p\)-divisible group structure on \(\mu_{p^\infty}\). Here we use the natural \(p\)-divisible group structure on the subgroup \(G\). Note that (46) is the same formula as in loc. cit.

**Lemma 3.38.** Let \(H \subset G\) be any finite group scheme of order \(p\). We have
\[
N_f g \circ f = \left( \prod_{\varpi \in F_f[f]} g(X[+]_f \varpi) \right) = \left( \prod_{\gamma \in H} g((\gamma + 1)Q - 1) \right)^{\#F_f[f]/p}
\]
\[
(47)
\]

**Proof.** First, by standard properties of \(N_f\) (see [15, Chapter I.1.2]) we have
\[
\log N_f g \circ f = \log \prod_{\varpi \in F_f[f]} g(X[+]_f \varpi) = \sum_{\varpi \in F_f[f]} \log g(X[+]_f \varpi) = \frac{\#F_f[f]}{p} \sum_{\gamma \in H} \log g((\gamma + 1)Q - 1).
\]

Now the Lemma follows from exponentiating. \(\Box\)

**Lemma 3.39.** Let \(H \subset G\) be any finite group scheme of order \(p\). Suppose \(g \in \mathcal{O}_G(G)^{x,N_f=\phi}\). Then we have
\[
\sum_{\gamma \in H} \log(g((\gamma + 1)Q - 1)) \equiv p \cdot \log g \pmod{p\mathcal{O}_{L_p}[[Q - 1]]}.
\]
\[
(48)
\]
Proof. We have

\[
\prod_{\gamma \in H} g((\gamma + 1)Q - 1)^{\#F_f/Q} \equiv N_f g \circ f = g^\phi \circ f \equiv g^{\#F_f} \pmod{\mathcal{O}_L[p][Q - 1]}
\]

(49)

\[
\implies \left( \prod_{\gamma \in H} g((\gamma + 1)Q - 1) - g^p \right)^{\#F_f/Q} \equiv 0 \pmod{\mathcal{O}_L[p][Q - 1]}
\]

If \( p \) is ramified in \( K \), then \( \#F_f = p \). Then (49) gives

\[
\prod_{\gamma \in H} g((\gamma + 1)Q - 1) - g^p \equiv 0 \pmod{\mathcal{O}_L[p][Q - 1]}
\]

(50)

Now suppose that \( p \) is inert in \( K \), so that \( \#F_f = p^2 \). Note that the ring \( \mathcal{O}_G(G) \otimes \mathcal{O}_L[p][Q - 1] \) has no zero divisors, and hence the congruence (49) again implies (50). Taking the logarithm of (50) immediately gives (48).

\[ \square \]

Definition 3.40. Let

\[
\mathcal{O}_G(G)^{\sim} = \mathcal{O}_L[p][Q - 1]^{\sim} := \{ h \in \mathcal{O}_G(G) : h = \tilde{h}' \text{ for some } h' \in \mathcal{O}_G(G_\eta) \}.
\]

Proposition 3.41. Let \( \pi \) be a uniformizer of \( \mathcal{O}_{K_p} \). We have

\[
\log : \mathcal{O}_G(G)^{N_f = \phi} \rightarrow \mathcal{O}_G(G)^{\sim}.
\]

Proof. In the \( p \) inert case, this follows immediately from the definition of \( \sim \) in (45) and dividing (48) by \( p \). When \( p \) is ramified, then dividing (45) by \( p \) shows that

\[
\log(g) \in p^{-1} \mathcal{O}_L[p][Q - 1].
\]

However, letting \( \partial = \frac{Qd}{dQ} \), we have

\[
\partial \log(g) = \left( \frac{Qd}{dQ} \right) \left( \log(g)(Q - 1) - \frac{1}{p} \log(g) \circ f(Q - 1) \right)
\]

(51)

\[
= Q \left( \frac{g'(Q - 1)}{g(Q - 1)} - \frac{f'(Q - 1)g(Q - 1)}{p} \right) \in \mathcal{O}_L[p][Q - 1].
\]

Here, the last inclusion follows from two observations. First, since \( g \in \mathcal{O}_G(G)^{\times} = \mathcal{O}_L[p][Q - 1]^{\times} \), we have \( g'/g \in \mathcal{O}_L[p][Q - 1] \). Second, since \( f(Q - 1) \equiv (Q - 1)^p \pmod{\mathcal{O}_L[p]} \) by (48), then (by differentiating) we have \( f'(Q - 1) \equiv 0 \pmod{\mathcal{O}_L[p]} \), and so \( \frac{f'(Q - 1)}{p} \in \mathcal{O}_L[p][Q - 1] \). Now from (51), we have

\[
\partial^j \log(g) \in \mathcal{O}_L[p][Q - 1]
\]

for all \( j \geq 1 \). But as one sees from (46), the map \( j \mapsto \partial^j \log(g) \) is a continuous map \( \mathbb{Z}_p^\times \rightarrow \mathcal{O}_L[p][Q - 1] \), and so taking a sequence \( \{ j_n \}_{n \geq 1} \) with \( j_n \rightarrow 0 \) in \( \mathbb{Z}_p^\times \), we obtain \( \log(g) \in \mathcal{O}_L[p][Q - 1] \), which is what we wanted.

\[ \square \]
Constructing the local measure. Given any $\beta \in U$, we seek to construct a $O_{L_p}$-valued measure on $G$ in the height 2 case using $\theta_{\alpha_0}$, and from this association get the desired map (10). From $\text{Col}(F_f, \alpha_{\infty})$ (see (9)) and the results of the previous section, we get a $\text{Gal}(\bar{L}_p/L_p)$-equivariant map

$$U \xrightarrow{\text{Col}(F_f, \alpha_{\infty})} O_{F_f[f^\infty]}^\times (F_f[f^\infty])^{N_f = \phi} \xrightarrow{\theta_{\alpha_0}} O_G^\times (G)^{N_f = \phi}.$$ 

Now postcomposing $\tilde{\log}$ with (52), we get a $G$-equivariant map

$$U \xrightarrow{\text{Col}(F_f, \alpha_{\infty})} O_{F_f[f^\infty]}^\times (F_f[f^\infty])^{N_f = \phi} \xrightarrow{\theta_{\alpha_0}} O_G^\times (G)^{N_f = \phi} \xrightarrow{\tilde{\log}} O_G(G)^{\sim},$$

where the last arrow uses Proposition 3.41.

Finally, we seek to find a natural map

$$O_G(G)^{\sim} \to \Lambda(G, O_{L_p}) := O_{L_p}[G],$$

which, postcomposed with (53), will give the desired map (10).

3.3.3. From power series in $Q - 1$ to measures on $O_{K_p}^\times$.

Definition 3.42. Recall that we let $p$ denote the unique prime of $O_K$ above $p$ (where, as before, we assume that $p$ is inert or ramified in $K$). Given any ideal $a \subset \mathbb{Z}_p$, let $\text{ord}_p(a) \in \mathbb{Q}$ be such that $a = p^{\text{ord}_p(a)}\mathbb{Z}_p$. Let

$$\epsilon = \left\lfloor \frac{1}{(p-1)\text{ord}_p(p)} \right\rfloor + 1 \in \mathbb{Z}_{\geq 1}$$

so that the $p$-adic logarithm induces an isomorphism

$$\log : 1 + p^\epsilon O_{K_p} \xrightarrow{\sim} p^\epsilon O_{K_p}.$$ 

Then (for example, by the structure theorem of finitely generated modules over PIDs) there is an isomorphism

$$O_{K_p}^\times \cong \Delta \times \Gamma$$

where $\Delta \subset O_{K_p}^\times$ is the torsion subgroup, and

$$\Gamma := 1 + p^\epsilon O_{K_p} \cong p^\epsilon O_{K_p}.$$ 

We recall the norm exact sequence

$$1 \to O_{K_p}^\times, \text{Nm}_{K_p/Q_p} = 1 \to O_{K_p}^\times \to \mathbb{Z}_p^\times$$

(with the last arrow a surjection if any only if $p$ is inert in $K$). Restricting this exact sequence to $\Gamma \subset O_{K_p}^\times$, we get

$$1 \to \Gamma, \text{Nm}_{K_p/Q_p} = 1 \to \Gamma \to 1 + p\mathbb{Z}_p.$$ 

We henceforth let

$$\Gamma_+ := \text{Nm}_{K_p/Q_p}(\Gamma) \subset 1 + p\mathbb{Z}_p.$$ 

Then we have an exact sequence

$$1 \to \Gamma, \text{Nm}_{K_p/Q_p} = 1 \to \Gamma \to \Gamma_+ \to 1.$$ 

Proposition 3.43. The exact sequence (57) splits, so that we get an isomorphism

$$\Gamma \cong \Gamma_+ \times \Gamma, \text{Nm}_{K_p/Q_p} = 1$$

where the projection onto the first factor is given by the norm $\text{Nm}_{K_p/Q_p}$.
Thus, since $\Gamma' \cong 1 + p\mathbb{Z}_p$, $\Gamma'$ is a free $\mathbb{Z}_p$-module. In particular, it is a projective $\mathbb{Z}_p$-module, and so the exact sequence (57) splits.

\[ \square \]

**Convention 3.44.** Henceforth, fix a splitting as in (58).

Using the typical Mahler basis \( \left( \frac{x}{n} \right) \) on $\mathbb{Z}_p$, for any $\mathbb{Z}_p$-algebra we get an identification

\begin{equation}
\Lambda(\mathbb{Z}_p, R) = R[\mathbb{Z}_p] \xrightarrow{\sim} R[T], \quad \mu \mapsto \sum_{n=0}^{\infty} \mu\left( \frac{x}{n} \right) T^n.
\end{equation}

**Proposition 3.45.** Let $Q = T + 1$. Under the identification (59), we have, in the notation of Definition 3.36,

\begin{equation}
\mathcal{O}_L[[Q - 1]] \cong \Lambda(\mathbb{Z}_p^\times, \mathcal{O}_L) \subset \Lambda(\mathbb{Z}_p, \mathcal{O}_L).
\end{equation}

**Proof.** This is standard, see for example the argument of [5, Lemma 8.3], or [45, Chapter I.1.3]. \( \square \)

**Definition 3.46.** We then extend (53) using the identification $\mathcal{O}_G(G) = \mathcal{O}_{L_p}[Q - 1]$ from (42):

\begin{equation}
\mu^0 : U \xrightarrow{\sim} \mathcal{O}_{F_f, f \in \mathcal{F}}(F_f[\mathbb{F}_p]) = \mathcal{O}_{G}(G) \xrightarrow{\log} \mathcal{O}_G(G) \xrightarrow{\sim} \mathcal{O}_{L_p}[Q - 1].
\end{equation}

We wish complete the above composition by finding a $\text{Gal}(\overline{\mathbb{K}}_p/K_p)$-equivariant map

\[ \mathcal{O}_{L_p}[Q - 1] \xrightarrow{\gamma} \Lambda(\mathcal{O}_p^\times, \mathcal{O}_{L_p}). \]

To this end, recall our fixed splitting in Convention 3.44 of the norm exact sequence (57) as in (58).

**Definition 3.47.** Henceforth, let

\[ \Gamma' := \Gamma^{\text{Nm}_{K_p/Q_p} = 1}. \]

**Definition 3.48.** Note that $\Gamma_+ \cong 1 + q\mathbb{Z}_p$, where $q = p$ if $p > 2$ and $q = 4$ if $p = 2$, and so $\Gamma_+ \cong \mathbb{Z}_p$. Thus, since $\Gamma \cong \mathcal{O}_{K_p}$ via the $p$-adic logarithm, \( \square \) shows that

\begin{equation}
\Gamma' \cong 1 + q\mathbb{Z}_p.
\end{equation}

Henceforth, fix such a trivialization. Let $\gamma'$ denote the topological generator of $\Gamma'$ corresponding to $1 + q$ under the above trivialization.

**Definition 3.49.** Henceforth, let $U_0 = \mathcal{O}_{K_p}^\times$ and $U_n = 1 + p^n\mathcal{O}_{K_p}$. Note that for $\delta \in (\mathcal{O}_{K_p}/p^n\mathcal{O}_{K_p})^\times$, the open subsets $\delta U_n \subset \mathcal{O}_{K_p}^\times$ form a basis of open subsets as $n \in \mathbb{Z}_{\geq 0}$ and $\delta \in (\mathcal{O}_{K_p}/p^n\mathcal{O}_{K_p})^\times$ vary. Given $\beta \in \mathcal{U}$, from the map (61) we have an associated $\mu^0(\beta)(Q - 1) \in \mathcal{O}_{L_p}[[Q - 1]]$. Recall the inverse of the reciprocity map $\kappa^{-1} : \mathcal{O}_{K_p}^\times \xrightarrow{\sim} \mathcal{G}$. We define

\begin{equation}
\mu_{\mathcal{O}_{K_p}^\times}(U_0) = \mu^0(\beta)(0).
\end{equation}

For any $n \in \mathbb{Z}_{\geq 1}$, we define

\begin{equation}
\mu_{\mathcal{O}_{K_p}^\times}(U_n) := \frac{1}{p^n} \sum_{j=0}^{p^n-1} \mu^0(\beta)(\zeta_{p^n}^j - 1)\zeta_{p^n}^{-j}.
\end{equation}

Furthermore, for any $\delta \in (\mathcal{O}_{K_p}/p^n\mathcal{O}_{K_p})^\times$, letting $\tilde{\delta} \in \mathcal{O}_{K_p}^\times$ be any lift, we define

\begin{equation}
\mu_{\mathcal{O}_{K_p}^\times}(\delta U_n) := \mu_{\mathcal{O}_{K_p}^\times}(\kappa^{-1}(\tilde{\delta}^{-1})\beta)((U_n)).
\end{equation}
In particular, we get a measure \( \mu \), seen as follows. We have, by (64),

\[
\mu_{\mathcal{O}^\times_{K_p}}(\kappa^{-1}(\tilde{\delta}^{-1})(\beta))(U_n) = \frac{1}{p^{2n}} \sum_{j=0}^{p^n-1} \mu^0(\kappa^{-1}(\tilde{\delta}^{-1})(\beta))(\zeta_{p^n}^j - 1)\zeta_{p^n}^{-j}
\]

However, since \( \mu_{p^n} \otimes \mathbb{Z}_p \mathcal{O}_{K_p}^{-1} = \mu_{p^n} \otimes \mathbb{Z}_p (\mathcal{O}_{K_p}/p^n\mathcal{O}_{K_p})\mathcal{O}_{K_p}^{-1} \), we see that \( [\kappa^{-1}(\tilde{\delta}^{-1})]|_G(\zeta_{p^n}^j - 1) \) only depends on \( \tilde{\delta} \pmod{p^n\mathcal{O}_{K_p}} \). Hence the above displayed expression only depends on \( \delta \), and hence (65) is well-defined. Writing \( U_n = \bigcup_{\delta \in \mathbb{U}_{n+1}} \delta U_{n+1} \), since \( \mu^0(\beta)(Q - 1) \in \mathcal{O}_{L_p}[Q - 1] \) and for any \( m \in \mathbb{Z}_{\geq 0} \),

\[
\frac{1}{p^2} \sum_{\delta \in \mathbb{U}_{n+1}} [\kappa^{-1}(\tilde{\delta})(\beta)]_G(\zeta_{p^n}^m) = \begin{cases} 0 & (m,p) = 1 \\ p|m & \end{cases},
\]

we have

\[
\frac{1}{p^2} \sum_{\delta \in \mathbb{U}_{n+1}} \mu^0(\kappa^{-1}(\tilde{\delta})(\beta))(\zeta_{p^n}^j - 1) = \begin{cases} 0 & (j,p) = 1 \\ \mu^0(\beta)(\zeta_{p^n}^j - 1) & \end{cases},
\]

and hence

\[
\sum_{\delta \in \mathbb{U}_{n+1}} \mu_{\mathcal{O}^\times_{K_p}}(\beta)(\delta U_{n+1}) = \mu_{\mathcal{O}^\times_{K_p}}(\beta)(U_n).
\]

In particular, we get a measure \( \mu_{\mathcal{O}^\times_{K_p}}(\beta) \in \Lambda(\mathcal{O}^\times_{K_p}, \mathcal{O}_{L_p}) \), which defines a map

\[(66) \quad \mu_{\mathcal{O}^\times_{K_p}} : \mathcal{U} \to \Lambda(\mathcal{O}^\times_{K_p}, \mathcal{O}_{L_p}).\]

**Proof of Theorem 3.2.** Using \( \kappa : \mathcal{G} \xrightarrow{\sim} \mathcal{O}^\times_{K_p} \), get an identification

\[(67) \quad \kappa^* : \Lambda(\mathcal{O}^\times_{K_p}, \mathcal{O}_{L_p}) \xrightarrow{\sim} \Lambda(\mathcal{G}, \mathcal{O}_{L_p}).\]

Now we let

\[(68) \quad \mu := \kappa^* \mu_{\mathcal{O}^\times_{K_p}} : \mathcal{U} \to \Lambda(\mathcal{G}, \mathcal{O}_{L_p})\]

which finally gives the desired map (10). \( \square \)

### 3.4. Moments of the local measure.

Given \( \beta \in \mathcal{U} \), the map (10) produces a measure

\( \mu_\beta := \mu(\beta) \in \Lambda(\mathcal{G}, \mathcal{O}_{L_p}) \).

From (61), (64) and (65), we see that \( \mu_\beta \) arises from the power series \( \mu^0(\beta)(Q - 1) \). The main result of this section is:

**Proposition 3.50.** For any \( k \in \mathbb{Z}_{\geq 0} \), we have

\[(69) \quad \mu_\beta(\kappa^k) = \partial^k \mu^0(\beta)(0)\]

where

\[\partial = \Omega_p \frac{d}{\omega_0}\]

and \( \omega_0 \) is the invariant differential fixed in Definition 3.7 (and which was used in the construction of \( \mu_\beta \)).
Proof. We first show that:

**Lemma 3.51.**

\[ \mu_\beta(\kappa^k) = \left( \frac{Qd}{dQ} \right)^k \mu^0(\beta)(0). \]

**Proof.** Recall that \([\cdot]_G\) denotes the \(O_{K_p}\)-module structure on \(G\). Notice that by (63), \(\mu^0(\beta)(Q - 1)\) can be recovered from \(\mu_\beta\) via the formula

\[ \mu^0(\beta)(Q - 1) = \mu_\beta([\kappa]_G(Q)). \]

Since \(\left( \frac{Qd}{dQ} \right)^k (Q^x) = Q^{x+k}\), we have \(\left( \frac{Qd}{dQ} \right)^k ([\kappa]_G(Q)) = [\kappa]_G(Q)^{k^k}\), and so

\[ \left( \frac{Qd}{dQ} \right)^k \mu_\beta([\kappa]_G(Q)) = \mu_\beta\left( \left( \frac{Qd}{dQ} \right)^k [\kappa]_G(Q) \right) = \mu_\beta([\kappa]_G(Q)^{k^k}), \]

which gives the assertion. □

Now by the previous discussion, we have completed the proof of the Proposition. In particular, we have

\[ \theta^*_{\alpha_0} Qd/dQ = \Omega_p d/\omega_0 \]

by (35).

□

3.5. Extending \(\mu\).

**Definition 3.52.** Recall that \(\text{Gal}(L_p/K_p) = \langle \phi \rangle\), where \(\phi^d = 1\). Let

\[ \tilde{G} := \text{Gal}(L_{p,\infty}/K_p) = \tilde{G} \times \langle \phi \rangle. \]

The above direct product decomposition exists by ramification theory (note that \(L_{p,\infty}/L_p\) is totally ramified, whereas \(L_p/K_p\) is unramified), and the fact that \(\text{Gal}(L_{p,\infty}/K_p)\) is abelian.

Note that there is a natural action of \(\tilde{G}\) on \(U\). We can thus extend the map (10) to a \(\tilde{G}\)-equivariant map

\[ \tilde{\mu} : U \to \Lambda(\tilde{G}, O_{L_p}) \]

following the procedure in [45, Chapter I.3.4].

**Definition 3.53.** Henceforth, for \(\beta \in U\), we often will adopt the notation

\[ \mu_\beta := \mu(\beta) \in \Lambda(G, O_{L_p}). \]

We adopt analogous notation for all restrictions and extensions of \(\mu\) (such as \(\tilde{\mu}\) below).

We first have the following lemma summarizing nice properties of \(\mu\).

**Lemma 3.54 (cf. Lemma I.3.4 of [45]).** We have

1. \(\mu_{\beta_1 + \beta_2} = \mu_{\beta_1} + \mu_{\beta_2}\),
2. for any \(\gamma \in \tilde{G}\), we have \(\mu_{\gamma(\beta)}(\gamma(U)) = \mu_\beta(U)\),
3. \(\mu_\beta\) depends only on the choice of \(\alpha_\infty : O_{K_p} \sim \to T_f F\),
4. if \(\alpha_\infty' = [\kappa(\sigma)]_f \alpha_\infty\) for \(\sigma \in \tilde{G}\), then the resulting measure is given by \(\mu'_{\beta}(U) = \mu_\beta(\sigma U)\).

**Proof.** This follows from standard properties of Coleman power series, see [45, Corollary I.2.3] and [8]. Note that (2), (3) and (4) follow immediately from [8] and our definitions (63), (64) and (65).

□
Definition 3.55. Define a map
\[ \tilde{\mu} : U \to \Lambda(\tilde{G}, O_{L_p}) \]
as follows. Suppose \( U \) is an open subset of \( \tilde{G} \) contained in the coset \( \gamma \tilde{G} \subset \tilde{G} \). Then we define
\[ (70) \quad \tilde{\mu}(\beta)(U) := \mu(\gamma^{-1}(\beta))(\gamma^{-1}U). \]
This is well-defined by Lemma 3.54.

3.6. The kernel of \( \tilde{\mu} \). In this section we study the kernel of \( \tilde{\mu} \), or more specifically, that of a slight variant of \( \mu \). We first study a slight modification of \( \mu \) which is a \( \Lambda(\tilde{G}, O_{L_p}) \)-homomorphism.

Definition 3.56. Henceforth, let
\[ \Lambda := \Lambda(\tilde{G}, \mathbb{Z}_p) = \mathbb{Z}_p[[\tilde{G}]], \quad \Lambda_{O_{L_p}} := \Lambda(\tilde{G}, O_{L_p}) = O_{L_p}[[\tilde{G}]]. \]

Definition 3.57. As in Section 2.1, let \( p_n \) denote the prime ideal of \( O_{L_p,n} \), and let
\[ U^1 := \lim_{\substack{\text{Nm} \to \infty}} \left( 1 + p_n O_{L_p,n} \right) \]
denote the tower of local principal units. The Coleman map (2.3) restricts to a \( \mathcal{G} \)-equivariant isomorphism
\[ \text{Col}(F_l, \alpha_{\infty}) : U^1 \xrightarrow{\sim} O_{F_l[f^\infty]}^{\times,1}(F_l[f^\infty])^{N_f = \phi} := \varepsilon^{-1}(1) \]
where, choosing an identification \( O_{F_l[f^\infty]}(F_l[f^\infty]) = O_{L_p}[X] \) as in (31), we have
\[ O_{F_l[f^\infty]}^{\times,1}(F_l[f^\infty]) = 1 + X O_{L_p}[X]. \]

Definition 3.58. Note that we get an induced map
\[ \mu^1 : U^1 \to \Lambda_{O_{L_p}} \]
by restricting \( \tilde{\mu} \) to \( U^1 \), which, since \( \mu^1 \) is \( \mathbb{Z}_p \)-linear and \( \mathcal{G} \)-equivariant, is a \( \Lambda \)-linear homomorphism. We can extend \( \otimes_{\mathbb{Z}_p} O_{L_p} \)-linearly to get a \( \Lambda_{O_{L_p}} \)-linear map
\[ \mu_{O_{L_p}}^1 : U^1 \otimes_{\mathbb{Z}_p} O_{L_p} \to \Lambda_{O_{L_p}}. \]

Proposition 3.59. The map \( \mu^1 : U^1 \to \Lambda_{O_{L_p}} \) is injective.

Proof. Recall that the kernel of the \( p \)-adic logarithm \( \log : 1 + p_\infty O_{L_p,\infty} \to L_{p,\infty} \) consists of the torsion part of \( 1 + p_\infty O_{L_p,\infty} \), which is simply the group of \( p \)-th power roots of unity. By the defining property of Coleman power series (6), we have that \( \log g_\beta = 0 \) if and only if \( \log g_\beta \) vanishes on \( \{\alpha_{dn}\}_{n \in \mathbb{Z}_{\geq 0}} \). Since \( g_\beta(\alpha_{nd}) = \beta_{nd} \), this happens if any only if \( \beta = (\zeta_p^n)_n \) is a tower of \( p \)-th roots of unity. By Lubin-Tate theory, \( U \)-contains non-trivial such \( \beta \) if and only if \( (F_l, O_{L_p}) = (\tilde{G}_m, \mathbb{Z}_p) \). So in all,
\[ \ker(\beta \mapsto \log g_\beta) = \begin{cases} \mathbb{Z}_p(1) & (F_l, O_{L_p}) = (\tilde{G}_m, \mathbb{Z}_p) \\ 0 & \text{else} \end{cases} \]

Now we examine the kernel of \( \log g_\beta \mapsto \tilde{\log} g_\beta \), in our situation when \( F_l \) has height 2. By (45), we have
\[ (71) \quad \tilde{\log} g_\beta = \log g_\beta - \frac{1}{\# F_l[f]} \sum_{\varpi \in F_l[f]} \log g_\beta(X[]+f\varpi). \]
From (71) and the defining property of Coleman power series (6), we have \( \tilde{\log} g_\beta = 0 \) if any only if
\[ \rho^{# F_l[f]} = \text{Nm}_{L_{p,n}/L_{p,n-1}}(\beta_n) \]
for all \( n \in \mathbb{Z}_{\geq 0} \). Hence the minimal polynomial of \( \beta_n \) over \( L_{p,n-1} \) is either

\[
x \# F_{\beta} [f] + (1)^{\# F_{\beta} [f]} \beta_n^{\# F_{\beta} [f]}, \quad \text{or} \quad x - \beta_n.
\]

This in turn implies that the \( \text{Gal}(L_{p,n}/L_{p,n-1}) \)-conjugates of \( \beta_n \) are \( \{ \gamma \# F_{\beta} [f] \beta_n \} \) for \( 0 \leq j \leq \# F_{\beta} [f] - 1 \), or \( \beta_n \in 1 + p_{n-1} \mathcal{O}_{L_{p,n-1}} \). Since this holds for all \( n \in \mathbb{Z}_{\geq 0} \), we have that

\[
L_{p,\infty} = L_p(\mu_{p^\infty}, (\beta)^{1/p^\infty})
\]

for some \( \beta \in \{ \beta_n \}_{n\in\mathbb{Z}_{\geq 0}} \). If \( \beta \in \mu_{p^\infty} \), then \((72)\) implies \( \text{Gal}(L_{p,\infty}/L_p) = \text{Gal}(L_p(\mu_{p^\infty})/L_p) \), which is isomorphic to a subgroup of \( \mathbb{Z}_p^\times \) via the cyclotomic character. However, since \( \text{Gal}(L_{p,\infty}/L_p) \sim \mathbb{O}_{K_p}^\times \), this implies \( \mathbb{O}_{K_p}^\times \) is isomorphic to a subgroup of \( \mathbb{Z}_p^\times \), a contradiction (for example, since \( 1 + p^2 \mathcal{O}_{K_p} \) has \( \mathbb{Z}_p \)-rank \( 2 \)). Hence \( \beta \notin \mu_{p^\infty} \), and \((72)\) implies that \( L_{p,\infty}/L_p \) is a Kummer extension, and we have an exact sequence

\[
0 \to \text{Gal}(L_p(\mu_{p^\infty}, (\beta)^{1/p^\infty})/L_p(\mu_{p^\infty})) \to \text{Gal}(L_{p,\infty}/L_p) \sim \mathbb{O}_{K_p}^\times \to \text{Gal}(L_p(\mu_{p^\infty})/L_p) \to 0.
\]

Again, \( \text{Gal}(L_p(\mu_{p^\infty})/L_p) \) is isomorphic to a subgroup of \( \mathbb{Z}_p^\times \), and by Kummer theory we have that \( \text{Gal}(L_p(\mu_{p^\infty}, (\beta)^{1/p^\infty})/L_p(\mu_{p^\infty})) \cong \mathbb{Z}_p \). Hence \((73)\) implies that we have an exact sequence

\[
0 \to \mathbb{Z}_p \to \mathbb{O}_{K_p}^\times \to \mathbb{Z}_p^\times.
\]

But now we have \( \mathbb{O}_{K_p}^\times \sim \mu(\mathcal{O}_{K_p}) \times (1 + p^\infty \mathcal{O}_{K_p}) \cong \mu(\mathcal{O}_{K_p}) \times \mathcal{O}_{K_p} \), where \( \mu(\mathcal{O}_{K_p}) \) strictly contains \( \mu(\mathbb{Z}_p) = \mu_{p-1} \). (Recall that \( \mu(R) \) denotes the group of roots of unity belonging to a ring \( R \).) Since \( \mu(\mathcal{O}_{K_p})/\mu_{p-1} \) is in the kernel of the map \( \mathbb{O}_{K_p}^\times \to \mathbb{Z}_p^\times \) in \((74)\), we hence have an inclusion \( \mu(\mathcal{O}_{K_p})/\mu_{p-1} \subset \mathbb{Z}_p \), which is a contradiction since \( \mathbb{Z}_p \) has no non-trivial torsion subgroup.

Hence, in all we have

\[
\ker(\beta \mapsto \log \beta) = 0
\]

in our height 2 situation. From the construction of \( \mu \) as summarized in Definition \[68\] we that the rest of the maps composed to define \( \mu^1 \) are injective, and so the Proposition is proved.

\[\square\]

3.7. The kernel and cokernel of \( \mu^1_{\mathcal{O}_{L_p}} \). We wish to study the kernel and cokernel of \( \mu^1_{\mathcal{O}_{L_p}} \).

**Definition 3.60.** Define

\[
\mathcal{M} := \ker(\mu^1_{\mathcal{O}_{L_p}}), \quad \mathcal{W} := \text{coker}(\mu^1_{\mathcal{O}_{L_p}}).
\]

These are \( \Lambda \mathcal{O}_{L_p} \)-modules, which fit into a tautological exact sequence

\[
0 \to \mathcal{M} \to \mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \xrightarrow{\mu^1_{\mathcal{O}_{L_p}}} \Lambda \mathcal{O}_{L_p} \to \mathcal{W} \to 0.
\]

It is well-known (see \[40\] Lemma 5.2(ii)) that \( \mathcal{U}^1 \) has \( \Lambda \)-rank \( [\mathcal{O}_{L_p} : \mathbb{Z}_p] \), and hence \( \mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \) has \( \Lambda \mathcal{O}_{L_p} \)-rank \( [\mathcal{O}_{L_p} : \mathbb{Z}_p] \). By Proposition \[3.59\] the image of \( \mu^1_{\mathcal{O}_{L_p}} \) has \( \Lambda \mathcal{O}_{L_p} \)-rank 1, and so \( \mathcal{M} \) has \( \Lambda \mathcal{O}_{L_p} \)-rank \( [\mathcal{O}_{L_p} : \mathbb{Z}_p] \) and \( \mathcal{W} \) must be \( \Lambda \mathcal{O}_{L_p} \)-torsion.
3.7.1. **The kernel $\mathcal{M}$.** We now study $\mathcal{M}$ in closer detail.

**Proposition 3.61.** There is a free $\Lambda$-submodule $U' \subset U^1$ of rank $r := [\mathcal{O}_{L_p} : \mathbb{Z}_p]$ such that under the inclusion

$$U' \subset U^1 \overset{\mu^1}{\hookrightarrow} \Lambda_{\mathcal{O}_{L_p}},$$

$U'$ is a principal ideal.

**Proof.** Recall that in our situation (when $F_L$ has height 2), $U^1$ has $\Lambda$-rank $r := [\mathcal{O}_{L_p} : \mathbb{Z}_p]$, so let $\beta_1, \ldots, \beta_r$ be a $\Lambda$-basis of a free $\Lambda$-submodule of $U^1$ of rank $r$. Then $\gamma_1 = \mu^1(\beta_1), \ldots, \gamma_r = \mu^1(\beta_r)$ is a $\Lambda$-basis of a free $\Lambda$-submodule of $\mu^1(U^1)$. Let $e_1, \ldots, e_r$ be a $\Lambda$-basis of $\Lambda_{\mathcal{O}_{L_p}}$, so that we have a matrix equation

$$\begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1r} \\ \vdots & \ddots & \vdots \\ \lambda_{r1} & \cdots & \lambda_{rr} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_r \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix}.$$

Multiplying on the left by $\Lambda$-linear elementary matrices, we can reduce the matrix on the left-hand side to row echelon form and arrive at an equation

$$\begin{pmatrix} \lambda'_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda'_{rr} \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_r \end{pmatrix} = \begin{pmatrix} \gamma'_1 \\ \vdots \\ \gamma'_r \end{pmatrix}$$

for some $\lambda'_{11}, \ldots, \lambda'_{rr} \in \Lambda$ and $\gamma'_1, \ldots, \gamma'_r \in \mu^1(U^1)$. (Recall that $\mu^1$ is $\Lambda$-linear.) Now note that by (76), we have

$$\lambda'_{11} \cdots \lambda'_{rr} \Lambda_{\mathcal{O}_{L_p}} \subset \mu^1(U^1),$$

and $\lambda'_{11} \cdots \lambda'_{rr} \Lambda_{\mathcal{O}_{L_p}}$ is clearly a principal ideal of $\Lambda_{\mathcal{O}_{L_p}}$. We now let

$$U' := (\mu^1)^{-1} \left( \lambda'_{11} \cdots \lambda'_{rr} \Lambda_{\mathcal{O}_{L_p}} \right) \subset U^1$$

which is the desired $U'$.

\[ \square \]

**Convention 3.62.** Suppose $M$ is a $\Lambda_{\mathcal{O}_{L_p}}$-module. Then $M \otimes_{\mathbb{Z}_p} \Lambda_{\mathcal{O}_{L_p}}$ is a $\Lambda_{\mathcal{O}_{L_p}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$-module. Note that there is an embedding $\Lambda_{\mathcal{O}_{L_p}} \cong \Lambda_{\mathcal{O}_{L_p}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \subset \Lambda_{\mathcal{O}_{L_p}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$, which gives another $\Lambda_{\mathcal{O}_{L_p}}$-action on $M \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$. In this situation, we will denote the action of $\lambda \in \Lambda_{\mathcal{O}_{L_p}}$ on $m \in M \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$ under this latter action by $\lambda x$.

**Definition 3.63.** Let $U' \subset U^1$ be as in Proposition 3.61. Then $U'$ is a free $\Lambda$-submodule of $U^1$ with a natural $\Lambda_{\mathcal{O}_{L_p}}$-action induced by the inclusion $U' \subset U^1 \overset{\mu^1}{\hookrightarrow} \Lambda_{\mathcal{O}_{L_p}}$. We then have a $\Lambda_{\mathcal{O}_{L_p}}$-eigendecomposition

$$U' \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} = \bigoplus_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p)} U'_\sigma,$$

where each $U'_\sigma$ is a free rank-1 $\Lambda_{\mathcal{O}_{L_p}}$-module where the $\Lambda_{\mathcal{O}_{L_p}}$-action $\cdot$ on $x \in U'_\sigma$ is given by

$$\lambda \cdot x = x \sigma(\lambda)$$

(in the notation of Convention 3.62).

Note that $U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$ is a $\Lambda_{\mathcal{O}_{L_p}}$-module, and $U' \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \subset U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$ is an inclusion of $\Lambda_{\mathcal{O}_{L_p}}$-submodules. We let

$$U_\sigma := \text{Sat}_{U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}}(U'_\sigma) := \left\{ x \in U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} : \text{there is } \lambda \in \Lambda_{\mathcal{O}_{L_p}} \text{ such that } \lambda \cdot x \in U'_\sigma \right\}.$$
Since $U'$ is a free $\Lambda$-module of rank $r$, for any $x \in U$ there is a $\lambda \in \Lambda$ with $\lambda \cdot x \in U'$. Hence (77) and (78) imply

$$U \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} = \sum_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p)} U_{\sigma}. \quad (79)$$

**Proposition 3.64.** We have

$$\mathcal{M} \subset \sum_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p), \sigma \neq 1} U_{\sigma}. \quad (80)$$

In particular, $\sum_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p), \sigma \neq 1} U_{\sigma}$ has $\Lambda_{\mathcal{O}_{L_p}}$-rank $r - 1$.

**Proof.** First, we show that

$$\mathcal{M} \cap (U' \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}) = \bigoplus_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p), \sigma \neq 1} U'_{\sigma}. \quad (81)$$

By (77), it suffices to show that $\mu_{\mathcal{O}_{L_p}}(U'_{\sigma}) = 0$ for any $\sigma \in \text{Gal}(L_p/\mathbb{Q}_p), \sigma \neq 1$. However, for such $\sigma$ and for any $x \in U'_{\sigma}$, we have for any $\lambda \in \Lambda_{\mathcal{O}_{L_p}},$

$$\lambda \mu_{\mathcal{O}_{L_p}}^1(x) = \mu_{\mathcal{O}_{L_p}}(\lambda \cdot x) = \mu_{\mathcal{O}_{L_p}}^1(x\sigma(\lambda)) = \sigma(\lambda)\mu_{\mathcal{O}_{L_p}}^1(x)$$

which cannot be true for $\lambda \notin \Lambda$.

Now we prove (80). Suppose $x \in \mathcal{M}$, i.e. $\mu_{\mathcal{O}_{L_p}}^1(x) = 0$. Then $\lambda \cdot x \in U'$ for some $\lambda \in \Lambda_{\mathcal{O}_{L_p}}$. Hence by (81) we have $x \in \sum_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p), \sigma \neq 1} U_{\sigma}$, which by the definition (78) implies that $x \in \sum_{\sigma \in \text{Gal}(L_p/\mathbb{Q}_p), \sigma \neq 1} U_{\sigma}.

\[\square\]

### 3.7.2. The Cokernel $W$

We now turn our attention to the cokernel $W$ of $\mu_{\mathcal{O}_{L_p}}^1$, and show that is a pseudo-null $\Lambda_{\mathcal{O}_{L_p}}$-module, isomorphic to $\mathcal{O}_{L_p}(\kappa)$.

Recall that our relative Lubin-Tate group $F_f$ for $L_p/K_p$ has

$$f(X) = \pi'X + \ldots \in \mathcal{O}_{L_p}[X]$$

with $\text{Nm}_{L_p/K_p}(\pi') = \xi \in \mathcal{O}_{K_p}$ where $v(\xi) = d = [L_p : K_p]$ (where $v : K_p^\times \to \mathbb{Z}_p$ is the normalized valuation, i.e. with $v(\pi) = 1$).

**Definition 3.65.** Fix an identification $\mathcal{O}_{F_f[f^\infty]}(F_f[f^\infty]) \cong \mathcal{O}_{L_p}[X]$ as in (31). By (42), we get an identification $\mathcal{O}_{F_f[f^\infty]}(F_f[f^\infty]) \cong \mathcal{O}_{L_p}[Q - 1]$, where

$$Q - 1 = (\theta^{-1})^*X \in \mathcal{O}_{\Gamma}(\mathcal{G}).$$

Let $0 \leq N \leq \infty$ be maximal such that

$$\text{Nm}_{K_p/\mathbb{Q}_p}(\xi)/\#F_f[f]^d \equiv 1 \pmod{p^N}. \quad (82)$$

Let $\mathbb{N} : \tilde{\mathcal{G}} \to (\mathbb{Z}/p^N\mathbb{Z})^\times$ be the character giving the action of $\tilde{\mathcal{G}}$ on $F_f[p^n]$. Note that for $\gamma \in \mathcal{G}$, we have

$$\mathbb{N}(\gamma) \equiv \kappa(\gamma) \pmod{p^N}. \quad (83)$$

As in [45, Chapter I.3.7], we define a homomorphism

$$j : \Lambda_{\mathcal{O}_{L_p}} \to (\mathcal{O}_{L_p}/p^N)(\kappa), \quad j(\mu) : = \mu(\mathbb{N}) = \int_{\tilde{\mathcal{G}}} \kappa d\mu \pmod{p^N}.$$

This is a surjective homomorphism of $\Lambda_{\mathcal{O}_{L_p}}$-modules.
Remark 3.66. Suppose that $F_f$ is self-dual in the sense of Definition 3.19. Then since $\mu_p \subset L_{p,\infty}$ (by the existence of the Weil pairing), it follows from local class field theory that $N = \infty$.

Theorem 3.67 (cf. [45], Chapter I.3.7). Assume that $F_f$ is self-dual (see Definition 3.19) so that $N = \infty$. Then we have an exact sequence of $\Lambda_{O_{L_p}}$-modules

\begin{equation}
0 \to M \to U^1 \otimes_{Z_p} O_{L_p} \xrightarrow{\mu_{O_{L_p}}^1} \Lambda_{O_{L_p}} j \to O_{L_p}(\kappa) \to 0.
\end{equation}

Proof. The first three arrows of (84) follow from the definitions of $M$ (Definition 3.60) and $\mu_{O_{L_p}}^1$ (Definition 3.58). We first show that $\mu_{O_{L_p}}^1(U^1 \otimes_{Z_p} O_{L_p}) \subset \ker(j)$. Recall that $d = \left| L_p : K_p \right|$, and the derivation $\partial = \frac{Q d}{dQ} = \Omega_p \frac{d}{d\omega_0}$ (using the identification (42) and (35)). By the construction of $\mu$ we have

\begin{equation}
\mu_{O_{L_p}}^1(\beta)(N) \mu_{O_{L_p}}^1(\beta)(\kappa) \sum_{i=0}^{d-1} \phi^i(1 - \phi) \left( \partial \mu_{O_{L_p}}^1(\beta)(0) \right) = (1 - \phi^d)^d \partial \mu_{O_{L_p}}^1(\beta)(0) = 0
\end{equation}

where the last equality follows because $\partial \mu_{O_{L_p}}^1(\beta)(0) \in O_{L_p}$ (since $\mu_{O_{L_p}}^1(Q - 1) \in O_{L_p}[Q - 1]$), and so $\phi^d$ acts trivially on it.

To show that $\ker(j) = \text{Im}(\mu_{O_{L_p}}^1)$, one follows the same argument as in the proof of Theorem 3.7, pp. 22-27 of loc. cit. One uses the power series representation $\mu_{O_{L_p}}^1(B) = \mu_{O_{L_p}}^1(\beta)(Q - 1) \in O_{L_p}[Q - 1]$ along with (46), and the necessary changes to Lemmas 3.10-3.13 of loc. cit.

\[\square\]

4. The Main Conjectures

In this section, we formulate and then prove Rubin-type Main Conjectures for imaginary quadratic fields in which $p$ is inert or ramified. The key is to use the kernel of $\mu_{O_{L_p}}^1$ as a $\Lambda$-adic local condition to define the relevant torsion $\Lambda_{O_{L_p}}$-module that will appear in the Main Conjecture (Conjecture 4.32).

4.1. Construction of the measure on the global Galois group.

Definition 4.1. Let $f \subset O_K$ be an integral ideal prime to $p$. Suppose that $w_i = 1$. Let $L := K(f)$, the ray class field of conductor $f$ over $K$. Henceforth fix an elliptic curve $A$ such that

1. $A$ has CM by $O_K$,
2. $A$ is defined over $K(f)$,
3. $A^{\text{tors}} \subset A(K^{\text{ab}})$,
4. $A$ has good reduction at all primes of $L$ not dividing $f$.

As remarked in [34, p. 4], the existence of such an elliptic curve is proven in [46, p. 216] and [45, Chapter II, Lemma 1.4]. Let $O_{K,\mathfrak{p}}$ denote the localization of $O_K$ at $\mathfrak{p}$ (i.e., inverting all elements outside of $\mathfrak{p}$). Then let $R$ be the integral closure of $O_{K,\mathfrak{p}}$ in $L$. We then fix a minimal generalized Weierstrass model of $E$ over $R$,

\[y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.\]

We let

\[\omega_A = \frac{dx}{2y + a_1 x + a_3} \in \Omega^1_{A/R}\]

be the usual holomorphic invariant differential attached to the above Weierstrass model. The pair $(A, \omega_A)$ determines a unique $O_K$-lattice with

\[\theta_{\infty, A} : \mathbb{C}/L \xrightarrow{\sim} A(\mathbb{C}).\]
Here, for all \( z \in \mathbb{C} \setminus L \), \( \theta_{\infty,A}(z) \) is the unique point with coordinates
\[
x(z) := \varphi_L(z) - b_2/12, \quad y(z) := (\varphi'(z) - a_1x(z) - a_3)/2
\]
where \( b_2 := a_1^2 + 4a_2 \). Let \( \hat{\Lambda} \) be the formal group of \( A \), with parameter \( t := -x/y \). We note that the function field of \( E/\mathbb{Q} \) is \( \mathbb{Q}(\varphi_L, \varphi'_L) \), and we denote the natural map \( \mathbb{Q}(\varphi_L, \varphi'_L) \to \mathbb{C}_p[t] \) (taken with respect to the embedding \( i_p \), fixed in (3)) obtained by expanding a rational function on \( E \) in terms of the formal parameter \( t \) by
\[
f \mapsto \hat{f}.
\]

**Definition 4.2.** Retain the notation of Definition 4.1. Fix an ideal \( \mathfrak{f} \subset \mathcal{O}_K \); we do not impose \( \mathfrak{f} \neq 1 \) or \( (\mathfrak{f}, p) = 1 \) unless otherwise specified. Given any ideal \( \mathfrak{c} \subset \mathcal{O}_K \), let \( K(\mathfrak{c}) \) be the ray class field over \( K \) of conductor \( \mathfrak{c} \). Henceforth let \( L = K(\mathfrak{f}) \), and for \( 0 \leq n \leq \infty \) let \( L_n = K(\mathfrak{f}p^n) \). As before, let \( A/L \) be an elliptic curve with complex multiplication by \( \mathcal{O}_K \), and such that \( L(A_{\text{tors}})/K \) is abelian. Equivalently, we have
\[
\psi_{A/L} = \phi \circ \text{Nm}_{L/K}
\]
where \( \psi_{A/L} : A_L^\times \to \mathbb{C}^\times \) is the Hecke character associated with \( A/L \) by the theory of complex multiplication and \( \phi : A_K^\times \to \mathbb{C}^\times \) is some Hecke character of infinity type \((1,0)\).

**Definition 4.3.** Given any algebraic Hecke character \( \chi : A_K^\times \to \mathbb{C}^\times \), we let \( \hat{\chi} : A_K^\times \to \mathbb{C}_p^\times \) denote its \( p \)-adic avatar. Given any \( p \)-adic algebraic Hecke character \( \chi : A_K^\times \to \mathbb{C}_p^\times \), we let \( \hat{\chi} \) denote its complex avatar.

**Definition 4.4.** Let
\[
G_n := \text{Gal}(L_n/K), \quad G_n^+ := \text{Gal}(L(\mu_p^n)/K), \quad G_\infty := \text{Gal}(L_\infty/K), \quad G_\infty^+ := \text{Gal}(L(\mu_p^\infty)/K).
\]
Let \( \hat{G}_\infty := \{ \chi : G_\infty \to \mathbb{C}_p^\times \} \) denote the group of (continuous) \( p \)-adic characters on \( G_\infty \), and similarly for the other above groups. When we wish to emphasize the dependence on the auxiliary conductor \( \mathfrak{f} \), we will write \( G_\infty(\mathfrak{f}) \), and similarly for other groups. Let
\[
\Phi_n := L_n \otimes_K K_\mathfrak{p} \cong \bigoplus_{\mathfrak{p} \mid \mathfrak{p}} L_{n,\mathfrak{p}}, \quad R_n = \mathcal{O}_{L_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}} \cong \bigoplus_{\mathfrak{p} \mid \mathfrak{p}} \mathcal{O}_{L_n,\mathfrak{p}}
\]
where \( \mathfrak{p} \mid \mathfrak{p} \) runs over prime ideals of \( \mathcal{O}_{L_n} \) above \( \mathfrak{p} \). (Recall that \( K_\mathfrak{p} = K_{\mathfrak{p}} \) with respect to our fixed embeddings (3).) For simplicity let \( \Phi_0 = \Phi \). Note that we have norm maps \( \text{Nm}_n : \Phi_n \to \Phi_{n-1} \) and \( \text{Nm}_n : R_n \to R_{n-1} \). Let
\[
\mathcal{U} := \lim_{\rightarrow} R_n^\times.
\]

**Definition 4.5.** In the notation of (86), let \( R_n^1 \) denote the pro-\( p \) part of \( R_n^\times \), i.e. the semi-local principal units. Then let
\[
\mathcal{U}^1 := \lim_{\rightarrow} R_n^1.
\]

We wish to construct a semilocal version of the map \( \mu : \mathcal{U} \to \Lambda(G, \mathcal{O}_{L_p}) \) in (10).

**Theorem 4.6.** Assume \((\mathfrak{f}, p) = 1\). There is a \( G_\infty \)-equivariant map
\[
\mu_{\text{glob}} : \mathcal{U}^1 \to \Lambda(G_\infty, \mathcal{O}_{L_p}).
\]

Similarly, there is a \( G_\infty^+ \)-equivariant map
\[
\mu_{\text{glob}}^+ : \mathcal{U}^1 \to \Lambda(G_\infty^+, \mathcal{O}_{L_p}).
\]
Proof. This follows from Theorem 3.2, applied to \( F_j = (\hat{A}, [p]) \) over \( L_p = L_{\wp} \) for any prime \( \wp | p \) of \( L \) (varying the embeddings \((\hat{A})\)), pulling back the resulting map \( \mu^1 \) via the natural projection (induced from \( i_p \) from (3)):

\[
i_p : U \to U,
\]

and taking the direct sum over \( \bigoplus_{\wp | p} \) of all the resulting maps. This gives (87). To get (88), take the image of (87) under the projection \( \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \to \Lambda(G_{\infty}^+, \mathcal{O}_{L_p}) \).

For \( w_j \neq 1 \), we have the following Proposition.

**Proposition 4.7** (cf. Proposition III.1.3 of [45]). There is a unique way to extend (87) to a map (89)

\[
\mu_{\text{glob}} = \mu_{\text{glob}, j} : U^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \Lambda(G_{\infty}, \mathcal{O}_{K(j)^\wp})[1/p]
\]

for any \( j \subset \mathcal{O}_K \) such that the following diagram is commutative

\[
\begin{array}{ccc}
U^1(f) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \xrightarrow{\mu_{\text{glob}, j}} & \Lambda(K(f^p \wp), \mathcal{O}_{K(j)^\wp})[1/p] \\
\downarrow \text{inclusion} & & \downarrow \text{cores} \\
U^1(g) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \xrightarrow{\mu_{\text{glob}, g}} & \Lambda(K(gp^\wp) / K, \mathcal{O}_{K(j)^\wp})[1/p] \\
\downarrow \text{Nm} & & \downarrow \text{res} \\
U^1(f) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \xrightarrow{\mu_{\text{glob}, f}} & \Lambda(K(f^p \wp) / K, \mathcal{O}_{K(j)^\wp})[1/p]
\end{array}
\]

for all \( j \| \wp \) with \((g, \wp) = 1 \) (see [45] Lemma III.1.2). If \( w_j = 1 \), then \( \mu_{\text{glob}, j} : U^1 \to \Lambda(G_{\infty}, \mathcal{O}_{K(j)^\wp}) \) and the diagram (90) holds without inverting \( p \).

Proof. One uses the same diagrams and argument as in [45] Lemma III.1.2 and Proposition III.1.3, except that the index of the image (i.e. \( w_j \)) of the norm map in the proof of Proposition 1.3 may be non-trivial, and hence it is necessary to invert \( p \) to make it surjective. In particular, \( \mu_{\text{glob}, j}(f) \) is explicitly defined by choosing \( g \subset \mathcal{O}_K \) with \( j \| g \) and \( g, \wp = 1 \) and \( w_g = 1 \) so that \( \mu_{\text{glob}, g} \) is already defined, and letting

\[
\mu_{\text{glob}, f} := \frac{1}{\# \text{Gal}(K(gp^\wp) / K(f^p \wp))} \text{res} \circ \mu_{\text{glob}, g} \circ \text{inclusion}.
\]

The fact that this map factors through \( U^1(f) \to \Lambda(K(f^p \wp), \mathcal{O}_{K(j)^\wp}) \) follows from the fact that \( \text{Gal}(K(g) / K(f)) \) acts trivially on \( U^1(f) \), (70) and (69).

**Definition 4.8.** Henceforth, let

\[
\mathbb{M} := \ker(\mu_{\text{glob}}),
\]

which is a finitely generated \( \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \)-module.

**Definition 4.9** (Elliptic functions). We recall the elliptic functions of Robert [37], following [34] Section 3. Suppose \( L \subset L' \) are two lattices in \( \mathbb{C} \) such that \((L' : L, 6) = 1 \), and let \( \varphi_L \) be the Weierstrass elliptic function attached to \( L \). Then the Robert elliptic function (37) is

\[
\psi(z; L, L') := \delta(L, L') \prod_{\rho \in \mathcal{Z}(L, L')} (\varphi_L(z) - \varphi_L(\rho))^{-1},
\]

where \( \delta(L, L') := \prod_{\rho \in \mathcal{Z}(L, L')} (\varphi_L(z) - \varphi_L(\rho)) \).
where $\delta(L, L')$ is the canonical $12^{th}$ root of $\Delta(L)[L':L]/\Delta(L')$ defined in [38]. As recalled in [34, p.3], these functions satisfy the usual distribution and norm compatibility relations:

$$\psi(z; L', a^{-1}L') = \prod_{\rho} \psi(z + \rho; L, a^{-1}L),$$

$$Nm_{K(m'/q)/K(m')}\left(\psi(\rho; L, a^{-1}L)\right)_{a|q} = \begin{cases} \psi(\rho; q^{-1}L, a^{-1}q^{-1}L) & q|m' \\ \psi(\rho; q^{-1}L, a^{-1}q^{-1}L)^{1-(q,m'/K)-1} & q \nmid m'. \end{cases}$$

Here, given an integral ideal $n \subset O_K$, $w_n$ denotes the number of roots of unity in $O_K^n$ which are congruent to 1 (mod $n$). Let $m \not\equiv O_K$ be a proper integral ideal with $(m, p) = 1$, and let $\Omega \in \mathcal{C}$ be a primitive $m$-torsion point of $\mathbb{C}/L$, i.e. with $\Omega = \Omega^{-1}L \cap O_K$. For any ideal $b$ with $(b, 6mfp) = 1$, we define

$$(91) \quad \psi_{\Omega, b}^L(z) := \psi(z + \Omega; L, b^{-1}L).$$

If we further fix an integral ideal $a$ such that $(a, 6mfp) = 1$ and $(b, 6amfp) = 1$, we define

$$(92) \quad \psi_{\Omega, a, b}^L(z) := \psi(z + \Omega; a^{-1}L, a^{-1}b^{-1}L).$$

**Proposition 4.10** (Proposition 3.1 of [34]). We have

$$\psi_{\Omega, b}^L(t), \psi_{\Omega, a, b}^L(t) \in O_{H_p}[t]^\times$$

where $H = L(A|m)$.

As recalled in loc. cit., these elliptic units satisfy the appropriate norm-compatibility properties. We now recall the following definition.

**Definition 4.11** (Elliptic units). We define distinguished norm-compatible systems of Robert elliptic units with respect to a lattice $L \subset O_K$ by

$$\xi_b := (i_p(\psi(\Omega; p^nL, b^{-1}p^nL)))_{n \in \mathbb{Z}_{\geq 0}}, \quad \xi_{a,b} := (i_p(\psi(\Omega; a^{-1}p^nL, b^{-1}a^{-1}p^nL)))_{n \in \mathbb{Z}_{\geq 0}} \in \mathbb{U}^1.$$

Recall our fixed embedding $i_p : \mathbb{Q} \subset \mathbb{C}_p$ from [3]. We then get an induced map $i_q : \Phi = \bigoplus_{q \mid p} L_q \rightarrow \mathbb{C}_p$, which maps one $L_q$ isomorphically onto a subfield $L_p \subset \mathbb{C}_p$, and maps all the other $L_q$’s to 0. We let $\tilde{\mu}_0$ denote the image of $\mu$ under $i_p$.

Note that we have distinguished elements from (4.11)

$$\xi_b, \xi_{a,b} \in \mathbb{U}.$$

Let $\xi_{\Omega, b}^L, \xi_{a,b}^L \in \mathbb{U}^1$ denote their projections onto semilocal principal units.

Recall by [42] that we can view $\tilde{\mu}_\beta \in O_{L_p}[Q - 1]$, in which case we denote it by $\tilde{\mu}_\beta(Q - 1)$.

**Definition 4.12.** Recall that $G_\infty = \text{Gal}(L_\infty/K) \cong \text{Gal}(L_\infty/L) \times \text{Gal}(L/K) \cong O_K^\times \times \text{Gal}(L/K)$. Given a character $\chi : G = \text{Gal}(L_\infty/K) \rightarrow \mathbb{Q}_p^\times$, let $\chi = \chi_0\chi'$ denote the decomposition with respect to the above decomposition. Let $n \in \mathbb{Z}_{\geq 0}$ be the smallest integer such that $\chi_0 : \text{Gal}(L_\infty/L) \cong O_K^\times \rightarrow \mathbb{Q}_p^\times$ factors through $\chi_0 : \text{Gal}(L_n/L) \cong (O_K/p^nO_K)^\times \rightarrow \mathbb{Q}_p^\times$. Define the Gauss sum

$$\tau(\chi) = \frac{1}{p^{2n}} \sum_{j=0}^{p^n-1} \chi_0(j)\zeta_p^{-j}. $$

**Lemma 4.13** (cf. [45] Lemma II.4.8). Suppose $\chi \in \hat{G}_n$ is of finite order. Recall the fixed orientation $(1, \zeta_p, \ldots, \zeta_p^n, \ldots)$ in Convention 3.18. Choose ideals $c \subset O_K$ prime to $fp$ whose Artin symbols $\sigma_c$ represent $G_n = \text{Gal}(L_n/K)$. Then we have

$$(93) \quad \tilde{\mu}_\beta(c^k\chi) = \tau(\chi) \sum_{c} (c^k\chi)(c^{-1}) \left(\frac{Qd}{dQ}\right)^k \tilde{\mu}_{\sigma_c}(\zeta_p^n - 1).$$
Definition 4.14. Recall our fixed CM elliptic curve $A$ from Definition 4.1. From the isomorphism \( \theta_{\infty, A} : \mathbb{C}/L \rightarrow A(\mathbb{C}) \), we get a differential \( 2\pi i dz \in \Omega^1_{A/\mathbb{C}} \) from the standard differential \( 2\pi i dz \) on \( \mathbb{C} \). Define \( \Omega_{\infty} \in \mathbb{C}^\times \) by
\[
\Omega_{\infty} \cdot 2\pi i dz = \omega_0.
\]

Definition 4.15. Given any Hecke character \( \epsilon \) of infinity type \((k, j)\) with conductor \( \mathfrak{f} \), and let \( \ell \in \mathbb{Z}_{\geq 0} \) be maximal such that \( p^\ell \mathfrak{f} \). Suppose that \( w_\ell = 1 \), by [15] Chapter II.4.7-II.4.8, there exists a Hecke character \( \phi \) of type \((1, 0)\), so that \( \epsilon = \phi^{k-j} \chi \), where \( \chi \) is a finite-order character. Let \( q \in \mathcal{O}_K \) be any ideal with \((q, p^\ell) = 1\), and \((q, L/K) = (p^\ell, L/K) \). We define
\[
\Omega_{\infty} = \phi(q \rho^\nu) \chi(q) \tau(\chi).
\]

Proof. This is the same calculation as in [15] Lemma II.4.8, using (64). \( \square \)

Theorem 4.16 (cf. Theorem II.4.11 of [15]). Suppose \( w_\ell = 1 \), \((p, \mathfrak{f}) = 1\). In the situation of Definition 4.1, for any Hecke character \( \epsilon : \mathbb{A}_K^\times / K^\times \rightarrow \mathbb{C}^\times \) of infinity type \((k, 0)\), \( k \geq 1 \), and conductor dividing \( fp^\infty \), we have
\[
\left( 1 - \frac{\epsilon(p)}{N(p)} \right) \left( \epsilon(b) - \epsilon(\mathfrak{b}) L_{fp}(\epsilon^{-1}, 0) \right),
\]
where \( N(m) \) denotes the positive generator of the ideal \( Nm_K/\mathbb{Q}(m) \).

Proof. This is the same series of calculations as in loc. cit., combined with Lemma 4.13. \( \square \)

We now define a special measure obtained from dividing \( \mu_{\text{glob}}(\zeta_b^{12}) \) by an appropriate twisting measure associated with \( b \).

Definition 4.17. Recall \( b, \mathfrak{f} \in \mathcal{O}_K \) as chosen in Definition 4.1 with \((p, \mathfrak{f}) = 1\) (but not necessarily \( w_\ell = 1 \)), and \( L = K(\mathfrak{f}) \). Recall that \( \sigma_b = (\mathfrak{b}, L_\infty/K) \) denotes the Artin symbol. Define the pseudomeasure
\[
\mu_{\text{glob}}(\mathfrak{f}) := \frac{1}{12} (\sigma_b - \mathbb{N}\mathfrak{b})^{-1} \mu_{\text{glob}}(\zeta_b^{12}),
\]
where \( \mathbb{N}\mathfrak{b} \) denotes the positive generator of \( \text{Nm}_{K/\mathbb{Q}}(\mathfrak{b}) \).

Proposition 4.18 (cf. Theorem II.4.12 of [15]). (1) Suppose \( \mathfrak{f} \neq \mathcal{O}_K \). In fact, under the above assumptions, the pseudomeasure \( \mu_{\text{glob}}(\mathfrak{f}) \) is a measure, i.e.
\[
\mu_{\text{glob}}(\mathfrak{f}) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p})[1/p], \quad \mu_{\text{glob}}(\mathfrak{f}) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \text{ if } w_\ell = 1.
\]
(2) If \( |\mathfrak{g}| \) and \( \overline{\mu}_{\text{glob}}(\mathfrak{g}) \) is the measure on \( G_{\infty}(\mathfrak{f}) \) induced by \( \mu_{\text{glob}}(\mathfrak{g}) \), then
\[
\overline{\mu}_{\text{glob}}(\mathfrak{g}) = \prod_{v | \mathfrak{p}, \mathfrak{v} \neq \mathfrak{f}} (1 - \sigma_v^{-1}) \mu(\mathfrak{f}).
\]
(3) If \( \mathfrak{f} = (1) = \mathcal{O}_K \), then for any \( \sigma \in G_{\infty} = \text{Gal}(L_\infty/K) = \text{Gal}(K(p^\infty)/K) \), \( (1 - \sigma) \mu_{\text{glob}}(1) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \).

Proof. This is given by essentially the same argument as in [15] Proof of Theorem II.4.12] (and Proposition 4.7 if \( w_\ell \neq 1 \)). First note that (98) follows from (95). The point (3) is follows from (97) because \( \mu_{\text{glob}}(1) = (1 - \sigma_v^{-1})^{-1} \mu_{\text{glob}}(v) \) for any prime \( v \). Hence it suffices to prove (97).

Let \( \mu_b := \mu_{\text{glob}}(\zeta_b^{12}) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \), and let \( \delta_b := (\sigma_b - \mathbb{N}\mathfrak{b}) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \). Then by (94), we have
\[
\delta_a \cdot \mu_b = \delta_b \cdot \mu_a.
\]
Essentially, one shows that the twisting measures \( \delta_b \) have greatest common divisor 1 among all \( b \subset \mathcal{O}_K \) as in Definition 4.1.

First, following loc. cit., let \( K_{\infty}/K \) be the the \( \mathbb{Z}_p^{\oplus 2} \)-extension in \( L_{\infty} \), say with Galois group \( \Gamma_0 \), and an isomorphism \( G_{\infty} = \Delta_0 \times \Gamma_0 \). Note that \( \mathcal{O}_{L_p}[\Gamma_0] \) is a unique factorization domain, since it is isomorphic to \( \mathcal{O}_{L_p}[T_1, T_2] \). Fix a character \( \theta \in \Delta_0 \). Then \( \theta(\delta_b) = \theta(\sigma_b|_{\Delta_0}) \cdot (\sigma_b|_{\Gamma_0}) \). One can show that the greatest common divisor of the \( \theta(\sigma_b) \) is 1. Since \( \mu_{p,\infty} \subset L_{p,\infty} \) (by self-duality of \( F_f \)), see Remark 3.66, the argument on pp. 77-78 in loc. cit. shows that the greatest common divisor of all the \( \theta(\sigma_b) \) divides \( \theta(\sigma) \) for any \( \sigma \in \Delta_0 \). But \( \delta(\sigma) \) is a unit, this gives the assertion.

Now since \( \theta(\delta_b) \) have greatest common divisor 1, applying \( \delta \) to \( [99] \), there must exist \( \mu_\theta \in \mathcal{O}_{L_p}[\Gamma_0] \) such that \( \theta(\delta_b) \cdot \mu_\theta = \theta(\mu_b) \) for any \( b \). Letting \( e_\theta = \frac{1}{#\Delta_0} \sum_{\sigma \in \Delta_0} \theta(\sigma)\sigma^{-1} \) denote the projector corresponding to \( \theta \in \hat{\Delta}_0 \), we then see that \( \mu = \sum_{\theta \in \Delta_0} \mu_\theta e_\theta \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \), and \( \delta_b \cdot \mu = \mu_b \).

In particular, \( \#\Delta_0 \cdot \mu \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \). However, proceeding by the same argument as on p. 78 of loc. cit. (specializing to one line \( \Gamma_0 \subset \Gamma_0 \)), one concludes that in fact \( \mu \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \).

Hence \( \mu = \mu_b/\delta_b \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \), and is independent of \( b \). When \( (p, 6) = 1 \), this gives the Proposition. If \( p = 2, 3 \), then it is known that \( \xi_b \in \mathcal{U}^1 \) is a 12th power, and so \( \mu_b \) is divisible by 12, and so we can repeat the above argument for \( \mu_b/12 \) to conclude. \( \square \)

**Definition 4.19** (Iwasawa module of elliptic units). For every \( n \in \mathbb{Z}_{\geq 0} \), let \( C_n = C_{(p^n)} \) be the group generated by the \( \xi_b \), \( (b, 6p) = 1 \), and by the roots of unity in \( L_n \). Assume \( \mathcal{f} = \mathfrak{m} \neq (1) \) so that the \( \xi_b \) are in fact units. Let \( \overline{C}_n \) denote the closure of \( C_n \) in \( R_n^\times \), and let \( C_n^1 \) denote the projection on the principal part of \( R_n^\times \). Then put

\[
\overline{\mathcal{C}} := \lim_{\longrightarrow \ n} \langle \overline{C}_n \rangle \subset \mathcal{U}^1.
\]

Let

\[
\mathcal{C}(\mathcal{f}) = \prod_{\mathfrak{g} | \mathcal{f}} \overline{\mathcal{C}}_\mathfrak{g}
\]

where \( \mathfrak{g} \subset \mathcal{O}_K \) runs over integral ideals dividing \( \mathcal{f} \).

**Definition 4.20.** Consider the image \( \mathcal{t}_p(\overline{\mathcal{C}}(\mathcal{f})) \subset \mathcal{U}^1 \). We henceforth define the \( \Lambda \)-module

\[
(\mathcal{L}_p) := \mu_1^1(\mathcal{t}_p(\overline{\mathcal{C}}(\mathcal{f}))).
\]

Using \( \Lambda \subset \Lambda_{\mathcal{O}_{L_p}} \), we have

\[
(\mathcal{L}_p)_\Lambda_{\mathcal{O}_{L_p}} = \mu_{\mathcal{O}_{L_p}}^1(\overline{\mathcal{C}}(\mathcal{f}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}).
\]

Proposition 3.67 immediately gives the following.

**Corollary 4.21.** The map \( \mu_{\mathcal{O}_{L_p}}^1 : \mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \to \Lambda_{\mathcal{O}_{L_p}} \), induces a pseudoisomorphism of \( \Lambda_{\mathcal{O}_{L_p}} \)-modules

\[
\mu_{\mathcal{O}_{L_p}}^1 : (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}) / (\mathcal{M}, \overline{\mathcal{C}}(\mathcal{f}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}) \to \Lambda_{\mathcal{O}_{L_p}} / ((\mathcal{L}_p)_\Lambda_{\mathcal{O}_{L_p}}).
\]

In particular, \( (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}) / (\mathcal{M}, \overline{\mathcal{C}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}) \) is a pseudo-null \( \Lambda_{\mathcal{O}_{L_p}} \)-module.

### 4.2. The \( p \)-adic Kronecker limit formula

We now state and prove special value formulas for our \( p \)-adic \( L \)-function. This slightly generalizes the special value formula (in the Eisenstein case) of [30, Chapter 9, Theorem 9.10].

**Definition 4.22.** Suppose that \( \mathfrak{f} \subset \mathcal{O}_K \) is an integral ideal with \( (\mathfrak{f}, p) = 1 \). Then we define a \( p \)-adic analytic function

\[
L_{p, \mathfrak{f}} : \hat{G}_\infty \to \mathcal{O}_{\mathfrak{c}_p}, \quad L_{p, \mathfrak{f}}(\chi) := \mu_{\text{glob}}(\mathfrak{f})(\chi^{-1})
\]

Here, \( \mu_{\text{glob}}(\mathfrak{f}) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p}) \) as in [97].
We recall Robert’s invariants. For more details in a concise and nice exposition, see [45 Chapter II.2.6].

**Definition 4.23** (Robert’s invariants). Let \( f \subset \mathcal{O}_K \) be an integral ideal with \( f \neq \mathcal{O}_K \). As before, let \( N(f) \) denote the positive generator of the ideal \( Nm_{K/Q}(f) \). Let \( \mathcal{C}(f) \) denote the ray class group modulo \( f \), so that Artin reciprocity gives \( \mathcal{C}(f) \cong \text{Gal}(K(f)/K) \). Let \( b \subset \mathcal{O}_K \), \( (b, 6f) = 1 \). For any \( \sigma \in \text{Gal}(K(f)/K) \), let

\[
\phi_\sigma(f) = \theta(1, f^{-1}f), \quad \sigma = (c, K(f)/K), \quad c \subset O_K.
\]

Here, given a lattice \( L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C} \), with \( \tau = \omega_1/\omega_2, \text{im}(\tau) > 0 \),

\[
\theta(z, L) = \Delta(L) \cdot e^{-6\eta(z,L)z} \cdot \sigma(z, L)^{12},
\]

where

\[
\sigma(z, L) = z \cdot \prod_{\omega \in L, \omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp \left(\frac{z}{\omega} + \frac{1}{2} \left(\frac{z}{\omega}\right)^2\right), \quad \Delta(L) = (2\pi i/\omega_2)^{12} q \prod_{n=1}^\infty (1 - q^n)^{24}, \quad q = e^{2\pi i \tau},
\]

and \( \eta \) is an \( \mathbb{R} \)-linear form on \( \mathbb{C} \) given by

\[
\eta(z, L) = \frac{\omega_1 \eta_1 - \omega_2 \eta_2}{2\pi i A(L)} z + \frac{\bar{\omega}_1 \eta_1 - \bar{\omega}_2 \eta_2}{2\pi i \bar{A}(L)} \bar{z}, \quad A(L) = (2\pi i)^{-1}(\omega_1 \bar{\omega}_2 - \bar{\omega}_1 \omega_2) = \pi^{-1} \text{Area}(\mathbb{C}/L),
\]

\[
\eta_1 = \omega_1 \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, m \neq 0} (m\omega_1 + n\omega_2)^{-2}, \quad \eta_2 = \omega_2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}, n \neq 0} (m\omega_1 + n\omega_2)^{-2}.
\]

**Theorem 4.24** (cf. Theorem II.5.2 of [45], Theorem 9.10 of [30]). Let \( g = fp^n \) where \( (f, p) = 1 \). Recall \( \mathcal{L}_{p,f} \) as in [20], and let \( \mathcal{L}_{p,g}(\chi) = \mathcal{L}_{p,f}(\chi)(1 - \chi(p)) \) if \( n > 0 \). Suppose that \( \chi \in \hat{G}_{\infty} \) is of finite order and conductor dividing \( g \), and if \( \chi = 1 \) then assume \( f \neq 1 \). Then we have

\[
(102) \quad \mathcal{L}_{p,g}(\chi) = -\frac{1}{12N(g)w_g} \cdot G(\chi^{-1}) \left(1 - \frac{\chi^{-1}(p)}{N(p)}\right) \cdot \sum_{c \in \mathcal{C}(g)} \chi(c) \cdot \log \phi_g(c).
\]

Here, \( G(\chi^{-1}) \) is defined in [94], and \( \phi_g \) is defined as in Definition 4.23.

**Proof.** Given our definition of \( \mu_{\text{glob}}(g) \) in [96], and the interpolation property [95], this argument is entirely analogous to the one given in [45] Proof of Theorem II.5.2].

\[\square\]

### 4.3. Formulation of the (two-variable) Main Conjectures.

In this section, we let \( L = K(f) \), where \( f \subset \mathcal{O}_K \), but do not impose any assumptions on \( f \) unless specified. In particular, we can let \( f = 1 \) (which will be the situation for our later applications to the Birch and Swinnerton-Dyer conjecture for elliptic curves with CM by imaginary quadratic fields of class number 1).

**Definition 4.25.** Recall the Galois group (see [85]) \( G_{\infty} = \text{Gal}(L_{\infty}/K) \). Let \( M_{\infty} \) denote the maximal abelian pro-\( p \) extension of \( L_{\infty} \) unramified outside (primes above) \( p \). Let \( N_{\infty} \) denote the maximal abelian pro-\( p \) extension of \( L_{\infty} \) unramified at all places of \( L_{\infty} \). Then we let

\[
\mathcal{X} := \text{Gal}(M_{\infty}/L_{\infty}), \quad \mathcal{Y} := \text{Gal}(N_{\infty}/L_{\infty}).
\]

Recalling that \( \text{Gal}(L_{\infty}/L) \sim \text{Gal}(L_{\infty}/L_p) \xrightarrow{\lambda} \mathcal{O}_K^\times \) via the reciprocity map, and that \( \Lambda_R = \Lambda(\text{Gal}(L_{\infty}/L_p), R) = R[\text{Gal}(L_{\infty}/L_p)] \) for a \( \mathbb{Z}_p \)-algebra \( R \), we can view \( \mathcal{X} \) and \( \mathcal{Y} \) as \( \Lambda_{\mathbb{Z}_p} \)-modules.

Hence, we can view \( \mathcal{X} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \) and \( \mathcal{Y} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \) as \( \Lambda_{\mathcal{O}_L} \)-modules. It is well-known (see [20] Remarks before Theorem 2) that \( \mathcal{X} \) has \( \Lambda_{\mathbb{Z}_p} \)-rank equal to \( d = [L_p : K_p] = r_2(L) \) (the number of pairs of complex embeddings of \( L \), while \( \mathcal{Y} \) is \( \Lambda_{\mathbb{Z}_p} \)-torsion. We wish to formulate a main conjecture for \( \mathcal{X} \), which will be facilitated by considering an appropriate \( \Lambda_{\mathcal{O}_L} \)-torsion quotient of \( \mathcal{X} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \). This
We will later show that the cyclotomic \( \mu_{\mathcal{O}_{L_p}}^1 : \mathcal{U} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \to \Lambda_{\mathcal{O}_{L_p}} \). See [103] for the precise formulation.

We will also wish to consider certain pure \( \mathbb{Z}_p \)-extensions of \( L_\infty \), and hence isotypic components of all the relevant \( \Lambda_{\mathcal{O}_{L_p}} \)-modules, so that we can study the associated Iwasawa invariants of these isotypic components. In this article, we will study only the \( \mu \)-invariant in-depth, as from its vanishing we will deduce a more general version of the Main Conjecture for \( \mathcal{Y}_\infty \) proven in [40, Theorem 4.1]. Note that we have a natural isomorphism \( \text{Gal}(L_\infty/L) \sim \text{Gal}(L_{p,\infty}/L_p) \xrightarrow{\sim} \mathcal{O}_{K_p}^\times \) induced by \( i_p \) from (3), since \( L_\infty/L \) is totally ramified. Let \( K \subset L' \subset L_\infty \) such that \( \text{Gal}(L_\infty/L') \cong 1 + p^r \mathcal{O}_{K_p} \). Let

\[
\Gamma := \text{Gal}(L_\infty/L') \cong 1 + p^r \mathcal{O}_{K_p}.
\]

Let

\[
\Delta := \text{Gal}(L'/K),
\]

where \( \epsilon \) is as in (54). Let \( K \subset K_\infty \subset L_\infty \) such that

\[
\Gamma' := \text{Gal}(K_\infty/K) \cong 1 + p^r \mathcal{O}_{K_p}.
\]

Let \( K \subset K_n \subset K_\infty \) be the unique finite subextension such that \( \text{Gal}(K_n/K) \cong (\mathbb{Z}/p^n)^{\oplus 2} \). If \( K(1) \cap K_\infty = K_t \), then the image of \( \Gamma \) in \( \Gamma' \) under restriction to \( K_\infty \) is \( (\Gamma')^{p^r} \). Fix a decomposition

\[
G_\infty = \Delta' \times \Gamma',
\]

where \( \Delta' = \text{Gal}(L_\infty/K_\infty) \) is a finite abelian group. Suppose that \( \chi \in \hat{\Delta'} \). Let \( L_{p,\chi} \) be the finite extension of \( L_p \) generated by the values of \( \chi \), and let \( \mathcal{O}_{L_{p,\chi}} \) be its valuation ring. Then given a \( \Lambda(G_\infty, \mathcal{O}_{L_p}) \)-module \( M \), we can define the \( \chi \)-isotypic component \( M_\chi \) as follows. Consider \( \Lambda(\Gamma', \mathcal{O}_{L_{p,\chi}}) \) as a \( \Lambda(G_\infty, \mathcal{O}_{L_p}) \)-module by letting \( \Delta' \) act through \( \chi \). Then let

\[
M_\chi := M \otimes_{\Lambda(G_\infty, \mathcal{O}_{L_p})} \Lambda(\Gamma', \mathcal{O}_{L_{p,\chi}}),
\]

which is the largest quotient of \( M \otimes_{\mathcal{O}_{L_p}} \mathcal{O}_{L_{p,\chi}} \) on which \( \Delta' \) acts through \( \chi \).

If \( p \nmid [L' : K] \) (recalling \( L = K(f) \), \( L' = K(p^r) \), \( f, p = 1 \)), then we have \( \Delta = \Delta' \), \( \Gamma = \Gamma' \), \( L_p = L_{p,\chi} \), and naturally we have isotypic decompositions

\[
\Lambda(G_\infty, \mathcal{O}_{L_p}) = \bigoplus_{\chi \in \Delta} \Lambda(G_\infty, \mathcal{O}_{L_p})_\chi, \quad M = \bigoplus_{\chi \in \Delta} M_\chi.
\]

However, for many applications the assumption \( p \nmid [L' : K] \) is cumbersome, which is why it is necessary to consider the equivariant main conjecture (i.e. a main conjecture involving \( \Lambda(G_\infty, \mathcal{O}_{L_p}) \)-modules).

**Definition 4.26.** We define the following fundamental \( \Lambda(G_\infty, \mathcal{O}_{L_p}) \)-modules

\[
\mathcal{U}' := (\mathcal{U} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})/(\mathcal{M}, \mathcal{E}(f) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}), \quad \mathcal{X}' := (\mathcal{X} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})/\text{rec}(\mathcal{M}).
\]

Here, \( \text{rec} : \mathcal{U} \to \mathcal{X} \) is the global reciprocity map, which we extend \( \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \)-linearly to a map \( \text{rec} : \mathcal{U} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \to \mathcal{X} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \). When we wish to emphasize the dependence on the auxiliary conductor \( f \), we write \( \mathcal{U}' = \mathcal{U}'(f), \mathcal{X}' = \mathcal{X}'(f) \), etc.

**Remark 4.27.** We will later show that the cyclotomic \( \mu \)-invariant of the ideal \( \mu_{\mathcal{O}_{L_p}}^1 (\mathcal{E}(f) \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}) \subset \Lambda(G_\infty, \mathcal{O}_{L_p}) \) is zero, and hence \( \mu_{\mathcal{O}_{L_p}}^1 (\mathcal{E}(f) \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}) \neq 0 \). In particular, \( \mathcal{M} \cap (\mathcal{E}(f) \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}) = 0 \), and \( \mathcal{U}' \) is indeed a \( \Lambda(G_\infty, \mathcal{O}_{L_p}) \)-torsion module.
Theorem 4.28 (Fundamental Exact Sequence). Suppose $F_f$ is height 2 and self-dual (see Definition 3.19). We have the following exact sequence of torsion $\Lambda(G_{\infty}, \mathcal{O}_{L_p})$-modules:

\[(103) \quad 0 \to (\mathcal{E}/\mathcal{C}(f)) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \to \mathcal{U}' \xrightarrow{\text{rec}} \mathcal{X}' \to \mathcal{Y} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \to 0.\]

Proof. This follows immediately from the definitions and Corollary 4.21. \qed

Proposition 4.29 (cf. Lemma III.1.10 of [45]). Suppose $\chi \in \hat{A}'$, and suppose that the conductor of $\chi$ is either $g$ or $gg^r$ for $0 \leq r \leq \epsilon$, where $g|f$. Recall $\mu_{\text{glob}}(g) \in \Lambda(G_{\infty}, \mathcal{O}_{L_p})$, as in [97], and let $\mu_{\text{glob}}(g; \chi) := \chi(\mu_{\text{glob}}(g))$. Then we have

\[(104) \quad \text{char}_{\Lambda(\Gamma', \mathcal{O}_{L_p}; \chi)}(U'_\chi) = \mu_{\text{glob}}(g; \chi)\Lambda(\Gamma', \mathcal{O}_{L_p}) \chi.\]

Proof. This essentially follows from (100), particularly the pseudo-nullity of $\text{coker}(\mu_{\mathcal{Q}_{L_p}}^1)$. \qed

Conjecture 4.30 (Rubin-type Main Conjecture). Suppose $F_f$ is height 2 and self-dual (see Definition 3.19). We have the following equality of determinants of torsion $\Lambda(G_{\infty}, \mathcal{O}_{L_p})$-modules:

\[(105) \quad \det_{\Lambda(G_{\infty}, \mathcal{O}_{L_p})}(U'_{\chi}) = \det_{\Lambda(G_{\infty}, \mathcal{O}_{L_p})}(\mathcal{X}').\]

Then for any $\chi \in \hat{A}'$ with $\chi \neq 1$ we have

\[(106) \quad \mu_{\text{glob}}(g; \chi)\Lambda(\Gamma', \mathcal{O}_{L_p}; \chi) \xrightarrow{\text{104}} \text{char}_{\Lambda(\Gamma', \mathcal{O}_{L_p}; \chi)}(U'_\chi) = \text{char}_{\Lambda(\Gamma', \mathcal{O}_{L_p}; \chi)}((\mathcal{X}')_{\chi}).\]

Remark 4.31. With a suitable modification (accounting for a “pole”), one can also formulate (and in many cases prove) a version of (106) for $\chi = 1$.

Conjecture 4.32 (Main Conjecture). We have the following equality of determinants of torsion $\Lambda(G_{\infty}, \mathcal{O}_{L_p})$-modules:

\[(107) \quad \det_{\Lambda(G_{\infty}, \mathcal{O}_{L_p})}((\mathcal{E}/\mathcal{C}(f)) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}) = \det_{\Lambda(G_{\infty}, \mathcal{O}_{L_p})}(\mathcal{Y} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}).\]

Furthermore, for any $\chi \in \hat{A}'$, we have

\[(108) \quad \text{char}_{\Lambda(\Gamma', \mathcal{O}_{L_p})} \left((\mathcal{E}/\mathcal{C}(f)) \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})\right)_{\chi} = \text{char}_{\Lambda(\Gamma', \mathcal{O}_{L_p})} \left(\mathcal{Y} \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}\right)_{\chi}.\]

Proposition 4.33. (1) Suppose that (107) holds. Then (105) holds.

(2) Suppose that (108) holds. Then (106) holds.

Proof. The first assertion follows immediately from the exact sequence (103) and the additivity of determinants in exact sequences. The second assertion follows from the first. \qed

The purpose of the next section is to establish certain properties on the Iwasawa invariants attached to $L_p$ and $\mathcal{X}$, specifically that the “algebraic $\mu$-invariant” (more precisely the $\mu$-invariants attached to certain isotypic components of $\mathcal{X}$) is (are) 0. Using an equivariant version of the Main Conjecture and certain reductions to the usual Main Conjecture due to Johnson-Leung and Kings [25], we will use this vanishing of the algebraic $\mu$-invariant to prove Conjecture 4.32 under suitable assumptions. In particular, this removes the technical assumption in [10] that $p \nmid \# \mathcal{O}_{K}^\times$, and hence allows us to access the arithmetic of complex multiplication elliptic curves with $j = 0, 1728$. 34
4.4. The cyclotomic algebraic $\mu$-invariant. We continue the notation of the previous section. First we recall the algebraic $\mu$-invariant.

Definition 4.34. Recall $\mathbb{U}$ as defined in [86], and recall we let $L = K(f)$ with $(f, p) = 1$, $w_l = 1$. Let $L \subset L^+_n \subset L(\mu_{p^{\infty}})$ be the intermediate extension with $\text{Gal}(L^+_n/L) \cong \mathbb{Z}/p^n$. Then let $L^+_\infty$ denote the compositum of the $L^+_n$, so that $\text{Gal}(L^+_\infty/L) \cong \mathbb{Z}$. For $0 \leq n \leq \infty$, let $M^+_n$ be the maximal pro-$p$ abelian extension of $L^+_\infty$ which is unramified outside $p$, and let $N^+_n$ be the maximal pro-$p$ abelian extension of $L^+_\infty$ which is unramified everywhere. The we let

$$X^+ := \text{Gal}(M^+_\infty/L^+_\infty) = \lim_{\to} \text{Gal}(M^+_n/L^+_n).$$

Let $\Gamma_+ := \text{Gal}(L^+_\infty/L) \cong \mathbb{Z}_p$, so that

$$\tag{109} (X^+)_{\Gamma^+_p} = \text{Gal}(M^+_n/L^+_n).$$

By the non-vanishing of the $p$-adic regulator ([7]), one can show that $\text{Gal}(M^+_n/L^+_\infty)$ is finite. Note that $X^+ \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p}$ is a $\Lambda(\text{Gal}(L^+_\infty/K), \mathcal{O}_{L_p})$-module. For any $\chi \in \hat{\Delta}$, we can consider the $\chi$-isotypic component $(X^+ \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})_\chi$.

Moreover, for each $0 \leq n \leq \infty$, let $K \subset K^+_n \subset K(\mu_{p^{\infty}})$ be the unique subfield with $\text{Gal}(K^+_n/K) \cong \mathbb{Z}/p^n$ if $n < \infty$, and $\text{Gal}(K^+_\infty/K) \cong \mathbb{Z}_p$ if $n = \infty$. Note that (by standard theory of cyclotomic extensions), $K^+_n/K$ is totally ramified. Let

$$\Gamma'_+ = \text{Gal}(K^+_\infty/K).$$

Note that if we let $t \in \mathbb{Z}_{\geq 0}$ such that $K^+_n \cap K(1) = K^+_t$, then we have a natural identification $\Gamma_+ = (\Gamma'_+)^t$.

Let $\chi \in \hat{\Delta}'$. We have the $\Lambda(\Gamma, \mathcal{O}_{L_p, \chi})$-module $(X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})_\chi$. We have a natural specialization map $\pi_+ : \Gamma \to \Gamma_+$, so letting

$$(X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})^+_\chi := (X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})_\chi \otimes_{\Lambda(\Gamma, \mathcal{O}_{L_p, \chi}), \pi_+} \Lambda(\Gamma_+, \mathcal{O}_{L_p, \chi}) = (X^+ \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})_\chi$$

denote the image of $(X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})_\chi$ under this specialization, we have a natural map

$$(X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})^+_\chi \to (X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})^+_\chi.$$

We note that $X^+$ is not a torsion $\Lambda(\Gamma, \mathcal{O}_{L_p, \chi})$-module (rather, by [20], Remarks after Theorem 1) it has rank $r_2(L) = d = [L : K]$). However, we will again consider a suitable quotient which is torsion, and study its $\mu$-invariant as a $\Lambda(\Gamma_+, \mathcal{O}_{L_p, \chi})$-module. Recall the torsion $\Lambda(G^\infty, \mathcal{O}_{L_p, \chi})$-module $X'$, which gives rise to a torsion $\Lambda(\Gamma, \mathcal{O}_{L_p})$-module $(X')^+$. Again, let $(X')^+$, $(X')^+_\chi$ denote the $\Gamma_+$-specializations of $X'$ and $(X')_\chi$. Then from the surjective quotient map $X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p} \to X'$, we get a surjective map (since tensoring is right-exact)

$$\tag{110} (X^+ \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})_\chi = (X \otimes_{\mathbb{Z}_p} \mathcal{O}_{L_p})^+_\chi \to (X')^+_\chi$$

where the right-hand side is a torsion $\Lambda(\Gamma_+, \mathcal{O}_{L_p})$-module. We will study the $\mu$-invariant of the right-hand side, i.e. the algebraic cyclotomic $\mu$-invariant, which we will do through a valuation calculation of specializations of the left-hand side. Using the strategy of Coates-Wiles [10], we then relate the valuation of specializations of the left-hand side to valuations of specializations of the $p$-adic $L$-function $L_{p,f}$ from (110), and hence to the analytic $\mu$-invariant. This culminates in Theorem 4.48. We will later show the analytic $\mu$-invariant is 0. This, combined with (110), will show that the algebraic cyclotomic $\mu$-invariant is 0.
4.5. Computation of algebraic Iwasawa invariants: a valuation calculation. We follow the strategy of [10] for studying

$$\text{ord}_p(\#(X^+)_{\mathcal{R}^U}) = \text{ord}_p(\#\text{Gal}(M^+_n/L^+_\infty)).$$

We need some lemmas and propositions which are refinements of results from loc. cit. We follow [45, Chapter III.2] throughout this section.

**Definition 4.35** ($p$-adic regulator). Let $F/K$ be any abelian extension of degree $r$. Let $\sigma_1, \ldots, \sigma_r$ be the embeddings $F \hookrightarrow \mathbb{C}_p$. (Note that all embeddings induce the same embedding $K_p \hookrightarrow \mathbb{C}_p$, since there is only one prime above $p$.) Let $E$ be a subgroup of finite index in $\mathcal{O}_F^\times$, and choose generators $e_1, \ldots, e_{r-1}$ for $E/E_{\text{tors}}$. Then the $p$-adic regulator of $E$ is defined to be

$$R_p(E) = \det(\log_\sigma(e_j))_{1 \leq i,j \leq r-1}.$$  

This is well-defined up to sign because $\log \text{Nm}_{F/K}(e) = 0$ for $e \in E$. As short-hand, let $R_p(F) = R_p(\mathcal{O}_F^\times)$.

**Theorem 4.36** ([7]). Assume $F/K$ is abelian (as is the case in Definition 4.35). Then $R_p(E) \neq 0$.

**Definition 4.37.** We follow the notation and presentation of [45, Chapter III.2.4]. Fix an abelian extension $F/K$ of degree $r$ as in Definition 4.35. For each prime $\mathfrak{p}$ of $F$ above $p$ (the unique prime of $K$ above $p$), let $w_{\mathfrak{p}}$ denote the number of $p$-power roots of unity in $F_{\mathfrak{p}}$. Let $\Phi = F \otimes_K K_p = \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}$, and let $U$ be the group of principal units in $\Phi$. Then the $p$-adic logarithm gives a homomorphism

$$\log : U \to \Phi$$

whose kernel has order $\prod_{\mathfrak{p}|p} w_{\mathfrak{p}}$, and whose image is an open subgroup of $\Phi$. Let $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. Let $E \subset \mathcal{O}_F^\times$ be a subgroup of finite index, and let $D = D(E) = E \cdot (1 + q)$, and let $\overline{D}$ be its closure in $\Phi^\times$, and $(\overline{D})$ the projection of $\overline{D}$ to $U$. We write $\Phi(F), U(F), E(F)$ when we refer to these objects for a specific $F$. Let $\Delta_p(F/K)$ denote the sum over $\mathfrak{p}|p$ of the relative discriminants of $F_{\mathfrak{p}}/K_p$. Given any number field $F$, let $h(F)$ denote its class number and let $h(F)[p^\infty]$ denote the $p$-part of the class number.

**Definition 4.38.** Suppose we are in the situation of Definition 4.34, i.e. with $F = L^+_n$. Then letting $U_n = U(L^+_n)$ and $E_n = E(L^+_n)$, Artin reciprocity gives

$$U_n/(\overline{E}_n) \cong \text{Gal}(M^+_n/N^+_n).$$

Recall that $N^+_n$ (resp. $M^+_n$) is the maximal pro-$p$ abelian extension unramified everywhere (resp. unramified outside $p$) of $L^+_n$. Note that $N^+_n \cap L^+_\infty = L^+_n$ and $L^+_\infty \subset M_n$. Let $Y_n \subset U_n$ be the subgroup such that

$$Y_n/(\overline{E}_n) \cong \text{Gal}(M^+_n/N^+_n L^+_\infty).$$

Recall that $L^+_n = L K^+_n$, and $K^+_n \subset K(\mu_{p^\infty})$ is such that $[K^+_n : K] = p^n$. In particular, $K^+_n/Q$ is an abelian extension. Consider the norm map $\text{Nm}_{L^+_n/Q} : U_n \to 1 + p\mathcal{O}_{K_p}$.

**Lemma 4.39** (cf. Lemma III.2.6 of [45]). We have $Y_n = \text{Nm}_{L^+_n/Q}[U_n]$.

**Proof.** If $u \in Y_n$, then since $U_n/Y_n \cong \text{Gal}(N^+_n L^+_\infty/N^+_n) \cong \text{Gal}(L^+_\infty/L^+_n)$, we have $(u, L^+_\infty/L^+_n) = 1$ (as usual, denoting the Artin symbol of an abelian extension $E'/E$ by $\langle \cdot, E'/E \rangle$). Then by the functoriality of the Artin symbol, we get $(u, L^+_\infty/L^+_n) = (\text{Nm}_{L^+_n/Q}(u), K^+_\infty/Q) = 1$. Hence the idèle $\text{Nm}_{L^+_n/Q}(u)$, which is $1$ outside of $p$, is necessarily $1$. The argument in reverse shows the converse.

Let $D_n = D(L^+_n)$, and similarly with $\overline{D}_n$. 36
Lemma 4.40. Assume that we are in the situation of Definition 4.34, i.e. with $F = L_n^+$. Let $p^a||[L : K]$, so that $p^{n+\delta}||[L_n^+ : K]$. Then we have $[\log(U_n) : \log((\mathcal{D}_n))] < \infty$, and in fact

$$\text{ord}_p \left( [\log(U_n) : \log((\mathcal{D}_n))] \right) = \text{ord}_p \left( \frac{\langle p^{n+\delta}R_p(L_n^+) \rangle}{w(L_n^+)\sqrt{\Delta_p(L_n^+/K) \prod_{\mathfrak{p}|p}(\mathfrak{p} \mathfrak{p})^{-1}}} \right),$$

where $\mathfrak{p}$ runs over all primes of $L_n^+$ above $p$.

Proof. This is [10, Lemma 8], the argument of which goes through using $\epsilon_d = 1 + q$ in place of $\epsilon_d = 1 + p$.

Corollary 4.41. Retain the situation of Lemma 4.40. Let $w(E)$ be the number of roots of unity in $E$. Then

$$\text{ord}_p \left( [U_n : \langle \mathcal{D}_n \rangle] \right) = \text{ord}_p \left( \frac{\langle p^{n+\delta}R_p(L_n^+) \rangle}{w(L_n^+)\sqrt{\Delta_p(L_n^+/K) \prod_{\mathfrak{p}|p}(\mathfrak{p} \mathfrak{p})^{-1}}} \right),$$

where $\mathfrak{p}$ runs over all primes of $L_n^+$ above $p$.

Proof. This is an immediate consequence of Lemma 4.40 using the snake lemma, see [10, Lemma 9] for details.

Proposition 4.42 (cf. Proposition III.2.17 of [15]). Retain the notation of Lemma 4.41. Let $p^f := [(K(1) \cap K_n^+) : K]$ and $p^e = [(L \cap K_n^+) : K]$. We have

$$(112) \quad \text{ord}_p \left( [M_n^+ : L_n^+] \right) = \text{ord}_p \left( \frac{\langle p^{n+\delta}R_p(L_n^+) \rangle}{w(L_n^+)\sqrt{\Delta_p(L_n^+/K) \prod_{\mathfrak{p}|p}(1 - \mathfrak{p} \mathfrak{p})^{-1}}} \right),$$

where the product on the right-hand side runs over primes of $L_n^+$ above $p$.

Proof. Let $D_n = E_n \cdot (1 + q)$. Then since $\text{Nm}_{L_n^+/K}(E_n) = 1$, we have $\text{Nm}_{L_n^+/Q}(\langle \mathcal{D}_n \rangle) = 1 + q^2p^{n+\delta-1}Z_p$. Now we have a diagram

$$0 \rightarrow \langle E_n \rangle \rightarrow \langle D_n \rangle \rightarrow 1 + q^2p^{n+\delta-1}Z_p \rightarrow 0$$

$(113)$

$$0 \rightarrow Y_n \rightarrow U_n \rightarrow 1 + q^2p^{n+\delta-1}Z_p \rightarrow 0$$

where the horizontal rows are exact and the vertical arrows are injective, by the above discussion and Lemma 4.39. Then $[L_n^+ : K]$ is exactly divisible by $p^{n+\delta}$, and so (112) follows from Lemma 4.41. (104). $[M_n^+ : L_n^+] = [N_n^+ : N_n^+L_n^+][N_n^+L_n^+ : L_n^+]$, and the fact that $[N_n^+L_n^+ : L_n^+] = [N_n^+ : L_n^+] = h(L_n^+)[p^\infty]$.

Definition 4.43. For $\chi \in \hat{\chi}$, recall the torsion $\Lambda(\Gamma', \mathcal{O}_L)$-modules $(\mathcal{X}')_\chi = \mathcal{X}'(f)_\chi$ and $\mathcal{U}'_\chi = \mathcal{U}'(f)_\chi$ from Definition 4.26. We let

$$f_\chi := \text{char}_{\Lambda(\Gamma', \mathcal{O}_L, \chi)}((\mathcal{X}')_\chi), \quad g_\chi := \text{char}_{\Lambda(\Gamma', \mathcal{O}_L, \chi)}(\mathcal{U}'_\chi).$$

In light of (104), we have

$$g_\chi = \mu_{\text{glob}}(\mathfrak{g}; \chi).$$
We consider the specializations, i.e. the images under the maps \( \mathcal{X}' \to (\mathcal{X}')_+^1 \) and \( \mathcal{U}' \to (\mathcal{U}')_+^1 \), which we denote by \( f_\chi^+ \) and \( g_\chi^+ \), respectively. Denote the \( \mu \)-invariants by \( \mu_{\text{inv}}(f_\chi^+) \) and \( \mu_{\text{inv}}(g_\chi^+) \), respectively. Similarly define the \( \lambda \)-invariants by \( \lambda_{\text{inv}}(f_\chi^+) \) and \( \lambda_{\text{inv}}(g_\chi^+) \).

**Corollary 4.44.** Let \( K_1^+ = K_1^+ \cap K(1) \). We have for all \( n \gg 0 \) and all \( \chi \in \hat{\Delta}' \),

\[
\mu_{\text{inv}}(f_\chi^+) \cdot p^{t+n-1}(p-1) + \lambda_{\text{inv}}(f_\chi^+ inc.) \\
\mu_{\text{inv}}(f_\chi^+) = 1 + \text{ord}_p \left( \frac{hR_p}{w\sqrt{\Delta}} (L_n^+/K) / \frac{hR_p}{w\sqrt{\Delta}} (L_n^+/K) \right).
\]

**Proof.** By (112), the right-hand side of (114) is \( \text{ord}_p ([M_{n+1}^+: L_\infty^+]) / [M_n^+: L_\infty^+] \). Now (114) follows from (109), (110) and Weierstrass preparation (using the standard definitions of \( \mu_{\text{inv}} \) and \( \lambda_{\text{inv}} \)). □

4.6. Computation of analytic Iwasawa invariants: application of analytic class number formula and \( p \)-adic Kronecker limit formula. We continue to follow [45, Chapter III.2].

**Lemma 4.45.** For any finite-order character \( \nu \in \hat{\Gamma}', \) define \( m_\nu = s \) if \( \nu((\Gamma_\nu)^{p-1}) = 1 \) but \( \nu((\Gamma_\nu)^{p-1}) \neq 1 \). Retaining the notation of Corollary 4.44, we have for all \( n \gg 0 \),

\[
\mu_{\text{inv}}(g_\chi^+) \cdot p^{t+n-1}(p-1) + \lambda_{\text{inv}}(g_\chi^+) = \text{ord}_p \left( \prod_{\nu, m_\nu = t+n} \nu(g_\chi^+) \right).
\]

**Proof.** This is just [45, Lemma III.2.9]. □

**Definition 4.46.** For any ramified character \( \epsilon \in \hat{G}_\infty = \hat{G}_\infty(f) \), let \( g = \text{cond}(\epsilon) \). Then in the same notation as for (102), define

\[
S_p(\epsilon) := -\frac{1}{12N(\mathfrak{g})w_\mathfrak{g}} \sum_{c \in \mathcal{O}(\mathfrak{g})} \epsilon^{-1}(c) \log \phi_\mathfrak{g}(c).
\]

Recall \( G(\epsilon) \) as defined in (94). Let

\[
S_n = \{ \epsilon \in \hat{\Gamma}' : p^n || \text{cond}(\epsilon) \}.
\]

**Proposition 4.47.** In the notation of (4.46), for all \( n \gg 0 \) we have

\[
\text{ord}_p \left( \prod_{\epsilon \in S_n} G(\epsilon)S_p(\epsilon) \right) = \text{ord}_p \left( \frac{hR_p}{w\sqrt{\Delta_p}} (L_n^+/K) / \frac{hR_p}{w\sqrt{\Delta_p}} (L_n^+/K) \right).
\]

**Proof.** This follows by precisely the same argument as in [45, Chapter III.2.11]. □

4.7. Relating the cyclotomic algebraic and analytic Iwasawa invariants: a fundamental inequality.

**Theorem 4.48.** The total \( \mu \)-invariants and \( \lambda \)-invariants of \( (\mathcal{X}')_+ \) and \( (\mathcal{U}')_+ \) satisfy the following inequality

\[
\sum_{\chi \in \Delta'} \mu_{\text{inv}}(f_\chi^+) \leq \sum_{\chi \in \Delta'} \mu_{\text{inv}}(g_\chi^+), \quad \sum_{\chi \in \Delta'} \lambda_{\text{inv}}(f_\chi^+) \leq \sum_{\chi \in \Delta'} \lambda_{\text{inv}}(g_\chi^+).
\]

**Remark 4.49.** As we will show later (Corollary 4.68),

\[
\sum_{\chi \in \Delta'} \mu_{\text{inv}}(g_\chi^+) = 0,
\]

and so equality for the \( \mu \)-invariants in (117) holds throughout.
Proof of Theorem 4.48. Write \( \nu \in S_{n+1} \) as \( \nu = \chi \rho \), where \( \chi \in \hat{\Delta}' \) and \( \rho \) is a character of \( \Gamma' \backslash \Gamma_+^{p^{t+n}} \) but not of \( (\Gamma_+)/(\Gamma_+^{p^{t+n}}) \). Write \( g = \text{cond}(\nu) = \mathfrak{g} \mathfrak{p}^n \), where \( (\mathfrak{g}, \mathfrak{p}) = 1 \). By (102), we have

\[
\rho(g_\chi^+) = \rho(g_\chi) = \rho(\mu(\mathfrak{g}_0; \chi)) = G(\nu)S_\rho(\nu)
\]

where the superscript “+” denotes specialization along \( \Gamma' \rightarrow \Gamma_+ \), and the first equality follows since \( \rho \) may be viewed as a character on \( \Gamma \backslash \Gamma_+^{p^{t+n}} \) using (58), and from the construction of the measure \( \mu_{\text{glob}} \). Running over all \( \nu \in S_{n+1} \) runs over all \( \chi \in \hat{\Delta}' \) and \( \rho \in \hat{\Gamma}_+ \) with \( m_\rho = t + n \), and so (116) gives

\[
(118) \prod_{\chi \in \hat{\Delta}'} \prod_{m_\rho = t+n} \rho(g_\chi^+) \sim p \cdot \frac{hR_\rho}{w\sqrt{\Delta_\rho}} \frac{(L_n^+)}{w\sqrt{\Delta_\rho}}(L_n^+),
\]

where “\( \sim \)” means “equality up to \( p \)-adic unit”. Now comparing (118) with (115), and then (114), gives (117).

4.8. Plan for proving the Main Conjecture. For the next few sections, we study the \( \mu \)-invariant of \( \mu_{\text{O}_{L_p}}(\xi) \) when \( \xi \in U^1 \) is a norm-compatible system of principal elliptic units. Our goal is to show that the \( \mu \)-invariant is 0, as work of Johnson-Leung-Kings [25] reduces Conjecture 4.32 to the vanishing of the \( \mu \)-invariant of \( \mu_{\text{O}_{L_p}}(\xi) \) on the cyclotomic line. To show this, we will use the method of Sinnott [48] as developed for elliptic units by Gillard-Robert [18] and [34].

4.9. Coleman power series of elliptic units. Here we recall the Coleman power series attached to \( \xi \), following [34].

Let \( F_f = (\hat{A}, [p]) \), where \([p] : A \rightarrow A/A[p] \) is the natural projection (here \( A[m] \) for an integral ideal \( m \subset O_K \) denotes the \( m \)-torsion). As is well-known, \((\hat{A}, [p]) \) is indeed a relative Lubin-Tate formal \( O_{K_p} \)-module with respect to the unramified extension \( L_p/K_p \) (recalling that \( L = K(f) \), \((f, p) = 1 \)). See [35] Lemma 1.10, for example. Then we can naturally fix [31] as

\[
(119) O_{F_{f}}(\mathfrak{f}^{\infty})(F_f[\mathfrak{f}^{\infty}]) = O_{L_p}[X],
\]

where as before \( X = -x/y \) is the formal parameter associated to the previously fixed Weierstrass model of \( A \). (See Definition 4.1.)

**Definition 4.50.** Let \( x \in \mathfrak{p} \) such that \( x \equiv 1 \pmod{f} \). We now fix a canonical \( \mathfrak{p}^{\infty} \)-level structure for \( T_fF_f \) as in [34], p. 9. Given an integral ideal \( a \subset O_K \) prime to \( f \), recall that we have the associated Artin symbol \( \sigma_a = (a, L/K) \in \text{Gal}(L/K) \) and isogeny

\[
[a] : A \rightarrow A/A[a] \cong A^{\sigma_a}.
\]

Then there is \( \Lambda(a, L) \in F \) such that

\[
[a]^\omega A^{\sigma_a} = \Lambda(a, L)\omega_A.
\]

We now let \( a_n := (xp^{-1})^n \) and let

\[
u_n := \Lambda(a_n, L)x^{-n}\Omega - \Lambda(a_n, L)\Omega = \Lambda(p^{-n}, L)(1 - x^n)\Omega.
\]

Let \( A^{(n)} := A^{\sigma_a} \). Then \( \theta_{\infty, A^{(n)}}(u_n) \) is a primitive \( p^n \)-torsion point of \( A^{(n)} \). Let \( m_{O_{C_p}} \subset O_{C_p} \) denote the maximal ideal of \( O_{C_p} \), and let \( \theta_{p, A^{(n)}} \) denote an isomorphism

\[
\theta_{p, A^{(n)}} : \hat{A}^{(n)}(m_{O_{C_p}}) \rightarrow \ker(A^{(n)}(C_p) \rightarrow A^{(n)}(\overline{\mathbb{F}}_p))).
\]

We define a sequence of generators

\[
\alpha_n := \theta_{p, A^{(n)}}^{-1}(\theta_{\infty, A^{(n)}}(u_n)) \in F_{f}[\phi^{-n}f^n], \quad \alpha_0 = 0.
\]
Then \( \phi^{-n} f(\alpha_n) = \alpha_{n-1} \), and \( \alpha_\infty = \lim_{n} \alpha_n : \mathcal{O}_K, \sim \rightarrow T_f F_f \) is a \( p^\infty \)-level structure. We henceforth define all Coleman power series with respect to this \( p^\infty \)-level structure.

**Proposition 4.51** (Propositions 4.1 and 4.2 of [34]). Suppose we are in the situation of Definition 4.11 so that since \( H = L(A[m]), (m, p) = 1, \) we have \( H_p = L_p \). Then in the notation of Theorem 2.2

\[
\text{Col}_{(F_f, \alpha_\infty)}(\xi_0) = \tilde{\psi}_{\Omega, a}(X) \in \mathcal{O}_{L_p}[X]^{\times, N_f = \phi}, \quad \text{Col}_{(F_f, \alpha_\infty)}(\xi_{a, b}) = \tilde{\psi}_{\Omega, a, b}(X) \in \mathcal{O}_{L_p}[X]^{\times, N_f = \phi}.
\]

4.10. **The logarithmic derivative of Coleman power series of elliptic units.** Again, we follow [34 Section 3.5].

**Definition 4.52.** Henceforth, fix the choice of \( \omega_0 \in \Omega_{1, f}^{1, f}/\mathcal{O}_{L_p} \) as in (3.7) to be the \( p \)-adic completion of \( \omega_A \in \Omega_{1, A}^{1, A}/\mathcal{O}_{L} \). Note that

\[
\omega_0 = \log'(X) dX
\]

where \( \log : \hat{A} \rightarrow \hat{\mathbb{G}}_a \) is the formal logarithm over \( L_p \) normalized so that \( \log'(0) = 1 \).

Let

\[
\partial_1 = \frac{d}{\omega_0} = \frac{d}{\log'(t) dt}.
\]

**Definition 4.53.** For shorthand, given \( \beta \in \mathcal{U} \), let

\[
g_\beta = \text{Col}_{(F_f, \alpha_\infty)}(\beta).
\]

Recall by [45], we have

\[
\tilde{\log}(g_\beta) = \log(g_\beta) - \frac{1}{q} \log(g_\beta) \circ f
\]

where \( q = \# F_f[f] \). Then in particular

\[
(120) \quad \partial_1(\log(g_\beta) \circ f) = \Omega_p \frac{f'(t)}{\log'(t)} \frac{g'_\beta}{g_\beta}(t).
\]

**Proposition 4.54.** Suppose that \( g_\beta = \hat{g} \) for some \( g \in \Omega_{1, \mathcal{Q}(\mathcal{O}_L, \omega')}. \) Then we have

\[
d \tilde{\log}(\hat{g}) = \frac{g'}{g} - \frac{f'(0)}{q} \left( \frac{g'}{g} \right)^q = \frac{g'}{g} - \frac{\Lambda(p, L)}{q} \left( \frac{g'}{g} \right)^q \mod p \mathcal{O}_{F_p}[[t]].
\]

**Proof.** This follows immediately from [69] and (120), and the fact that \( f \) lifts the \( q \)-power Frobenius. \( \square \)

**Definition 4.55.** Recall that the function field of \( A/L \) is \( L(x, y) \). Let \( a, b \) be as in Definition 4.11 and let \( m = \mathfrak{f} \) so that \( \mathcal{R}_{\Omega, a, b}^L \) is defined over \( L \). We define

\[
V := \psi_{\Omega, a, b}^L
\]

and

\[
\mathcal{R}_{\Omega, a, b}^L := \left( \frac{V'}{V} - \frac{\Lambda(p, L)}{q} \left( \frac{V'}{V} \right)^q \right) \circ \theta^{-1}_{\infty, A} \in L(x, y).
\]

Let \( \mathfrak{P} \) be the unique prime of \( \mathcal{O}_L \) above \( p \). \( \mathcal{O}_{L,(\mathfrak{P})} \) be the localization of \( \mathcal{O}_L \) at \( \mathfrak{P} \). We have the following commutative diagram of reduction and formal completion maps

\[
\begin{array}{ccc}
\mathcal{O}_{L,(\mathfrak{P})}[x, y]_{(\mathfrak{P})} & \xrightarrow{\text{red}_1} & \mathbb{F}_q(x, y) \\
\downarrow \text{C}_A & & \downarrow \text{C}_A \\
\mathcal{O}_{L_p}[t]_{(\mathfrak{P})} & \xrightarrow{\text{red}_2} & \mathbb{F}_q((t))
\end{array}
\]

(121)
Here, the vertical arrows are induced by expanding \(x\) and \(y\) in terms of the formal parameter \(t\), and the subscripts (\(\mathfrak{p}\)) denote the localizations at the primes \(\mathfrak{p}O_L(\mathfrak{p})[x, y]\) and \(\mathfrak{p}O_{L_p}[t]\).

**Theorem 4.56** (Proposition 5.1 of [34]). Suppose \(\Omega\) is as in Definition 4.11 and \(a, b\) are as in Definition 4.11. Furthermore, suppose we have a set \(\mathcal{D} \subset O_K\) of nonzero elements all prime to \(6p\) and such that the induced map \(\mathcal{D} \to (O_K/\mathfrak{p})\) is injective. Then the reductions

\[
\text{red}_2(\partial_1(\log \hat{\psi}_{d\Omega,a,b}^L)), \quad d \in \mathcal{D}
\]

are linearly independent over \(\mathbb{F}_q\).

**Proof.** This is the same proof as in loc. cit., mutatis mutandis, which we will recall (nearly verbatim) here for convenience. By Propositions 4.51 and 4.54 we have by (121)

\[
\text{red}_2(\partial_1(\log \hat{\psi}_{d\Omega,a,b}^L)) = \text{red}_2 \circ C_A(\mathcal{R}_{d\Omega,a,b}^L) = c_A \circ \text{red}_1(\mathcal{R}_{d\Omega,a,b}^L).
\]

Since \(c_A\) is injective, we only need to show that the \(\text{red}_1(\mathcal{R}_{d\Omega,a,b}^L)\) are linearly independent. Note that these are rational functions on \(A/\mathbb{F}_q\), and one can see that the divisors of \(\text{red}_1(\mathcal{R}_{d\Omega,a,b}^L)\) and \(\text{red}_1(\mathcal{R}_{d\Omega,a,b}^L)\) (computed on p. 12 of loc. cit.) are disjoint unless \(d \equiv d' \pmod{\mathfrak{p}}\). \(\square\)

4.11. **The Iwasawa-Mellin-Leopoldt transform.** We continue the previous notation and the notation of Definition 3.42. Recall that \(L_n = K([\mathfrak{p}]^n), \ G_n = \text{Gal}(L_n/K)\) for \(0 \leq n \leq \infty\). Note that

\[
\varepsilon := [\text{ord}_p(\mathfrak{p})] = \begin{cases} 2 & p = 2 \\ 1 & p > 2 \end{cases}
\]

In particular, we can fix an isomorphism

\[
G_\infty = \text{Gal}(K_\infty^+/K) \times \text{Gal}(L_\infty/K_\infty^+),
\]

where \(K_\infty^+ \subset K(\mu_{p,\infty})\) is the unique subfield with \(\text{Gal}(K_\infty^+/K_\infty) \cong 1 + p^\varepsilon \mathbb{Z}_p\). We have

\[
\text{Gal}(K_\infty^+/K) \cong \text{Gal}(K(\mu_{p,\infty})/K(\mu_{p,\infty})) \cong \text{Gal}(K_p(\mu_{p,\infty})/K_p(\mu_{p,\infty})) = \text{Gal}(L_p(\mu_{p,\infty})/L_p(\mu_{p,\infty}))
\]

\[
\kappa_1 \sim 1 + p^\varepsilon \mathbb{Z}_p \subset \Gamma \subset \mathcal{O}_{K_p}^\infty \cong \text{Gal}(L_p,\infty/L_p).
\]

We follow the exposition in [34, Section 6.2].

**Definition 4.57.** Recall that

\[
G_\infty^+ := \text{Gal}(L(\mu_{p,\infty})/K).
\]

Then (122) induces

\[
G_\infty^+ = \text{Gal}(K_\infty^+/K) \times \text{Gal}(L(\mu_{p,\infty})/K_\infty^+).
\]

Suppose we are given a measure

\[
\mu \in \Lambda(G_\infty^+, \mathcal{O}_{L_p}).
\]

Choose a topological generator \(\gamma\) of \(1 + p^\varepsilon \mathbb{Z}_p\). Fix an isomorphism

\[
\kappa_1 : \text{Gal}(K_\infty^+/K) \sim 1 + p^\varepsilon \mathbb{Z}_p.
\]

Let \(c := \kappa_1(\gamma)\). The logarithm, as before, induces a homeomorphism

\[
\ell : 1 + p^\varepsilon \mathbb{Z}_p \sim \mathbb{Z}_p
\]

defined by \(\ell(c^\alpha) = x\). Let \(\alpha := \ell \circ \kappa : \text{Gal}(K_\infty^+/K) \to \mathbb{Z}_p\). Let \(\chi \in G_\infty\) be any \(p\)-adic character on \(G_\infty\). Then we obtain a measure

\[
\nu := \alpha \in \Lambda(\mathbb{Z}_p, \mathcal{O}_{L_p}).
\]
Using (59), we let
\[ G_\mu(\chi, Y) \in \mathcal{O}_{L_p}[Y] \]
be the power series associated to \( \nu \). If \( \chi = \chi_0 \chi_1 \) where \( \chi_0 \) is trivial on \( \text{Gal}(K_\infty^+/K) \), then
\[ G_\mu(\chi, Y) = G_\mu(\chi_0, \chi_1(\gamma)^{-1}(1 + Y) - 1). \]
Suppose that \( \kappa_*(\chi^{-1} \mu) = \mu'|_{1 + p^r \mathbb{Z}_p} \) where \( \mu' \in \Lambda(Z_p, \mathcal{O}_{L_p}) \) is supported on \( Z_p^\times \). (Here, “\( 1 + p^r \mathbb{Z}_p \)” denotes restriction to the open subgroup \( 1 + p^r \mathbb{Z}_p \subset Z_p^\times \), viewing \( \mu' \in \Lambda(Z_p, \mathcal{O}_{L_p}). \) Then, by definition,
\[ G_\mu(\chi, Y) = \sum_{n=0}^{\infty} \mu' \left( \binom{1 + p^r \mathbb{Z}_p}{x/n} \right) Y^n \]
where \( 1_{1 + p^r \mathbb{Z}_p} \) is the characteristic function of the open subgroup \( 1 + p^r \mathbb{Z}_p \subset Z_p^\times \). Loosely speaking, the power series \( G_\mu(\chi, Y) \) associated to \( \alpha_*(\chi^{-1} \mu) \) is obtained by “restricting the power series associated to \( \mu' \in \Lambda(Z_p^\times, \mathcal{O}_{L_p}) \)
\[ \sum_{n=0}^{\infty} \mu' \left( \binom{x}{n} \right) Y^n \]
to the open subgroup \( 1 + p^r \mathbb{Z}_p \subset Z_p^\times \).

Suppose that \( \mu = \mu^+_\text{glob}(\beta) \) for some system of semilocal units \( \beta \in \mathbb{U} \). Then one can express \( G_\mu(\chi_0, X) \) in terms of the Iwasawa-Leopoldt-Mellin transforms in the sense employed by Gillard [16]. Recall the splitting (58).

**Definition 4.58.** Given a measure \( \mu \in \Lambda(Z_p, \mathcal{O}_{L_p}) \), we associated a measure
\[ \alpha := \sum_{\delta \in \Delta_p} \tau_\delta \ast \mu \]
where \( \tau_\delta : Z_p^\times \to Z_p^\times \) is multiplication by \( \delta \), and \( \Delta_p \) denotes the roots of unity in \( Z_p^\times \), so that
\[ Z_p^\times = \Delta_p \times (1 + p^r \mathbb{Z}_p). \]
We then let \( \text{IML}(\mu) \) denote the power series associated to the measure \( (\ell_\ast \alpha)|_{1 + p^r \mathbb{Z}_p} \), where \( |_{1 + p^r \mathbb{Z}_p} \) denotes restriction to the open subgroup \( 1 + p^r \mathbb{Z}_p \subset Z_p^\times \).

Recall that \( i_p \) from (3) gives a projection \( i_p : \mathbb{U} \to \mathcal{U} \). By construction of \( \mu^+ : \mathcal{U} \to \Lambda(Z_p^\times, \mathcal{O}_{L_p}) \) from (53), we have that \( \mu^+(i_p(\beta)) \in \Lambda(Z_p^\times, \mathcal{O}_{L_p}) \).

**Definition 4.59.** Let \( \mathcal{R} \subset G^+_\infty \) be a full set of representatives of \( G^+_\infty \) modulo \( \text{Gal}(L(\mu_{p^\infty})/L) \).

Note that since \( L(\mu_{p^\infty})/L \) is totally ramified at primes above \( p \), we have the natural identification
\[ \text{Gal}(L(\mu_{p^\infty})/L) = \text{Gal}(L_p(\mu_{p^\infty})/L_p) \]
under the previously fixed embedding \( i_p \) from (3). Then we can write
\[ \mu^+_{\text{glob}} = \sum_{\sigma \in \mathcal{R}} (\tau_\sigma^{-1} \circ \kappa^{-1}) \ast \mu^+(i_p(\sigma(\beta))). \]

**Definition 4.60.** Let \( \omega : Z_p^\times \to \Delta_p \) denote the Teichmüller character, so that \( x = \omega(x)c^\ell(x) \) for any \( x \in Z_p^\times \), and by convention \( \omega(x) = 0 \) for any \( x \in p\mathbb{Z}_p \). Recall the differential operator \( \partial = \frac{d}{dx} = \Omega_p \frac{d}{d\mathbb{Z}_p} = \Omega_p \partial_1 \), and that by (69), the measure associated to \( \partial \mu^+(i_p(\beta))/(Q - 1) \) is \( \kappa \mu^+(i_p(\beta)) \), for any \( \beta \in \mathbb{U} \). Now we let
\[ \mu^1(\beta) := \omega^1 \kappa \mu^+(i_p(\beta)), \quad G^1_\beta(Y) := \text{IML}(\mu^1(\beta))(c^{-1}(1 + Y) - 1). \]
Proposition 4.61 (Lemma 6.1 of [34]). Suppose $\beta \in \mathbb{U}$ and $\chi = \chi_0$ is any $p$-adic character of $G_{\infty}^+$ (with $\chi_1 = 1$). Let $i \in \mathbb{Z}$ such that $\chi^{-1} \circ \kappa^{-1} = \omega^i$. Then

$$G_{\mu_{\text{glob}}(\beta)}(\chi_0, Y) = \sum_{\sigma \in \mathcal{R}} \chi_0(\sigma)(1 + Y)^{-\langle \sigma \rangle} G_{\mu(\sigma(\beta))}^i(Y)$$

where

$$\langle \cdot \rangle = \ell \circ \kappa_1.$$ 

Proof. This is a straightforward computation, using (125) and standard identities involving $G_{\mu}(\chi_0, Y)$ (see the discussion before [34, Lemma 6.1]).

4.12. Definition of the $\mu$-invariant and key theorem (after Sinnott). Let $\zeta_p$ be any primitive $p$-th root of unity. Following the notation of Definition 3.36 given a power series $f(Y)$, we let

$$\tilde{f}(Y) = f(Y) - \frac{1}{p} \sum_{j=0}^{p-1} f(\zeta_p^j(Y + 1) - 1),$$

so that if $f(Y)$ is associated to a measure on $\mathbb{Z}_p$ via (59), then $\tilde{f}(Y)$ is its restriction to $\mathbb{Z}_p^\times$.

Definition 4.62. For any

$$f = \sum_{n=0}^{\infty} a_n Y^n \in \mathcal{O}_{L_p}[Y],$$

we let

$$\mu(f) := \min_{n \in \mathbb{Z}_p} \text{ord}_p(a_n).$$

We call this the $\mu$-invariant of $f$. When $f = G_{\mu}(\chi, X)$ for a measure $\mu$ on $G_{\infty}^+$, we let

$$\mu_{\chi}(\mu) := \mu(f)$$

and call $\mu(\mu)$ the $\mu$-invariant of $\mu$.

Recall the identifications

$$F_{\tilde{f}}[f^{\infty}] = G, \quad \mathcal{O}_{L_p}[X] = \mathcal{O}_{F_{\tilde{f}}[f^{\infty}]}(F_{\tilde{f}}[f^{\infty}]) = \mathcal{O}_{\tilde{G}}(G) = \mathcal{O}_{L_p}[Q - 1], \quad Q - 1 = \theta_{\alpha_0}(X)$$

made in (42). As before, given $a \in \mathbb{Z}_p$, let $[a]_G(Y) = Q^n - 1$.

Definition 4.63. Let $\mathcal{F}_1$ be a finite subset of $\mathbb{Z}_p^\times$ such that for any $(a, b) \in \mathcal{F}_1^2$, if $\zeta a/b \in \mathcal{O}_K^\times$ for some $\zeta \in \Delta_p$, then $a = b$. In other words, $\mathcal{F}_1$ is a set of elements of $\mathbb{Z}_p^\times$ multiplicatively independent over $\Delta_p \cdot \mathcal{O}_K^\times$. Let $\mathcal{F} = \Delta_p \cdot \mathcal{F}_1$, and let $\Delta'_p \subset \Delta_p$ be a full set of representatives of $\Delta_p/(\mu(K) \cap \Delta_p)$.

Theorem 4.64 (cf. Theorem 6.1 in [34]). Let $M$ be any finite extension of $L_p$, and let $f_a$, $a \in \mathcal{F}_1$, be a set of elements belonging to

$$\mathcal{O}_M[X] \cap M(x, y)$$

(the intersection of formal power series in $Q - 1$ over $\mathcal{O}_M$, identified via (119) and (127) with the ring of regular functions on the formal group $\mathbb{A}$ over $\mathcal{O}_M$, and rational functions on $\mathbb{A}/M$). Then we have

$$\mu \left( \sum_{a \in \mathcal{F}_1} IML(f_a \circ [a]_G) \right) = \min_{a \in \mathcal{F}_1} \mu \left( \sum_{\delta \in \Delta_p \cap \mu(K)} \tilde{f}_a \circ [\delta]_G \right).$$

Proof. This is essentially the same proof as of Theorem 6.1 in loc. cit., except we replace "$\mu(k)$" (in the notation of loc. cit.) with $\mu(K) \cap \Delta_p$. We give details on the most important steps here for the convenience of the reader, following loc. cit.
As remarked in loc. cit., by the remark after the statement and proof of [48, Theorem 1], through taking an appropriate average, it suffices to consider the case where \( \bar{f}_a = f_a \) (since this does not change \( IML(f_a \circ [a]_G) \), and \( f_a \circ [\delta]_G = f_a \) for all \( \delta \in \mu(K) \). In this case, we need to prove

\[
(129) \quad \mu \left( \sum_{a \in F_1} IML(f_a \circ [a]_G) \right) = \min_{a \in F_1} \mu(\#(\Delta_p \cap \mu(K))f_a).
\]

Let \( \mu_a \in \Lambda(Z_p, \mathcal{O}_{l_p}) \) denote the measure associated with \( f_a \) (via (59)). For any \( \delta \in \Delta_p \), note that \( \delta(1 + p^rZ_p) \subset Z_p^\times \) is an open subset. Then

\[
(\tau_{\delta, *}\mu_a)|_{1 + p^rZ_p} = \tau_{\delta, *}(\mu_a|_{\delta(1 + p^rZ_p)}).
\]

Now given \( \delta \in \Delta_p \), let \( f_{a, \delta} \) denote the power series associated with \( \mu_a|_{\delta(1 + p^rZ_p)} \). Now we have

\[
(130) \quad \mu \left( \sum_{\delta \in \Delta_p, a \in F_1} f_{a, \delta} \circ [a]_G \right) = \mu \left( \sum_{\delta \in \Delta_p} \sum_{a \in F_1} f_{a, \delta} \circ [a]_G \right) = \mu \left( \#(\Delta_p \cap \mu(K)) \right) \sum_{(\delta,a) \in \Delta'_p \times F_1} f_{a, \delta} \circ [a]_G.
\]

Hence, to prove (128), by (129) and (130), we need to prove

\[
\mu \left( \sum_{(\delta,a) \in \Delta'_p \times F_1} f_{a, \delta} \circ [a]_G \right) = \min_{a \in F_1} \mu(f_a).
\]

Without loss of generality, we may assume that \( \min_{a \in F_1} \mu(f_a) = 0 \).

Assume for the sake of contradiction that \( \mu \left( \sum_{(\delta,a) \in \Delta'_p \times F_1} f_{a, \delta} \circ [a]_G \right) > 0 \). Then by [17 Proposition 1.4.7], we have \( \mu \left( \sum_{(\delta,a) \in \Delta'_p \times F_1} f_{a, \delta} \circ [a]_G \right) > 0 \). By the same (formal) argument as in loc. cit. (which follows the proof of [16 Proposition 2.3.3]), there exists a finite extension \( M' \) of \( M \) such that \( g_a \in \mathcal{O}_{M'}[X] \cap M'(x,y) \). We further finitely extend \( M' \) if necessary to ensure that \( L_p \subset M' \).

Then reducing \( g_a \) modulo the maximal ideal of \( \mathcal{O}_{M'} \), we get

\[
\mu \left( \sum_{(\delta,a) \in \Delta'_p \times F_1} f_{a, \delta} \circ [a]_G \right) = 0
\]

where \( f_{a, \delta} \) denotes the reduction of \( f_{a, \delta} \), and \([\cdot]_A\) denotes multiplication in the group law on the reduction \( \bar{A} \) of \( A \) (using the previously fixed Weierstrass model). Hence, by [17 Théorème 1.1.1], for any \( \delta' \in \Delta'_p \) we have that \( \sum_{\delta \in \mu(K), \delta'} f_{a, \delta} \circ [a\delta]_G \) is constant, and so \( f_{a, \delta} \) is constant. Letting \( \pi_{M'} \) denote a local uniformizer of \( \mathcal{O}_{M'} \) and letting \( \nu_0 \) denote the Dirac measure at 0, as remarked in [34] Proof of Theorem 6.1) (see also [17 Proof of Théorème 1.5.1]), it follows from an argument of Sinnott that for any \( \delta \in \mu(K) \cdot \delta' \),

\[
\mu_a|_{\delta(1 + p^rZ_p)} = \overline{c_a}\nu_0 + \pi_{M'}\mu_a'
\]

where \( c_a \in \mathcal{O}_{M'} \), \( \overline{c_a} \) is its reduction, and for some \( \mathcal{O}_{M'} \)-valued measure \( \mu'_a \) on \( Z_p \). The support of the left-hand-side of the above equality is \( \delta(1 + p^rZ_p) \), while \( \nu_0 \) is supported on \( \{0\} \), which is only possible if \( c_a \equiv 0 \pmod{\pi_{M'}} \). Now since \( \mu_a \) is invariant by \([\delta]_G \) for \( \delta \in \mu(K) \), we see that \( \mu_a|_{\mu(K) \cdot \delta(1 + p^rZ_p)} = \mu_a|_{\mu(K) \cdot \delta(1 + p^rZ_p)} \equiv 0 \pmod{\pi}_{M'} \). Since this holds for any \( \delta' \in \Delta'_p \), then get \( \mu_a \equiv 0 \pmod{\pi}_{M'} \). Now since the above argument holds for any \( a \in F_1 \), this contradicts the assumption \( \min_{a \in F_1} \mu(f_a) = 0 \), and we are done.
4.13. Vanishing of the (analytic) \(\mu\)-invariant. We now apply the linear independence result Theorem 4.56 and Theorem 4.64 to measures arising from elliptic units, in order to show that their \(\mu\)-invariants are 0. We continue to follow [34].

**Definition 4.65.** Let \(I\) be a set of integral ideals of \(\mathcal{O}_K\) which are prime to \(fp\), and which are in bijection with the class group \(\text{Gal}(K(1)/K)\) via the Artin reciprocity map. For each \(j \in \{0, \ldots, s - 1\}\), we let \(O_j \subset \mathcal{O}\) be the subset consisting of \(c \in I\) such that \(\langle \sigma_c \rangle \equiv j \pmod{s}\) (in the notation of (126)), where \(\sigma_c := (c, L(\mu_{p^\infty})/K)\).

Henceforth, let \(\Xi \subset \mathcal{O}_K\) denote a full set of representatives of \((\mathcal{O}_K/f)^\times \pmod{\mu(K)}\).

**Theorem 4.66** (cf. Proposition 6.1 of [34]). Let \(a, b \subset \mathcal{O}_K\) be proper integral ideals of \(\mathcal{O}_K\). Recall \(\mu_{\text{glob}}^+\) as defined in (88), \(\xi_{a,b}\) as defined in Definition 4.11, and the \(\mu\)-invariant \(\mu\) as defined in Definition 4.62. We have

\[
\mu(G_{\mu_{\text{glob}}^+(\xi_{a,b})}(\chi_0, X)) = 0, \quad \mu_{\chi_0}(\mu_{\text{glob}}^+(\xi_{a,b})) = 0.
\]

**Proof.** The argument is essentially the same as in loc. cit. For the convenience of the reader, we give details for the main steps. Let \(R'\) be a full set of representatives of \(\text{Gal}(L(\mu_{p^\infty})/K(1))/(\mod{I_j})\), where \(I_j := \text{Gal}(L(\mu_{p^\infty})/L)\). Now choose \(R\) as in (4.59), so that \(R = \bigsqcup_{j=0}^{s-1} R_j\), where \(R_j = I_j R'\). By Lemma 4.61, we have

\[
G_{\mu_{\text{glob}}^+(\xi_{a,b})}(\chi_0, X) = \sum_{j=0}^{s-1} (1 + X)^{-j} S_j((1 + X)^j - 1)
\]

where for \(0 \leq j \leq s - 1\),

\[
S_j(x) = \sum_{\sigma \in R_j} \chi_0(\sigma)(1 + X)^{(j - (\sigma)) / s} G_{\mu_{\text{glob}}^+(\xi_{a,b})}((1 + X)^j).
\]

Now by [17] Lemma 2.10.2], we are reduced to showing that \(S_0\) is coprime with \(p\), or equivalently that \(S_0(c(1+X) - 1)\) is coprime with \(p\). By the same formal reductions as in [34] Proof of Proposition 6.1], by Theorem 4.64 and Lemma 4.61, it suffices to prove that for some \(\delta \in I_0\), we have

\[
\chi_0(\sigma_0) e^{- (\sigma_0) / s} \left( \sum_{\delta \in \mu(K) \cap \Delta_p, b \in \Xi} \chi_0(\sigma_b) \partial(\log^L_{\mu_{\text{glob}}^+(\xi_{a,b})} \circ [\delta]_A) \right)
\]

is coprime with \(p\), where \([\cdot]_A\) denotes multiplication in the group law on \(A\). Now note that the set of elements \(\{\delta b\} = (\mu(K) \cap \Delta_p) \Xi \subset \mathcal{O}_K\) maps injectively into \((\mathcal{O}_K/f)^\times\), and so Theorem 4.56 shows that the above power series has at least one coefficient coprime with \(p\). Hence we are done. 

We now can show vanishing of the cyclotomic analytic \(\mu\)-invariant.

**Corollary 4.67.** Recall the notation of Definition 4.43. Then for any \(\chi \in \Delta'\), we have

\[
\mu_{\text{inv}}(g_{\chi}^+) = 0.
\]

**Proof.** This follows since by construction,

\[
(\sigma_b - N(b)) g_{\chi}^+ = \chi((\sigma_b - N(b)) \mu_{\text{glob}}^+(\xi_{a,b}^+)) = \chi(((\sigma_b - N(b)) \mu_{\text{glob}}^+(\xi_b)),
\]

and the right-hand side has \(\mu_{\chi}(\mu_{\text{glob}}^+(\xi_b)) = 0\) by (131). 

□

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4.14. **Vanishing of the (algebraic) \( \mu \)-invariant.** We shall now prove the vanishing of the cyclotomic algebraic \( \mu \)-invariant, using Corollary 4.67 and (117).

**Corollary 4.68.** Recall the notation of Definition 4.43. Then for any \( \chi \in \hat{\Delta}' \), we have
\[
\mu_{\text{inv}}(f^+_\chi) = 0.
\]
In particular, \((\lambda')^+\) has no non-trivial \(\mathcal{O}_{L_p}\)-torsion, and so is finitely \(\mathcal{O}_{L_p}\)-generated.

**Proof.** This follows immediately from (117) and Corollary 4.67. \(\square\)

4.15. **Proof of the Main Conjecture from vanishing of the cyclotomic algebraic \( \mu \)-invariant.** In this section, we prove the Main Conjecture (Conjecture 4.32) from the result of [25], using as input the vanishing of the cyclotomic algebraic \( \mu \)-invariant

**Theorem 4.69.** When \( \omega_1 = 1 \), we have that (107) is true, and (108) is true for any \( \chi \in \hat{\Delta}' \). When \( \omega_1 \neq 1 \), we have that these statements are true over \( \Lambda(G_\infty, \mathcal{O}_{L_p})[1/p] \) after tensoring all modules and rings with \( \otimes_{Z_p} \mathbb{Q}_p \).

**Proof.** Suppose first \( \omega_1 = 1 \). This is immediately given by the results of [25], extending all coefficients by \( \otimes_{Z_p} \mathcal{O}_{L_p} \) whenever appropriate, and our results on the vanishing of the algebraic \( \mu \)-invariant. By Corollary 4.68, we have that \((\lambda')^+\) is a finitely generated \(\mathcal{O}_{L_p}\)-module. Recall that \(N^+_n\) is the maximal unramified pro-\(p\) abelian extension of \(L^+_n\), and \(\mathcal{Y}^+ = \lim_{\rightarrow} \text{Gal}(N^+_n/L^+_n)\). Then by the fundamental exact sequence, we have a surjection \((\lambda')^+ \to \mathcal{Y}^+ \otimes_{Z_p} \mathcal{O}_{L_p}\), and so \(\mathcal{Y}^+ \otimes_{Z_p} \mathcal{O}_{L_p}\) is a finitely-generated \(\mathcal{O}_{L_p}\)-module, and hence \(\mathcal{Y}^+\) is a finitely-generated \(\mathbb{Z}_p\)-module. Now Corollary 5.10 and Lemma 5.11 of loc. cit. imply Conjecture 5.5 of loc. cit., which by [25, Section 5.2], gives the Theorem. When \( \omega_1 \neq 1 \), the statement follows from the diagram (90) and the definition of \( \mu_{\text{glob}}(g) \).

**Corollary 4.70.** If \( \omega_1 = 1 \), we have that (107) is true. Moreover (108) is true for any \( \chi \in \hat{\Delta}' \). If \( \omega_1 \neq 1 \), we have that these statements are true over \( \Lambda(G_\infty, \mathcal{O}_{L_p})[1/p] \) after tensoring all modules and rings with \( \otimes_{Z_p} \mathbb{Q}_p \).

**Proof.** This follows immediately from Theorem 4.69 and Proposition 4.33 (tensoring with \( \otimes_{Z_p} \mathbb{Q}_p \) if \( \omega_1 \neq 1 \)). \(\square\)

5. **Descent, Rank 0 converse and results on BSD**

Let \( K \) be an imaginary quadratic field of class number 1, and \( E/K \) any elliptic curve with CM by \( \mathcal{O}_K \). Denote its associated algebraic Hecke character of type \((1,0)\) by \( \lambda \) and denote its conductor by \( g \). We now show the results of the previous section to \( L = K(1) = K \) (see Proposition 4.7, \( L_n = K(p^n) \), so that \( G_\infty = \text{Gal}(L_\infty/K) \cong \Gamma \times \Delta \), where \( \Gamma \cong \text{Gal}(L_\infty/L') \) and \( \Delta = \text{Gal}(L'/K) \). Note that \( K(p^n) = K(E[p^n]) \) (where the last equality follows from the theory of complex multiplication). Let \( \lambda : \text{Gal}(L_\infty/K) \to \mathcal{O}_{K_p}^{\times} \) be the local reciprocity character, and let \( \chi_E : \text{Gal}(L'/K) \to \mathcal{O}_{K_p}^{\times} \) be its restriction to \( \Delta \).

Descent in the rank 0 case essentially entails replacing the map “\( \delta \)” in [30, Section 11.4] with the map \( (\lambda/\chi_E)^{-1} \mu^1 : (U^1)_{\chi_E} \to \Delta(\text{Gal}(L_{p,\infty}/K_p), \mathcal{O}_{L_p})_{\chi_E} \), and following the same arguments. In essence, \( (\lambda/\chi_E)^{-1} \mu^1 \) is a \( \Lambda(\text{Gal}(L_{p,\infty}/K_p), \mathcal{O}_{L_p})_{\chi_E} \)-adic lifting of \( \delta \), which is itself simply the “first moment” of the explicit local reciprocity law.
5.1. The Selmer group via local class field theory.

**Definition 5.1.** Let \( T(\lambda) := \mathcal{O}_{L_p}(\lambda), V(\lambda) := T(\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, W(\lambda) := V(\lambda)/T(\lambda) = T(\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p. \) We define

\[
S_n(\lambda) = \ker \left( \prod_v \text{loc}_v : H^1(L(E[p^n]), W(\lambda)) \to \prod_v H^1(K(E[p^n]), E) \right)
\]

where the product runs over all places of \( K(E[p^n]) \). Let \( S_{\infty}(\lambda) = \lim_{\rightarrow} S_n(\lambda) \). In particular, we see that \( S_{\infty}(\lambda) \) is a \( \Lambda(G_\infty, \mathcal{O}_{L_p}) \)-module (note that \( \mathcal{O}_{L_p,\lambda} = \mathcal{O}_{L_p} \)). Following the notation of [40], we define a relaxed Selmer group

\[
S'_n(\lambda) = \ker \left( \prod_{v\mid p} \text{loc}_v : H^1(K(E[p^n]), W(\lambda)) \to \prod_{v\mid p} H^1(K(E[p^n]_v), E) \right),
\]

where the product runs over all places of \( K(E[p^n]) \) not dividing \( p \).

**Definition 5.2.** Given a group \( H \) and an \( H \)-module \( M \), let \( M^H \) denote the invariants by \( H \).

**Proposition 5.3** (Theorem 11.2 of [40]). For all \( n \geq 2 \), the restriction map \( H^1(K_n, W(\lambda)) \to H^1(K_\infty, W(\lambda)) \) induces isomorphisms

\[
S'_n(\lambda)^{\Delta} \cong \Hom(\mathcal{X}, W(\lambda))^{\Gal(K(E[p^n])/K(E[p^n])) \times \Delta} = \Hom(\mathcal{X}_{\chi_E}, W(\lambda))^{\Gal(K(E[p^n])/K(E[p^n])) \times \Delta},
\]

**Proof.** This is well-known. See [40] Proof of Theorem 11.2 for an outline with detailed references. \( \square \)

**Lemma 5.4** (cf. Lemma 11.6 of [40]). We have an isomorphism

(132) \[ H^1(K_p(E[p^\infty]), W(\lambda))^{\Delta} \cong \Hom((\mathcal{U}^1)^{\chi_E}, W(\lambda))^{\Delta} \]

as \( \Gal(K_p(E[p^\infty])/K_p(E[p^\infty])) \)-modules.

**Proof.** Since \( \Gal(K_p/K_p) \) acts on \( W(\lambda) = E[p^\infty] \) through \( \lambda \), we have that \( \Gal(K_p(E[p^\infty])/K_p(E[p^\infty])) \) acts trivially on \( W(\lambda) \), and so

(133) \[
H^1(K_p(E[p^\infty]), W(\lambda))^{\Delta} = \Hom(\Gal(K_p(E[p^\infty])/K_p(E[p^\infty])), W(\lambda))^{\Delta}
= \Hom(\Gal(K_p(E[p^\infty]))^{ab}/K_p(E[p^\infty]))^{\chi_E}, W(\lambda))^{\Delta}.
\]

By local class field theory, since \( K = K(1) \), we have an isomorphism

(134) \[ \mathcal{U}^1 \times \mu(\mathcal{O}_{K_p}) \times \hat{\mathbb{Z}} = \mathcal{U} \times \hat{\mathbb{Z}} \cong \Gal(K_p(E[p^\infty])^{ab}/K_p(E[p^\infty])). \]

By Lemma [40] 11.5, we have that \( \chi_E \) is non-trivial on the decomposition subgroup of \( \Delta \) above \( p \), and so \( \hat{\mathbb{Z}}_{\chi_E} = 1 \). Clearly, \( \Gal(K_p(E[p^\infty])/K_p) \) acts trivially on \( \mu(\mathcal{O}_{K_p}) \), and so \( \mu(\mathcal{O}_{K_p})^{\chi_E} = 1 \). Hence (134) gives an isomorphism

(135) \[ (\mathcal{U}^1)^{\chi_E} \cong \Gal(K_p(E[p^\infty])^{ab}/L_p,\infty)^{\chi_E}. \]

This, along with (133), gives (132). \( \square \)

Let \( M_n/K(E[p^n]) \) denote the maximal pro-\( p \) abelian extension of \( K(E[p^n]) \) unramified outside of places above \( p \). Let \( M_{p,\infty} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} M_n \) and \( \mathcal{X}_p = \Gal(M_{p,\infty}/K_p(E[p^\infty])) \). The \( \chi_E \)-part \( (\mathcal{U}^1)^{\chi_E} \mapsto (\mathcal{X}_p)^{\chi_E} \) of the local reciprocity map \( \mathcal{U}^1 \mapsto \mathcal{X}_p \) induces a map

\[
\Hom((\mathcal{X}_p)^{\chi_E}, W(\lambda)) \to \Hom((\mathcal{U}^1)^{\chi_E}, W(\lambda)).
\]
We denote this map by $f \to f|_{(\mathcal{U}^1)_{XE}}$. We have the following commutative diagram (cf. [40, p. 62]):

$$(136) \quad \begin{array}{c}
\text{Hom}((\lambda)^{\infty}, W(\lambda)) \\
\downarrow \beta \\
\phi \downarrow \\
\text{Hom}((\mathcal{U}^1)_{XE}, W(\lambda))
\end{array}
$$

where the middle row is exact by the local descent exact sequence. Given $f \in \text{Hom}((\lambda)^{\infty}, W(\lambda))$, denote by $f_p$ the image under the natural map $\text{Hom}((\lambda)^{\infty}, W(\lambda)) \to \text{Hom}((\lambda)^{\infty}, W(\lambda))$ given by the natural inclusion $X_p \subset X$ (induced by $i_p$ from [40]). By Proposition 5.3 and (136), we have

$$(137) \quad S^{\infty}(\lambda) = \{ f \in \text{Hom}((\lambda)^{\infty}, W(\lambda)) : f_p|_{(\mathcal{U}^1)_{XE}} \in \text{im}(\phi) \}.
$$

We have a map $(\mu_{XE}) := (\mu^1_{glob})_X : (\mathcal{U}^1)_{XE} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \Lambda(\text{Gal}(E[p^\infty]), E_{Kr})_{XE}[1/p]$ induced by taking the $\chi_E$-component of $[89]$ (with $\mathfrak{f} = 1$, so that $\mathcal{U}^1 = \mathcal{U}^1$).

**Definition 5.5.** Given a measure $\mu \in \Lambda(G, R)$ and a continuous function $f$ on $G$, we define the twist of $\mu$ by $f$ by

$$f^*\mu(g) = \mu(fg)$$

for any continuous function $g$ on $G$.

**Definition 5.6.** Define

$$(138) \quad \delta := (\lambda/\chi E)^* (\mu_{XE}) : (\mathcal{U}^1)_{XE} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \Lambda(\text{Gal}(E[p^\infty]), E_{Kr})_{XE}[1/p].
$$

Here, recall our notation for pullback: for any function $f$ on $\text{Gal}(E_{Kr}/K_p)$, and any $u \in (\mathcal{U}^1)_{XE}$,

$$(\lambda/\chi E)^* (\mu_{XE})(u)(f) = (\mu_{XE})(u)((\lambda/\chi E)f).$$

5.2. **Interlude: technical lemmas relating $\mathcal{U}^1$ to good reduction twists.** Recall the fixed CM elliptic curve $A/L$ from Definition 4.1 so that $u_1 = 1$, $(\mathfrak{f}, p) = 1$, $L = K(\mathfrak{f})$, and $F_\mathfrak{f} = A$ is a relative Lubin-Tate group for the unramified extension $L_p/K_p$. Henceforth make the following choice.

**Choice 5.7.** Recalling that $\mathfrak{g}$ is the conductor of $\lambda$, let $\mathfrak{g}_0$ denote the prime-to-$p$ part of $\mathfrak{g}$. Choose $\mathfrak{f}$ as above with $\mathfrak{g}_0/\mathfrak{f}$.

This ensures that $K(E[p^n]) \subset K(\mathfrak{f}p^n) = L(A[p^n])$ (where the first equality follows because $K(E[p^n])/K$ is ramified at primes dividing $\mathfrak{g}p$, and the last equality follows from [45] Proposition II.1.16). Denote the associated tower of local units by $\mathcal{U}'$, and the associated tower of principal local units by $(\mathcal{U}')^1$. Recalling that $\mathcal{U}^1$ denotes the tower of local units attached to $E$, by the previous sentence, we have $\mathcal{U}^1 \subset (\mathcal{U}')^1$. Moreover, the norm from $L(A[p^n]) \to K(E[p^n])$ induces a norm map $Nm : (\mathcal{U}')^1 \to \mathcal{U}^1$. Denote the type $(1, 0)$ Hecke character associated with $A$ by $\lambda_A$, and viewed as a Galois character on $\text{Gal}(L(A[p^\infty])/K)$ let $\chi_A = \lambda_A|_{\text{Gal}(L(A[p^\infty])/K_\infty)}$, where $K_\infty/K$ denotes the unique $\mathbb{Z}_p^{\infty}$-extension of $K$.

**Proposition 5.8.** We have

$$(139) \quad \lambda/\chi E = \lambda_A/\chi A$$

as characters $\text{Gal}(K_\infty/K) \to \mathcal{O}_{K_p}^\times$.
Proof. By the theory of complex multiplication (Hilbert Theorem 5.11), we see that $\lambda/\chi_E$ and $\lambda_A/\chi_A$ both map $\text{Gal}(K_\infty/K)$ into $1 + p^r \mathcal{O}_{K_p}$ with finite cokernel. Moreover, since $\lambda$ and $\lambda_A$ differ by a finite-order character (both are of infinite type $(1,0)$), we have that $\lambda/\chi_E$ and $\lambda_A/\chi_A$ differ by a finite-order character. Since the images of both $\lambda/\chi_E$ and $\lambda_A/\chi_A$ lie in the torsion-free group $1 + p^r \mathcal{O}_{K_p}$, this finite-order character must be trivial, and so we have $\lambda/\chi_E = \lambda_A/\chi_A$ on all of $\text{Gal}(K_\infty/K)$.

\section*{5.3. Description of the Selmer group via Wiles’s explicit reciprocity law.}

\begin{proposition} \label{prop:selmer_description}
Let
\[
\delta' : \mathcal{U}^1 \to \text{Hom}(E(K_{p,\infty}) \otimes_{\mathcal{O}_{K_p}} K_p/\mathcal{O}_{K_p}, W(\lambda)), \quad \delta'_A : (\mathcal{U}')^1 \to \text{Hom}(A(L_{p,\infty}) \otimes_{\mathcal{O}_{K_p}} K_p/\mathcal{O}_{K_p}, W(\lambda_A))
\]
be the natural Kummer maps precomposed with the local Artin maps (cf. [45 Chapter 1.4]). Note that since $A(L_{p,\infty}) = E(L_{p,\infty})$, there is a trace map $\text{Tr}_{L_{p,\infty}/K_{p,\infty}} : A(L_{p,\infty}) = E(L_{p,\infty}) \to E(K_{p,\infty})$. There is commutative diagram of $\Lambda(\text{Gal}(K_\infty/K), \mathcal{O}_{K_p}[1/p])$-equivariant maps (cf. (90))
\[
\begin{array}{ccc}
\mathcal{U}^1 & \xrightarrow{\delta'} & \text{Hom}(E(K_{p,\infty}) \otimes_{\mathcal{O}_{K_p}} K_p/\mathcal{O}_{K_p}, W(\lambda)) \\
\downarrow & & \downarrow \text{Tr}_{L_{p,\infty}/K_{p,\infty}} \\
(\mathcal{U}')^1 & \xrightarrow{\delta'_A} & \text{Hom}(A(L_{p,\infty}) \otimes_{\mathcal{O}_{K_p}} K_p/\mathcal{O}_{K_p}, W(\lambda_A)) \\
\downarrow \text{Nm} & & \downarrow \text{incl} \\
\mathcal{U}^1 & \xrightarrow{\delta'} & \text{Hom}(E(K_{p,\infty}) \otimes_{\mathcal{O}_{K_p}} K_p/\mathcal{O}_{K_p}, W(\lambda)),
\end{array}
\]
where the first right vertical arrow is pullback by $\text{Tr}_{L_{p,\infty}/K_{p,\infty}} : A(L_{p,\infty}) \to E(K_{p,\infty})$, the second right vertical arrow is pullback by the inclusion $\text{incl} : E(K_{p,\infty}) \subset E(L_{p,\infty}) = A(L_{p,\infty})$, and the left vertical arrows and right vertical arrows are isomorphisms of $\Lambda(\text{Gal}(K_\infty/K), \mathcal{O}_{K_p}[1/p])$-modules.

Proof. This follows immediately from the restriction/corestriction functorialities of the local reciprocity map.
\end{proposition}

\section*{5.3. Description of the Selmer group via Wiles’s explicit reciprocity law.}

\begin{proposition} \label{prop:selmer_description2}
Let
\[
\delta W(\lambda) = \delta E[p^\infty] := \{ \text{maps } u \mapsto \delta(u)v : v \in W(\lambda) = E[p^\infty] \}.
\]
We have
\[
\text{im}(\phi)^G \supset \delta W(\lambda)^G = \text{Hom}((\mathcal{U}^1)_E/\ker(\delta), W(\lambda))^G
\]
with the left-hand side containing the right-hand side with index a finite power of $p$.
\end{proposition}

Proof. The equality in (141) follows immediately from the definitions. When $E$ has good reduction, the containment is a consequence of the explicit local reciprocity law of Wiles as stated in [45 Theorem 1.4.2]. Indeed, first assume that $E/\mathcal{O}_{K_p}$ has good reduction. We have a natural map $\delta' : (\mathcal{U}^1)_E \to \text{Hom}(E(L_{p,\infty}) \otimes_{\mathcal{O}_{K_p}} K_p/\mathcal{O}_{K_p}, W(\lambda))_E$ given by the Kummer pairing. Taking $G_\infty$-invariants, by loc. cit. we have
\[
(\delta')^G = ((\beta_n)_{n \in \mathbb{Z}_{\geq 0}} = \left( x \otimes \pi^{-k} \mapsto \left[ \frac{d \log(\beta_0)}{\log E(x)} \right]_E (\alpha_k) \right).
\]
Using the fact that $\theta_{\alpha_0}^* Q_d = \Omega_p \cdot\frac{d}{\log E(x)}$ by (93), we have that $\ker(\delta^G) \supset \ker((\delta')^G)$. Now (141) and the finite index assertion follow after noting that $\log_E : E[p^r \mathcal{O}_{K_p}] \to p^r \mathcal{O}_{K_p}$ for all $r \gg 0$. 49
Now if $E/\mathcal{O}_{K_p}$ has bad reduction, using (140) and the above argument for a good reduction twist $A/\mathcal{O}_{K_p}$, we again see that $\ker(\delta^{G_{\infty}}) \supset \ker((\delta')^{G_{\infty}})$, which gives (141) in this case. \hfill \square

A key consequence of (141) is the following.

**Theorem 5.11.** There is a natural containment
\begin{equation}
S_0(\lambda)^{G_{\infty}} \supset \text{Hom}((\mathcal{X'})_{\chi_E}, W(\lambda))^{G_{\infty}}
\end{equation}
where the left-hand side contains the right-hand side with finite $p$-power index, and where $\mathcal{X}'$ is defined as in Definition 4.43. Hence,
\begin{equation}
\#S_0(\lambda)^{G_{\infty}} \sim \#\text{Hom}((\mathcal{X'})_{\chi_E}, W(\lambda))^{G_{\infty}}
\end{equation}

where “$\sim$” denotes equality up to a finite power of $p$. (Here, we follow the usual convention that one side is infinite if an only if the other side is infinite.)

**Proof.** This follows immediately from (141) and (137). \hfill \square

**Proposition 5.12.** We have
\begin{equation}
\#(\text{Hom}((\mathcal{U'})_{\chi_E}, W(\lambda))^{\text{Gal}(L_{\infty}/K)}) \sim \#(\text{Hom}((\mathcal{U'})_{\chi_E}, W(\lambda))^{\text{Gal}(L_{\infty}/K)}).
\end{equation}

**Proof.** This follows from the same arguments as in [40, Proof of Theorem 11.16], using the Rubin-type Main Conjecture (106) for $\chi = \chi_E$ and the fact that $\mathcal{X}$ has no nonzero pseudo-null submodules by [20, Theorem 2]. \hfill \square

Theorem 5.11 has the following consequence for rank 0 BSD for elliptic curves $E/\mathbb{Q}$ with CM by $\mathcal{O}_K$.

**Theorem 5.13.** In the setting of Theorem 5.11, write
\[ \mathcal{L} = \frac{L(\overline{\lambda}, 1)}{\Omega_{\infty}}, \]
where $\Omega_{\infty}$ is the usual Néron period associated with $E$, and in particular $\mathcal{L} \in \mathbb{Q}$. Then we have
\begin{equation}
\#S_0(\lambda) \sim \#(\mathcal{O}_{K_p}/\mathcal{O}_{K_p}),
\end{equation}
where “$\sim$” denotes equality up to a finite power of $p$. In particular,
\[ L(E/\mathbb{Q}, 1) = L(\overline{\lambda}, 1) \neq 0 \iff \#S_0(\lambda) < \infty. \]

**Proof.** We have
\begin{equation}
\#S_0(\lambda) \sim \#(\text{Hom}((\mathcal{X'})_{\chi_E}, W(\lambda))^{G_{\infty}})
\end{equation}
\begin{equation}
\sim \#(\text{Hom}((\mathcal{U'})_{\chi_E}, W(\lambda))^{G_{\infty}}).
\end{equation}

Now we have
\[ \#(\text{Hom}((\mathcal{U'})_{\chi_E}, W(\lambda))^{G_{\infty}}) = \#\{ \phi \in \text{Hom}((\mathcal{U'})_{\chi_E}, W(\lambda))^{G_{\infty}} : \phi(\mathcal{C}(g)_{\chi_E}) = 0 \} \]
\[ = \{ v \in W(\lambda) : ((\lambda/\chi_E)^* \mu_{\text{glob}}(g; \chi_E))v = 0 \} \]
\[ = \{ v \in W(\lambda) : \left( 1 - \frac{\lambda(p) \mathcal{N}(p)}{N(p)} \right) \mathcal{L}v = 0 \} \]
\[ = \#(\mathcal{O}_{K_p}/\left( 1 - \frac{\lambda(p) \mathcal{N}(p)}{N(p)} \right) \mathcal{L} \mathcal{O}_{K_p}). \]

Now (145) immediately follows. \hfill \square

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Remark 5.14. Note that \( S_0(\lambda) = \text{Sel}_p\infty(E/\mathbb{Q}) \). Hence Theorem 5.13 implies that if \#\text{Sel}_p\infty(E/\mathbb{Q}) < \infty, i.e. corank_{\mathbb{Z}_p}(\text{Sel}_p\infty(E/\mathbb{Q})) = 0, then \( L(E/\mathbb{Q}, 1) = L(\lambda, 1) \neq 0 \), i.e. \( \text{ord}_s = 1 L(E/\mathbb{Q}, s) = 0 \). As mentioned before, it is possible to make the constant \( c(E) \) explicit, and hence with more work establish the \( p \)-part of the BSD formula for \( E/\mathbb{Q} \) in this case.

Together with the \( 2\infty \)-Selmer distribution results of Smith [49], we have the following result establishing that 50% of quadratic twists of certain elliptic curves with CM by \( K = \mathbb{Q}(i) \) and with full rational 2-torsion have analytic rank 0. Given an elliptic curve \( E \) mentioned before, it is possible to make the constant \( c(E) \) explicit, and hence with more work establish the \( p \)-part of the BSD formula for \( E/\mathbb{Q} \) in this case.

Corollary 5.15. Suppose \( E/\mathbb{Q} \) has full rational 2-torsion (i.e. \( E[2] = E[2](\mathbb{Q}) \)), that \( E \) admits no cyclic rational 4-isogeny (i.e. \( E(\mathbb{Q}) \) does not contain a cyclic group of order 4), and that \( E \) has CM by \( \mathcal{O}_K \) for \( K = \mathbb{Q}(i) \). We have that

\[
\lim_{X \to \infty} \frac{\# \{ 0 < |d| < X : d \text{ squarefree}, \text{ord}_s = 1 L(E^d/\mathbb{Q}, s) = 0 \}}{\# \{ 0 < |d| < X : d \text{ squarefree} \}} = \frac{1}{2}.
\]

**Proof.** By the Selmer distribution results of [49], we have that corank_{\mathbb{Z}_2}(\text{Sel}_2\infty(E^d/\mathbb{Q})) = 0 for 50% of fundamental discriminants \( d \) (with respect to the ordering by increasing \( |d| \)). Now the assertion follows from Theorem 5.13. \( \square \)

In particular, the above theorem has the following consequence for the congruent number family \( E^d : y^2 = x^3 - d^2 x \).

Corollary 5.16. Consider \( E^d : y^2 = x^3 - d^2 x \). Then \( \text{ord}_s = 1 L(E^d/\mathbb{Q}, s) = 0 \) for 100% of squarefree \( d \equiv 1, 2, 3 \pmod{8} \), ordering by increasing \( |d| \).

**Proof.** This follows from the result of [49] that corank_{\mathbb{Z}_2}(\text{Sel}_2\infty(E^d/\mathbb{Q})) = 0 for 100% of squarefree \( d \equiv 1, 2, 3 \pmod{8} \). \( \square \)

Remark 5.17. This verifies one half of Goldfeld’s conjecture (Conjecture 8.8) for the congruent number family, i.e. that “50% of quadratic twists of \( E : y^2 = x^3 - x \) have analytic rank 0”. We will later show (Corollary 8.9) that the other half of Goldfeld’s conjecture is true for \( E \), i.e. that “50% of quadratic twists of \( E \) have analytic rank 1”. This is the first instance of the verification of Goldfeld’s conjecture for any quadratic twist family for an elliptic curve over \( \mathbb{Q} \).

6. Factorizations of \( p \)-adic \( L \)-functions and Selmer groups

In this section, we show a factorization along the anticyclotomic line of the \( p \)-adic \( L \)-function of a certain CM form into two \( p \)-adic Hecke \( L \)-functions, which are special cases of the construction from the previous sections. This factorization will be crucial for relating the Iwasawa theory of elliptic units to the Iwasawa theory of Heegner points, and hence for obtaining rank 1 converse theorems.

Henceforth, let \( E/\mathbb{Q} \) be an elliptic curve with CM by an imaginary quadratic field \( K \), so that \( K \) has class number 1. Let \( \lambda \) be the Hecke character over \( K \) associated with \( E/K \), of infinity type \((1,0)\), and \( \theta_\lambda \) its associated theta series. Let \( \chi \) be any Hecke character over \( K \), and let \( \chi^*(x) = \chi(\bar{x}) \) (and as before \( \bar{x} \) denotes the complex conjugate of \( x \)). Now let \( g = \theta_\psi \), and let \( F = \mathbb{Q}(g, \chi) \) denote the finite extension of \( \mathbb{Q} \) generated by the Hecke eigenvalues of \( g \) and by the values of \( \chi \). Note that we have the following compatibility of the central character \( w_g \) of \( g \) and \( \chi \):

\[
w_g \cdot \chi|_{\mathbb{A}_K^\times} = 1.
\]

Note that we have the factorization of \( L \)-series

\[
L(g \times \chi, s) = L(\lambda, s)L(\psi^* \chi, s)
\]

(147)
Definition 6.1. Let $V_g$ denote the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation associated with $g$, so that we have
\begin{equation}
V_g|_{\text{Gal}(\overline{\mathbb{K}}/K)} = F_p(\psi) \oplus F_p(\psi^*), \quad V_{g,\chi} := V_g|_{\text{Gal}(\overline{\mathbb{K}}/K)} \otimes \mathcal{O}_{F_p} \chi = F_p(\lambda) \oplus F_p(\psi^*\chi).
\end{equation}
Let $T_g \subset V_g|_{\text{Gal}(\overline{\mathbb{K}}/K)}$ and $T_{g,\chi} \subset V_{g,\chi}$ be $\text{Gal}(K/K)$-invariant lattices such that
\[ T_g \cong \mathcal{O}_{F_p}(\psi) \oplus \mathcal{O}_{F_p}(\psi^*), \quad T_{g,\chi} \cong \mathcal{O}_{F_p}(\lambda) \oplus \mathcal{O}_{F_p}(\psi^*\chi) \]
and let $W_g = V_g/T_g$ and $W_{g,\chi} = V_{g,\chi}/T_{g,\chi}$.

Henceforth, for any $p$-adic Galois character $\rho$, let
\[ T(\rho) := \mathcal{O}_{F_p}(\rho), \quad V(\rho) := F_p(\rho), \quad W(\rho) := V(\rho)/T(\rho). \]

Definition 6.2. Assume that we have
\[ \epsilon_p(g,\chi)\chi_p\eta_p(-1) = +1. \]
Then $g$ is a modular form on a Shimura curve defined by a quaternion algebra split over $\mathbb{Q}$ which is split at $p$, and we have a continuous function
\[ \mathcal{L}_p(g \times \chi) : \hat{\Gamma}^- \to \mathbb{C}_p, \]
where
\[ \Gamma^- = \text{Gal}(K^-_\infty/K), \]
and where $\mathcal{L}_p(g \times \chi)$ is the continuous (square root) Rankin-Selberg $p$-adic $L$-function constructed in [30, Chapter 8], associated with the Rankin-Selberg self-dual pair $(g, \chi)$. Strictly speaking, $\mathcal{L}_p(g \times \chi)$ depends on certain choices; we can choose a trivialization $\mathcal{O}_{K_p} \cong \mathbb{Z}_p^{\oplus 2}$ as in Choice 8.6 of loc. cit., so that our fixed $p^\infty$-level structure $\alpha_\infty$ determines a $\Gamma(p^\infty)$-level structure $\alpha_\infty = \alpha_{\infty,1} \oplus \alpha_{\infty,2}$ with $\langle \alpha_{\infty,1}, \alpha_{\infty,2} \rangle_{\text{Weil}} = t$ (letting $\hat{\alpha}_\infty$ denote the $p^\infty$-level structure on the dual $\hat{A}$ induced by the given principal polarization $A \cong \hat{A}$) and hence determines a point $(A, \alpha_\infty)$ on the $\Gamma(p^\infty)$-level Shimura curve $X$ attached to $B$, which satisfies the aforementioned conditions for the construction of $\mathcal{L}_p(g \times \chi)$. Let $\Omega_p' \in (\mathcal{O}_{B_{dR,X}}^+)(A, \alpha_\infty)$ be defined so that
\[ \omega_{\text{can}}(A, \alpha_\infty) = \frac{\omega_0}{\Omega_p'}, \]
where $\omega_{\text{can}}(A, \alpha_\infty) \in (\omega \otimes \mathcal{O}_{B_{dR,X}}^+)(A, \alpha_\infty)$ is the canonical differential characterized, as in Definition 4.34 of loc. cit., by
\begin{equation}
\langle \iota_{\text{dR}}(\omega_{\text{can}})(A, \alpha_\infty), \alpha_\infty t^{-1} \rangle_{\text{Weil}} = 1
\end{equation}
where, letting $\pi : A_\infty \to X$ denote the universal object with dual $\hat{\pi} : \hat{A}_\infty \to X$,
\[ \langle \cdot, \cdot \rangle : R^1\pi_*\hat{\mathcal{L}}_{p,\hat{A}_\infty} \times R^1\hat{\pi}_*\hat{\mathcal{L}}_{p,\hat{A}_\infty} \to \mu_{p^\infty,X} \]
is the universal Weil pairing, and $\iota_{\text{dR}}$ is the de Rham comparison inclusion of (3.33) in Chapter 3 of loc. cit. Recall the natural projection $\vartheta : \mathcal{O}_{B_{dR,X}}^+ \to \hat{O}_X^+$ (denoted by $\theta$ in loc. cit.). Then (149) together with [24], [34] and [35] imply
\begin{equation}
\vartheta(\omega_{\text{can}}(A, \alpha_\infty)) = \theta_{\alpha_0}^* \frac{dQ_{\alpha_0}}{Q_{\alpha_0}}, \quad \vartheta(\Omega_p') = \Omega_p = t_\pi, \quad \vartheta \left( \frac{dQ_{\text{dR}}}{Q_{\text{dR}}} \right) = \text{KS} \left( \frac{dQ_{\otimes 2}}{Q} \right),
\end{equation}
where $\text{KS} : (\omega \otimes \mathcal{O})_{\otimes 2} \sim \Omega^1 \otimes \mathcal{O} \hat{O}$ is the Kodaira-Spencer isomorphism described in loc. cit.
**Definition 6.3.** Recall $G_\infty = \text{Gal}(L_\infty/K)$ and $\Delta' = \text{Gal}(L_\infty/K_\infty)$. Henceforth, for any character $\rho \in \hat{G}_\infty$ with $\rho|\Delta' \neq 1$ of conductor $g$, let

$$L_p(\rho) := \rho^*\mu_{\text{glob}}(g) \in \Lambda(\text{Gal}(L_\infty/K), \mathcal{O}_{K_p}).$$

Note that via the natural projection $\Lambda(\text{Gal}(L_\infty/K), \mathcal{O}_{L_p}) \to \Lambda(\text{Gal}(K_\infty/K), \mathcal{O}_{L_p})$, we can view $L_p(\rho) \in \Lambda(\text{Gal}(K_\infty/K), \mathcal{O}_{L_p})$. We will often consider $\rho = \lambda$ and $\rho = \psi^*\chi$.

The following factorization of $p$-adic $L$-functions will be essential in our proof of the rational Heegner point Main Conjecture (Conjecture 7.70).

**Theorem 6.4.** Let $\chi$ be as in Proposition 7.2 below, so that in particular $F_p = K_p$. For any $\psi \in \hat{\Gamma}^-$ of Hodge-Tate weights $(0,0)$ or $(-1,-1)$, we have the following equality:

$$C \cdot L_p(g \times \chi)^2(\psi) = L_p(\lambda)^-(\psi)L_p(\psi^*\chi)^-(\psi),$$

where the superscript of “$-$” on the right-hand side denotes specialization to the anticyclotomic line $\Gamma^- \subset \text{Gal}(K_\infty/K)$, and where $C \in \mathcal{O}_p^\times$ is a constant.

In particular, $L_p(g \times \chi) \neq 0$, and so $L_p(\lambda)^-(\psi)L_p(\psi^*\chi)^-(\psi) \in \Lambda(\Gamma^-, \mathcal{O}_{K_p})[1/p]$ is not zero.

**Proof.** By the interpolation property established in [30, Theorem 8.22] of loc. cit., (147), and (95), we get the equality for (151) when $\psi$ has Hodge-Tate weight $(0,0)$, up to the factor $(\Omega_p/\vartheta(\Omega'_p))^2$.

However, (150) implies that $\Omega_p = \vartheta(\Omega'_p)$, and so (151) for weight $(0,0)$ follows.

For (151) in the Hodge-Tate weight $(-1,-1)$, we note that in our situation, $k = 2$, $j = -1$, and

$$d_{2,-1,\Delta} = \left(\frac{g_{4\text{rd}}}{d_{4\text{rd}}}\right)^{-1} \left(\frac{Qd}{dQ}\right)^{-2},$$

using the natural identification $(\omega \otimes \mathfrak{O})^{\otimes 2} = \Omega^1 \otimes \check{\mathfrak{O}}$ induced by the Kodaira-Spencer isomorphism. Now (151) follows in the weight $(-1,-1)$ case by taking the limit as $j \to -1$ in $\mathbb{Z}_p^\times$ in [30, Theorem 8.22], and using (152) to show that this latter limit is equal to the limit of the product of (95) for $L_p(\lambda)^-(\psi)$ and $L_p(\psi^*\chi)^-(\psi)$ as $k \to 0$ in $\mathbb{Z}_p^\times$.

Finally, note that the main result of [39] implies that all but finitely many of the $L$-values in the interpolation range of $L_p(g \times \chi)$ for $k = 2$, $j = 0$, are not zero, and hence $L_p(g \times \chi) \neq 0$.

**Definition 6.5.** Henceforth, let

$$L_p(g \times \chi) := C \cdot L_p(\lambda)^-(\psi)L_p(\psi^*\chi)^-(\psi) \in \Lambda_{\text{loc}}[1/p].$$

Then $L_p(g \times \chi) \neq 0$, and $L_p(g \times \chi)(\psi) = L_p(g \times \chi)(\psi)$ for $\psi \in \hat{\Gamma}^-$ of Hodge-Tate weights $(0,0)$ or $(-1,-1)$, by Theorem 6.4.

**Definition 6.6.** By the theory of complex multiplication, there exists an abelian variety $B/K$ with $\text{End}(B/K) = \mathcal{O}_F$, and such that the $L$-series of $B$ satisfies the following factorization:

$$L(B,s) = \prod_{\sigma:F \to \mathbb{C}} L((g \times \chi)\sigma, s),$$

where the product runs over all field embeddings of $F \to \mathbb{C}$.

**Definition 6.7.** Henceforth, let $\Lambda_{\text{loc}} = \Lambda(\Gamma^-, \mathcal{O}_{F_p}) = \mathcal{O}_{F_p}[\Gamma^-]$. Moreover, given a $\text{Gal}(\bar{K}/K)$-representation $V$ with coefficients in $F_p$, a $\text{Gal}(\bar{K}/K)$-invariant lattice $T \subset V$, and $W = V/T$, let

$$S^\text{rel}(W) = \ker \left( \prod_v \text{loc}_v : H^1(K, W \otimes \mathcal{O}_{F_p}, \Lambda_{\text{loc}}) \to \prod_{v \mid p} H^1(K_v, W \otimes \mathcal{O}_{F_p}, \Lambda_{\text{loc}}) \right).$$
where, letting $I_v$ denote the inertia subgroup of $\text{Gal}(\overline{K}/K_v)$,
\[ H^1_{ur}(K_v, V \otimes \mathcal{O}_F \Lambda_{\overline{\mathcal{O}}_p}) = \ker \left( H^1(K_v, V \otimes \mathcal{O}_F \Lambda_{\overline{\mathcal{O}}_p}) \to H^1(I_v, V \otimes \mathcal{O}_F \Lambda_{\overline{\mathcal{O}}_p}) \right), \]
and $H^1_{ur}(K_v, W \otimes \mathcal{O}_F \Lambda_{\overline{\mathcal{O}}_p})$ is the image of $H^1_{ur}(K_v, T \otimes \mathcal{O}_F \Lambda_{\overline{\mathcal{O}}_p})$ under the natural map on cohomology induced by the projection $V \to V/T = W$. Similarly define $S^{rel}(V)$ and $S^{rel}(T)$, where the unramified local conditions for the latter are defined by the preimage under the map on cohomology induced by the inclusion $T \to V$ of the unramified local conditions for $V$. Let $\mathcal{A}^{rel}(W) = \text{Hom}(S^{rel}(W), F_p/\mathcal{O}_F)$, and similarly with $\mathcal{A}^{rel}(V)$ and $\mathcal{A}^{rel}(T)$.

We have the following factorization of Selmer groups.

**Theorem 6.8.** We have
\[ S^{rel}(V_{\theta,\chi}) \cong S^{rel}(V(\lambda)) \oplus S^{rel}(V(\psi^*\chi)), \]
\[ S^{rel}(W_{\theta,\chi}) \cong S^{rel}(W(\lambda)) \oplus S^{rel}(W(\psi^*\chi)), \]
and
\[ \mathcal{A}^{rel}(W_{\theta,\chi}) \cong \mathcal{A}^{rel}(W(\lambda)) \oplus \mathcal{A}^{rel}(W(\psi^*\chi)). \]

**Proof.** This follows immediately from the definitions and (148). \qed

7. A Heegner Point Main Conjecture for Supersingular Abelian Varieties of $GL_2$-type

In this section, we formulate an appropriate version of Perrin-Riou’s Heegner point Main Conjecture for supersingular CM abelian varieties. In certain cases, we relate this Main Conjecture to the Rubin-type Main Conjecture ([105] and [106]) proven in the previous sections (Corollary 4.70).

**Assumption 7.1.** Again, let $E/\mathbb{Q}$ be an elliptic curve with CM by $\mathcal{O}_K$ where $K$ is an imaginary quadratic field of class number 1, and let $\lambda$ be the type $(1,0)$ Hecke character associated with $E/K$. Assume that the root number of $L(\lambda, s) = L(E/\mathbb{Q}, s)$ is $-1$.

**7.1. A choice of twist of $\lambda$.** We now consider a certain twist of $\lambda$ by a Hecke character $\chi$ over $K$, such that (in particular) the associated theta-series $\theta_{\psi}$ will have level prime to $p$ when viewed as a modular form over $K$. Recall that given a Hecke character $\chi$ of conductor $f \subset \mathcal{O}_K$, then we can view $\chi$ as a character on ideals $a$ prime to $f$ via
\[ \chi(a) = \prod_{v \mid f, \infty} \chi_v(a_v) \]
where $a_v \in \mathcal{O}_{K_v}$ is any uniformizer of $a\mathcal{O}_{K_v}$.

**Proposition 7.2.** There exist infinitely many finite order Hecke characters $\chi : \Lambda_K^\times \to \mathbb{C}^\times$ such that, letting $\psi = \lambda/\chi$,
\begin{enumerate}
  \item $\chi$ is valued in $K^\times$,
  \item $\psi$ is unramified at $p$, i.e. $\chi|_{\mathcal{O}_K^\times} = \lambda|_{\mathcal{O}_K^\times}$,
  \item $L(\psi^*\chi, 1) \neq 0$.
\end{enumerate}

**Proof.** The first condition is clearly satisfied for infinitely many $\chi$; for example, see [41] Corollary 5.22. By the theory of complex multiplication, $\lambda|_{\mathcal{O}_K^\times} \subset K^\times$, so one can moreover choose $\chi$ to satisfy (2) by restricting to the (infinitely) many Dirichlet characters $\chi : (\mathcal{O}_K/f)^\times \to \mathcal{O}_K^\times$.

We claim that condition (3) holds for all but finitely many $\chi$ satisfying (1) and (2) above. Indeed, by [39] we have that $L(\lambda^*\chi_0, 1) \neq 0$ for all but finitely many anticyclotomic characters $\chi_0$. By [22]
Lemma 5.31], we have that any anticyclotomic \( \chi_0 \) can be written as \( \chi/\chi^* \) for some Hecke character \( \chi \). Clearly, as \( \chi \) varies through Dirichlet characters \( \mathbb{A}_\mathbb{Q} \rightarrow \mathcal{O}_K^\times \) satisfying (1) and (2), \( \chi/\chi^* \) varies through infinitely many anticyclotomic characters. Hence \( L(\psi^*\chi,1) \neq 0 \) for all but finitely many \( \chi \).

**Definition 7.3.** Note that the associated theta series \( g = \theta_\psi \) has central character \( \chi^{-1}|_{\mathbb{A}_\mathbb{Q}} \) when viewed as a modular form on \( GL_2(\mathbb{A}_\mathbb{Q}) \). Recall that we have an associated abelian variety \( A_g/\mathbb{Q} \) of \( GL_2 \)-type. Let \( F_g \) denote the finite extension of \( K \) obtained by adjoining the Hecke eigenvalues of \( g \), so that naturally \( \text{End}(A_g) = \mathcal{O}_{F_g} \). Note that if \( \chi \) is chosen as in Proposition 7.2, then \( F_g = K \). In any case, \( F_g \subset F \) (recalling that \( F \) is the finite extension of \( K \) obtained by adjoining the Hecke eigenvalues of \( g \) and values of \( \chi \)), and \( \mathcal{O}_{F_g} \cong \text{End}(A_g) \). We consider the self-dual Rankin-Selberg pair \( (g, \chi) \). Then we have the Serre tensor product

\[ B := A_g \otimes_{\mathcal{O}_{F_g}} \mathcal{O}_F(\chi) \]

associated with \( (g, \chi) \) where we consider \( g \) over \( K \), or more precisely its base change from \( \mathbb{Q} \) to \( K \). Here, \( \mathcal{O}_F(\chi) \) denotes the free \( \mathcal{O}_{F_g} \)-module space \( \mathcal{O}_F \) with the action of \( \text{Gal}(K^{ab}/K) \) given by multiplication through \( \chi \).

**Convention 7.4.** To lessen notation, henceforth we use the same notation to denote an algebraic Hecke character, its \( p \)-adic avatar and its associated \( p \)-adic Galois character. The particular avatar of the character in use should be clear from context.

### 7.2. Heegner points

In this section, for \( \chi \) satisfying Proposition 7.2, we define Heegner points on \( B \) as follows.

**Definition 7.5.** Let \( D/\mathbb{Q} \) be an indefinite quaternion algebra ramified at \( \ell \mid \infty \) such that the local root numbers \( \epsilon_\ell \) satisfy

\[ \epsilon_\ell(D) = \epsilon_\ell(g \times \chi)(\chi|_{\mathbb{A}_\mathbb{Q}})\eta(-1), \]

where \( \eta \) denotes the quadratic character associated with \( K/\mathbb{Q} \). In this setting, we have a family of Shimura curves \( X = \lim_{\to} X_U \), running over open compact subgroups \( U \subset D^\times(\mathbb{A}_\mathbb{Q},f) \). Let \( J_U \) denote the Albanese of \( X_U \), so that using the Hodge class we have an Abel-Jacobi map

\[ X \to J := \lim_{\to} J_U \]

defined over \( \mathbb{Q} \).

**Proposition 7.6.** In addition to the properties listed in Proposition 7.2, we can choose \( \chi \) to have

\[ \epsilon_p(D) = \epsilon_p(g,\chi)(\chi|_{\mathbb{A}_\mathbb{Q}})\eta_p(-1) = \epsilon_p(g,\chi) = +1, \]

i.e. \( D \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p) \).

**Proof.** Let \( \pi \) be the automorphic representation on \( GL_2(\mathbb{A}_\mathbb{Q}) \) associated with \( g = \theta_\psi \). Recall that the central character \( w_g \) of \( g \) is \( \chi^{-1}|_{\mathbb{A}_\mathbb{Q}} \), and so by property (1) of Proposition 7.2 we have

\[ w_g,p(-1) = (\chi|_{\mathbb{A}_\mathbb{Q}})^{-1}(-1) = \eta_p(-1), \]

and so \( \epsilon_p(g,\chi) = +1 \). Note that we have \( \epsilon_p(g,\chi) = \epsilon_p(\lambda)\epsilon_p(\psi^*\chi) \) (see the properties of epsilon factors given in [54]), where \( \epsilon_p(\chi') \) denotes the local root number at \( p \) of a Hecke character \( \chi' \) over \( K \). By [52] Formula for \( \rho_0(e) \), p. 2.19], we have \( \epsilon_p(\psi^*\chi) = \epsilon_p(\chi) \) (since \( \psi \) is unramified at \( p \)) and \( \epsilon_p(\chi) = \epsilon_p(\lambda) \) (by property (2) of \( \chi \) given in Proposition 7.2), and so we have \( \epsilon_p(g,\chi) = \epsilon_p(\chi)^2 = \epsilon_p(E/Q)^2 = +1. \)

□
Definition 7.7. Henceforth, fix an embedding $K_p \subset M_2(\mathbb{Q})$ such that $K_p \cap M_2(\mathbb{Z}) = \mathcal{O}_{K_p}$. Recall that $g = \theta_\psi$, and that the central character of $g$ is $\omega_g = \chi|_{\mathcal{A}_\mathcal{A}}^-$. In particular, $\omega_g$ is nontrivial. By our choice of $\chi$ given in Proposition 7.2 we have that $\alpha := \psi(p)$ is a well-defined value since $\psi$ is unramified at $p$, and hence there exists a nonzero vector

$$\phi \in \pi_g := \text{Hom}^0(X, A_g) := \text{Hom}(X, A_g) \otimes \mathbb{Z} \mathbb{Q},$$

where $A_g$ is the quotient of the $J_U$ cut out by $g$ via the Eichler-Shimura construction, such that

$$U_p \phi = \alpha \phi.$$

By Proposition 7.6 the quaternion algebra $D$ on which $g$ lives is split at $p$, i.e. we can identify $D \otimes \mathbb{Q}_p = M_2(\mathbb{Q}_p)$. For any $m \in \mathbb{Z}_{\geq 0}$, define the Iwahori subgroup

$$U_{p,m} = \left\{ g \in GL_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^m} \right\} \subset GL_2(\mathbb{Z}_p).$$

Henceforth, fix a point $P \in X^{K_{\mathbb{Q}}}$; using the moduli-theoretic interpretation of $X$, we can view $P$ as a tuple consisting of a CM abelian surface (in fact a “false elliptic curve”) $A$ together with a Rosatti idempotent $e$, a trivialization $\alpha : Z_p^{\oplus 2} \xrightarrow{\sim} eT_p A$ of its Tate module, tame level structure, polarization, and other data, see [3] for an exposition. Recall that we let $K_m^-/K$ denote the degree $p^n$ subextension of the anticyclotomic $Z_p$-extension $K_{\infty}/K$, i.e. $\text{Gal}(K_m^-/K) \cong \mathbb{Z}/p^n$. For our purposes, it will be enough to suppress all tame data and only consider the data $(A, \alpha)$ when considering $P$. We can furthermore fix $P$ so that it satisfies [30] Choice 8.6 (i.e. an appropriate trivialization $\mathcal{O}_{K_p} \cong \mathbb{Z}_p^{\oplus 2}$ as in loc. cit.). Note that $P$ induces a point $P_U$ on $X_U$ for every compact open subgroup $U \subset B(\mathcal{A}_f)_{\times}$ (where here $\mathcal{A}_f \subset \mathcal{A}_\mathcal{Q}$ denotes the finite rational adéles); for any $P \in X_{U'}$ where $U \subset U'$ as open compact sugroups of $B_f^\times = (B \otimes \mathbb{Q} \mathcal{A}_f)_{\times}$, let $\phi(P)_U = \phi(P_U)$. We now define a family of Heegner points

$$P_{\chi,n}(\phi) := \int_{\text{Gal}(\overline{K}/K_m^-)} \phi(\sigma)\mathcal{U}_{p,n} \otimes \chi(\sigma) d\sigma \in B(K_m^-)_{\mathbb{Q}} := B(K_m^-) \otimes \mathbb{Z} \mathbb{Q}.$$

where the Haar measure $d\sigma$ on $\text{Gal}(\overline{K}/K)$ is chosen to have total volume 1. Note that for $0 \leq m \leq n$, we have natural trace maps

$$\text{Tr}_{n,m} : B(K_m^-)_{\mathbb{Q}} \rightarrow B(K_m)_{\mathbb{Q}}, \quad y \mapsto \sum_{\sigma \in \text{Gal}(K_m^-/K_m)} y^\sigma.$$

We now make the following assumption, which will be essential for our purposes of formulating a height $2 \pm$ theory of big logarithm maps and Selmer groups, and is sufficient for the applications we have in mind when $K$ is an imaginary quadratic field of class number 1 in which $p$ ramifies (since in this case $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-p})$ for some prime $p$).

Assumption 7.8. In addition to Assumption 7.1 henceforth assume that $p$ ramifies in $K$. Henceforth, fix a character $\chi$ as in Proposition 7.2, so that in particular $F_g = F = K$. Let $\psi := \psi$.

Definition 7.9. Recall that for any number field or local field $L$ containing the values of a character $\chi'$ on $\text{Gal}(K_{ab}/K)$, $L(\chi')$ is the $L$-vector space of dimension 1 on which $\text{Gal}(K_{ab}/K)$ acts via $\chi'$, that $\text{End}(A_g) \cong \mathcal{O}_{F_g} = \mathcal{O}_K$, and that $B = A_g \otimes \mathcal{O}_K \mathcal{O}_F(\chi)$. For any number field $L$ containing $K$, have a factorization

$$H^1(L, V_{g,\chi}) = H^1(L, V(\lambda)) \oplus H^1(L, V(\psi^* \chi))$$
induced by \([148]\). Viewing \(P_{\chi,n} \in B(K_n^-)_\mathbb{Q} \to H^1(K_n^-, T_{g,\chi})\) via the Kummer map, we then have projections onto the two factors

\[
\begin{align*}
\pi_1 : H^1(K_n^-, V_{g,\chi}) &\to H^1(K_n^-, V(\lambda)), \\
\pi_2 : H^1(K_n^-, V_{g,\chi}) &\to H^1(K_n^-, V(\psi^* \chi))
\end{align*}
\]

and corresponding images

\[
\begin{align*}
P_{\chi,n}(\phi)^1 &= \pi_1(P_{\chi,n}(\phi)) \in H^1(K_n^-, V(\lambda)), \\
P_{\chi,n}(\phi)^2 &= \pi_2(P_{\chi,n}(\phi)) \in H^1(K_n^-, V(\psi^* \chi)).
\end{align*}
\]

In the next proposition, we study the compatibility of the family \(\{P_{\chi,n}(\phi)\}\) under the above trace maps.

**Definition 7.10.** Let \([\_]_g\) denote the formal \(O_{F_{g,p}} = O_{K_p}\)-module multiplication on \(A_g\); recall that \(\text{End}(A_g) \cong O_{F_g} = O_K\). Let that \([\_]_g\) commutes with the natural Gal(\(K/K\))-action on \(A_g\).

**Proposition 7.11.** Suppose \((g, \chi)\) is as above. Then for \(n \geq 0\), we have

\[
\begin{align*}
\text{Tr}_{n+1,n}(P_{\chi,n+1}(\phi)^1) &= [\alpha]_g P_{\chi,n}(\phi)^1, \\
\text{Tr}_{n+1,n}(P_{\chi,n+1}(\phi)^2) &= [\psi^*(p)]_g P_{\chi,n}(\phi)^2 = [\alpha]_g P_{\chi,n}(\phi)^2.
\end{align*}
\]

**Proof.** Recall our fixed \(P \in X^{K^n}\), and that we identify \(P\) with the isomorphism class of the pair \((A, \alpha)\) (suppressing tame data). Let \([\_]_A\) denote the formal \(O_{K_p}\)-module structure on the CM false elliptic curve \(A\), and let \(\pi\) denote a uniformizer of \(O_{K_p}\). Let \(\alpha_n = \alpha \mod p^n : O_K/p^n \cong (\mathbb{Z}/p^n)^\oplus \to eA[p^n]\). Let \(a_{p,n}(g/K)\) denote the Hecke eigenvalue at the prime \(p\) of \(O_K\) of the base change of the modular form \(g\) to \(K\). Then under the projections \([156]\), we have for any \(\sigma \in \text{Gal}(K^{ab}/K_{n+1}^-)\),

\[
\begin{align*}
\sum_{\sigma' \in \text{Gal}(K_{n+1}^-/K_n^\pm)} \pi_1 \left( \phi((A, \alpha_n)^{\sigma'}) U_{p,n+1} \otimes \chi(\sigma\sigma') \right) \\
= [\alpha]_g \sum_{\sigma' \in \text{Gal}(K_{n+1}^-/K_n^\pm)} \pi_1 \left( \phi((A, [\pi] A \alpha_n)^{\sigma'}) U_{p,n} \otimes \chi(\sigma\sigma') \right) \\
= [\alpha]_g \sum_{\sigma' \in \text{Gal}(K_{n+1}^-/K_n^\pm)} \sum_{\sigma'' \in \text{Gal}(K_{n+1}^-/K_n^\pm)} \pi_2 \left( \phi((A, [\pi] A \alpha_n)^{\sigma''}) U_{p,n+1} \otimes \chi(\sigma\sigma'') \right) \\
= [\psi^*(p)]_g \sum_{\sigma' \in \text{Gal}(K_{n+1}^-/K_n^\pm)} \pi_2 \left( \phi((A, [\pi] A \alpha_n)^{\sigma'}) U_{p,n+1} \otimes \chi(\sigma\sigma') \right)
\end{align*}
\]

Now we have, for any \(\sigma \in \text{Gal}(K^{ab}/K_{n+1}^-)\),

\[
\begin{align*}
\sum_{\sigma' \in \text{Gal}(K_{n+1}^-/K_n^\pm)} \left( \pi_1 \left( \phi((P_{\sigma'}) U_{p,n+1} \otimes \chi(\sigma)) U_{p,n+1} \right)^{\sigma'} \right) &= \sum_{\sigma' \in \text{Gal}(K_{n+1}^-/K_n^-)} \pi_1 \left( \phi((A, \alpha_n)^{\sigma'} \otimes \chi(\sigma)) \right) \\
= [\alpha]_g \sum_{\sigma \in \text{Gal}(K_{n+1}^-/K_n^-)} \pi_1 \left( \phi((A, [\pi] A \alpha_n)^{\sigma}) \otimes \chi(\sigma\sigma') \right) \\
= [\alpha]_g \sum_{\sigma \in \text{Gal}(K_{n+1}^-/K_n^-)} \pi_1 \left( \phi((A, \alpha_n)^{\sigma}) \otimes \chi(\sigma\sigma') \right),
\end{align*}
\]

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where the penultimate equality follows because \( \phi((A,[\pi]_A\alpha_n))\) is invariant under the action of \( U_{p,n} \). We have a similar identity after replacing \( \pi_1 \) by \( \pi_2 \). Now (157) immediately follows from (154).

**Remark 7.12.** Using our fixed \( p^{\infty} \)-level structure \( \alpha \) (fixed by our choice of \( P \in X^{K^x} \) in Definition 7.7), we have canonical liftings \( \tilde{P}_{\chi,n}(\phi)^1 \in H^1(K_n,V(\lambda)) \) and \( \tilde{P}_{\chi,n}(\phi)^2 \in H^1(K_n,V(\psi^*\chi)) \) such that cores \( \text{cores}_{K_n/K_n}(\tilde{P}_{\chi,n}(\phi)^i) = \tilde{P}_{\chi,n}(\phi)^i \), induced by pulling back \( \tilde{P}_{\chi,n}(\phi)^i \) via the image \( (A,\alpha) \mapsto (A,\alpha)_{U_{p,\infty}} \) under the projection \( X \to X_{U_{p,\infty}} \). It is then clear, from the same calculations as in Proposition 7.11 that

\[
\begin{align*}
\text{Tr}_{n+1,n}(\tilde{P}_{\chi,n+1}(\phi)^1) &= [\alpha]_g \tilde{P}_{\chi,n}(\phi)^1, \\
\text{Tr}_{n+1,n}(\tilde{P}_{\chi,n+1}(\phi)^2) &= [\psi^*(p)]_g \tilde{P}_{\chi,n}(\phi)^2 = [\alpha]_g \tilde{P}_{\chi,n}(\phi)^2.
\end{align*}
\]

**7.3. Norm-compatible + families from Heegner points.** From the previous section we get a family \( \{P_{\chi,n}(\phi)\} \). However, from (157) we see that this family is not quite norm-compatible. In this section we define norm-compatible plus families of Heegner points, closely related to \( \{P_{\chi,n}(\phi)\} \), which are norm compatible. For the remainder of this paper, we make the following assumption for simplicity, which we expect to relax in the future.

**Definition 7.13.** Henceforth, let \( \Lambda_{\overline{O}_{K_p}, n} := \Lambda(\text{Gal}(K_n^-/K), \overline{O}_{K_p}) \) and \( \overline{\Lambda} := \Lambda(\Gamma^-, \overline{O}_{K_p}) \). Similarly, let \( \Lambda_{O_{K_p}, n} := \Lambda(\text{Gal}(K_n/K), O_{K_p}) \) and \( \overline{\Lambda}_{O_{K_p}} := \Lambda(\text{Gal}(K_\infty/K), O_{K_p}) \).

**Proposition 7.14.** We have that

\[
H^1(K_\infty^-, T_{g,\chi}) \xrightarrow{(148)} H^1(K_\infty^-, T(\lambda)) \oplus H^1(K_\infty^-, T(\psi^*\chi))
\]

is a free \( \overline{\Lambda}_{O_{K_p}} \)-module.

**Proof.** We note that each of the 1-dimensional \( \text{Gal}(\overline{K}/K) \)-representations \( T(\lambda) \) and \( T(\psi^*\chi) \) is irreducible. Hence the argument of [3] Lemma 5.3 applies verbatim to each of \( H^1(K_\infty^-, T(\lambda)) \) and \( H^1(K_\infty^-, T(\psi^*\chi)) \), and hence each factor of \( H^1(K_\infty^-, T_{g,\chi}) \) is a free \( \overline{\Lambda}_{O_{K_p}} \)-module.

**Definition 7.15.** By our choice of \( \chi \) from Proposition 7.2, we have that \( \psi : \mathcal{A}_K^\times \to K^x \) is a type \((1,0)\) Hecke character which is unramified at \( \mathfrak{p} \). Let \( A_g/\mathbb{Q} \) denote the associated abelian variety with complex multiplication by \( O_K \), so that \( A_g/K \) has complex multiplication by \( O_K \otimes \mathbb{Z} \). Let \( \hat{A}_g \) denote the formal completion at \( \mathfrak{p} \) of a good integral model of \( A_g \), which is a two-dimensional formal group over \( O_{K_p} \). We have a \( \text{Gal}(\overline{K}/K) \)-equivariant splitting

\[
A_g[p^{\infty}] \cong W(\psi) \oplus W(\psi^*)
\]

which respects the CM action by \( O_K \times O_K \). This shows that \( \hat{A}_g \) is a product of one-dimensional formal \( O_{K_p} \)-modules \( F_\psi \oplus F_{\psi^*} \) over \( O_{K_p} = O_{K_p} \). Henceforth, let \( [\cdot]_g \) denote the \( O_{K_p} \)-module structure on the formal group \( F_\psi \) corresponding to the factor \( W(\psi) \) in the above decomposition of \( A_g[p^{\infty}] \). Note that by Lubin-Tate theory, \( \text{Gal}(K(W(\psi))/K) \cong O_{K_p}^X \) via the reciprocity map. Let \( \Delta = \text{Gal}(K(W(\psi))/K_\infty) \).

**Definition 7.16.** Recall the \( O_{K_p} \)-module structure \( [\cdot]_g \) on \( F_\psi \); we can view \([\cdot]_g \) as a power series \([\cdot]_g(X)\) in a formal parameter \( X \) on \( F_\psi \). By the Weierstrass preparation theorem, the roots of \([\pi^m]_g \) (viewed as a power series) are given by a distinguished polynomial. Let \( \pi_m(X) \) denote the
The distinguished polynomial associated with the power series \([\pi^m]_g(X)/[\pi^{m-1}]_g(X)\). We define
\[
\omega^+_n(X) = X \prod_{1 \leq m \leq n, n \text{ even}} \pi_m(X), \quad \omega^-_n(X) = X \prod_{1 \leq m \leq n, n \text{ odd}} \pi_m(X),
\]
\[
\tilde{\omega}^+_n(X) = \omega^+_n(X)/X, \quad \tilde{\omega}^-_n(X) = \omega^-_n(X)/X.
\]
Note that we have
\[
(160) \quad \omega^+_n(X) = \omega^+_n(X)/X = \omega^+_n(X)/X = [\pi^m]_g(X) = [p^n]_g(X).
\]
For brevity, we will often denote \(\omega^+_n(X)\) simply by \(\omega^+_n\), and similarly for \(\pi_n, \omega^-_n, \tilde{\omega}^+_n\) and \(\tilde{\omega}^-_n\).

**Definition 7.17.** Henceforth fix an identification
\[
\text{Gal}(K(W(\psi)[p^\infty])/K) \cong \text{Gal}(K_{\infty}/K) \times \Delta,
\]
where under the Lubin-Tate reciprocity isomorphism \(O_{K_p}^\times \cong \text{Gal}(K(W(\psi))/K)\), \(1 + p' O_{K_p}\) maps isomorphically to \(\text{Gal}(K_{\infty}/K)\) and \(O_{K_p}/p^\times\) maps isomorphically to \(\Delta\). Note that fixing a \(O_{K_p}\)-generator \(\gamma\) of the free \(O_{K_p}\)-module \(\text{Gal}(K_{\infty}/K) \cong 1 + p' O_{K_p}\), we can identify
\[
(162) \quad \Lambda(\text{Gal}(K(W(\psi)[p^n])/K), O_{K_p}) \cong O_{K_p}[\Delta][X]/([p^n]_g(X))
\]
by sending \(\gamma - 1 \mapsto X\). Henceforth, let \(\Delta_{O_{K_p,n}} = \Lambda(\text{Gal}(K_n/K), O_{K_p})\), so that there is a natural projection \(\Lambda(\text{Gal}(K(W(\psi)[p^n])/K), O_{K_p}) \to \Delta_{O_{K_p,n}}\).

Note that there is a natural projection \(\Delta_{O_{K_p,n}} \to \Delta_{O_{K_p,n}}\) given by the natural quotient \(\text{Gal}(K_n/K) \to \text{Gal}(K_{\infty}/K)\); this, of course, naturally makes any \(\Delta_{O_{K_p,n}}\)-module into a \(\Delta_{O_{K_p,n}}\)-module.

Henceforth, let \(K_{\pi^m} \subset K(W(\psi))\) denote the subextension with \([K_{\pi^m} : K] = p^m\) and \(K_{\pi^m} \subset K_{[m+1]}\). Hence \(K_{\pi^m} = K_m\).

**Lemma 7.18** (cf. Lemma 5.2 of [8]). The classes \(P_{\chi,n}(\phi)^1, P_{\chi,n}(\phi)^2\) lie in the images of the natural maps
\[
H^1(K_{\infty}, V(\lambda)) \to H^1(K_n, V(\lambda)), \quad H^1(K^{-\infty}, V(\psi^*\chi)) \to H^1(K^{-\infty}, V(\psi^*\chi)),
\]
respectively.

**Proof.** This is proven analogously as in the proof of Lemma 5.2 of loc. cit., using [157]. \(\square\)

We now show that \(P_{\chi,n}(\phi)^i\) is “divisible” by \(\alpha^n\) in \(\Delta_{O_{K_p,n}}[1/p]/\omega^+_n\), or more precisely by \(\tilde{\omega}_{2n}\).

**Theorem 7.19** (cf. Proposition 5.4 and preceding discussion of [8]). Let \(i = 1, 2\).

1. For any \(n \in \mathbb{Z}_{\geq 0}\), we have
   \[
   \tilde{\omega}_{2n} P_{\chi,n}(\phi)^i = 0.
   \]

2. For any \(n \in \mathbb{Z}_{\geq 0}\), there exist classes
   \[
   P^+_{1,n} \in H^1(K_n, V(\lambda))/\omega^+_n, \quad P^+_{2,n} \in H^1(K_n, V(\psi^*\chi))/\omega^+_n
   \]
such that \(\tilde{\omega}_{2n} P^+_{1,n} = P_{\chi,n}(\phi)^i\).

3. The classes \(P^+_{i,n}\) are compatible under the natural projections (induced by corestriction)
   \[
   H^1(K_n, V(\lambda))/\omega^+_n \to H^1(K_{n-1}, V(\lambda))/\omega^+_n, \quad H^1(K_n, V(\psi^*\chi))/\omega^+_n \to H^1(K_{n-1}, V(\psi^*\chi))/\omega^+_n.
   \]


Proof. By the freeness result of Proposition 7.14, we have identifications $H^1(K_n^-, V(\lambda)) \cong \Lambda_{\mathcal{O}_K}[1/p]$ and $H^1(K_n^-, V(\psi^*\chi)) \cong \Lambda_{\mathcal{O}_K[p]}[1/p]$ for some $r_1, r_2 \in \mathbb{Z}_{\geq 0}$. By Lemma 7.18, we have that the classes $P_{\lambda,n}(\phi)^i$ are the images of some classes under $H^1(K_n^-, V(\lambda)) \to H^1(K_n^-, V(\lambda))$ and $H^1(K_n^-, V(\psi^*\chi)) \to H^1(F_n^-, V(\psi^*\chi))$. Hence, we may view $P_{\lambda,n}(\phi)^i \in \Lambda_{\mathcal{O}_K[p]}[1/p]$. We can consider the projection of $P_{\lambda,n}(\phi)^i$ in each factor of $\Lambda_{\mathcal{O}_K[p]}[1/p]^{-r_i}$ separately, and so without loss of generality view $P_{\lambda,n}(\phi)^i \in \Lambda_{\mathcal{O}_K[p]}[1/p]$.

(1): Note that (157) implies (by an easy induction) that we have the following equality in $\Lambda_{\mathcal{O}_K[p]}[1/p]$: 

$$\omega_n^+ P_{\lambda,n}(\phi)^i = \pi_n \omega_n^+ P_{\lambda,n}(\phi)^i = \pi_n \omega_n^+ T_{\lambda,n-1}(P_{\lambda,n}(\phi)^i) $$

$$= \pi_n \omega_n^+(\alpha_{[g]} P_{\lambda,n-1}(\phi)^i = \pi_n \omega_n^+ T_{\lambda,n-2}(P_{\lambda,n-1}(\phi)^i)$$

$$= \ldots = \pi_n \omega_n^+ \pi_{n-1} \ldots \omega_1^+ P_{\lambda,0}(\phi)^i = [p^n]_g P_{\lambda,0}(\phi)^i = 0,$$

where the last equality follows because $[p^n]_g = 0$ in $\Lambda_{\mathcal{O}_K[p]}[1/p]$.

(2): This follows immediately from (160), which implies that there is a natural identification $\Lambda_{\mathcal{O}_K[p]}[1/p] \cong \Lambda_{\mathcal{O}_K[p]}[1/p]$ induced by multiplication by $\omega_n^+$, and (162).

(3): This follows immediately from (2) and (157). \hfill \Box

Definition 7.20. Using Theorem 7.19, we now define

$$P^+_1 := \lim_n P^+_1, n \in \lim_n H^1(K_n^-, V(\lambda))/\omega_n^+ = H^1(K_n^-, V(\lambda))$$

$$P^+_2 := \lim_n P^+_2, n \in \lim_n H^1(K_n^-, V(\psi^*\chi))/\omega_n^+ = H^1(K_n^-, V(\psi^*\chi))$$

7.4. Kummer theory and the bridge between $GL_1$ and $GL_2$.

Definition 7.21. Recall the Serre tensor product $B = A \otimes_{\mathcal{O}_F} \mathcal{O}_F$. For any $n \geq m$, we have natural trace maps $Tr_{K_n^+, K_m} : B(K_n^+, \mathcal{O}_F) \to B(K_m^+, \mathcal{O}_F)$. Define

$$B^+(K_{\pi^{n+1}, p}) := \{ P \in B(K_{\pi^{n+1}, p}) : Tr_{K_{\pi^{n+1}, p}, K_{\pi^{n+2}, p}} (P) \in B(K_{\pi^{n+1}, p}) \forall 0 \leq m < n, \text{ even m} \},$$

Definition 7.22. As before, recall that

$$B^+(K_{\pi^{n+1}, p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \hookrightarrow H^1(K_{\pi^{n+1}, p}, W_{g, \chi})$$

$$= H^1(K_{\pi^{n+1}, p}, W(\lambda)) \oplus H^1(K_{\pi^{n+1}, p}, W(\psi^*\chi))$$

via the Kummer map. Recall also the projectors $\pi_1$ and $\pi_2$ from (155). We define

$$H^1_+(K_{\pi^{n+1}, p}, W_{g, \chi}) := B^+(K_{\pi^{n+1}, p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p,$$

and define

$$H^1_+(K_{\pi^{n+1}, p}, T_{g, \chi}) \subset H^1(K_{\pi^{n+1}, p}, T_{g, \chi})$$

to the the orthogonal complement of $H^1_+(K_{\pi^{n+1}, p}, W_{g, \chi}) \subset H^1(K_{\pi^{n+1}, p}, W_{g, \chi})$ under the $\mathcal{O}_K[p]$-bilinear Tate local duality pairing

$$H^1(K_{\pi^{n+1}, p}, T_{g, \chi}) \times H^1(K_{\pi^{n+1}, p}, W_{g, \chi}) \to K_\mathcal{O}_F$$

(recall that $\chi$ and $\psi^*\chi$ are self-dual characters). We also define

$$H^1_+(K_{\pi^{n+1}, p}, W(\lambda)) := \pi_1(H^1_+(K_{\pi^{n+1}, p}, W_{g, \chi})) \subset H^1(K_{\pi^{n+1}, p}, W(\lambda))$$
and
\[ H^1_+(K_{\pi^{n+1},p}(W(\psi^*\chi))) := \pi_2(H^1_+(K_{\pi^{n+1},p}, W_{g,\chi})) \subset H^1(K_{\pi^{n+1},p}, (W(\psi^*\chi))), \]
and define
\[ H^1_+(K_{\pi^{n+1},p}, T(\lambda)) \subset H^1(K_{\pi^{n+1},p}, T(\lambda)), \quad H^1_+(K_{\pi^{n+1},p}, T(\psi^*\chi)) \subset H^1(K_{\pi^{n+1},p}, T(\psi^*\chi)) \]
to be the orthogonal complements of
\[ H^1_+(K_{\pi^{n+1},p}, W(\lambda)) \subset H^1(K_{\pi^{n+1},p}, W(\lambda)), \quad H^1_+(K_{\pi^{n+1},p}, W(\psi^*\chi)) \subset H^1(K_{\pi^{n+1},p}, W(\psi^*\chi)), \]
respectively, under the $O_{K_p}$-bilinear Tate local duality pairings
\[ H^1(K_{\pi^{n+1},p}, T(\lambda)) \times H^1(K_{\pi^{n+1},p}, W(\lambda)) \to K_p/O_{K_p} \]
and
\[ H^1(K_{\pi^{n+1},p}, T(\psi^*\chi)) \times H^1(K_{\pi^{n+1},p}, W(\psi^*\chi)) \to K_p/O_{K_p}, \]
respectively. Note that we have
\[ H^1_+(K_{\pi,n}, T(\lambda)) = \text{Hom}_{O_{K_p}}(H^1(K_{\pi^{n+1},p}, W(\lambda))/\pi_1(B^+(K_{\pi^{n+1},p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p), K_p/O_{K_p}) \]
and similarly with $\lambda$ replaced by $\psi^*\chi$. Make similar definitions for rational coefficients, i.e. $V(\lambda)$ and $V(\psi^*\chi)$.

**Definition 7.23.** For $0 \leq n \leq \infty$, we define
\[ H^1_+(K_{\pi^{2n},p}, T(\lambda)) \subset H^1(K_{\pi^{n},p}, T(\lambda)) \]
to be the image of $H^1_+(K_{\pi^{2n},p}, T(\lambda))$ under the corestriction map
\[ \text{cores}_{K_{\pi^{2n},p}/K_{\pi^{n},p}}: H^1(K_{\pi^{2n},p}, T(\lambda)) \to H^1(K_{\pi^{n},p}, T(\lambda)) \]
(using the fact that $K_{\pi^{n},p} \subset K_{\pi^{2n},p}$). Similarly define $H^1_+(K_{\pi^{n},p}, T(\psi^*\chi))$ and $H^1_+(K_{\pi^{n},p}, T_{g,\chi})$. Again, make similar definitions for rational coefficients, i.e. $V(\lambda)$ and $V(\psi^*\chi)$.

**Proposition 7.24.** Letting $K_{\pi,\infty} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} K_{\pi,n}$, we have
\[ P_1^+ \in H^1_+(K_{\pi,\infty}, T(\lambda)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1_+(K_{\pi,\infty}, V(\lambda)), \]
\[ P_2^+ \in H^1_+(K_{\pi,\infty}, T(\psi^*\chi)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1_+(K_{\pi,\infty}, V(\psi^*\chi)). \]

**Proof.** This follows immediately from the definitions and Remark 7.12. \qed

**Definition 7.25.** We define
\[ z_+ := P_1^+ \oplus P_2^+ \in H^1_+(K_{\pi,\infty}, T_{g,\chi}). \]

**Definition 7.26.** Recall our fixed elliptic curve $E/\mathbb{Q}$ with CM by class number 1 imaginary quadratic field $K$, and $\lambda: \mathbb{A}_K^\infty \to \mathbb{C}^\times$ the type $(1,0)$ Hecke character associated with $E$. For $0 \leq n \leq \infty$, consider the composition
\[ E(K_p(\hat{E}[p^n])) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \to H^1(K_p(\hat{E}[p^n]), E[p^n]) \xrightarrow{\sim} \text{Hom}(\text{Gal}(K_p/K_p(\hat{E}[p^n])), E[p^n]) \]
\[ \to \text{Hom}(1 + p_nO_{K_p}(E[p^n]), E[p^n]) \]
\[ \xrightarrow{\sim} \text{Hom}_{O_{K_p}}((1 + p_nO_{K_p}(E[p^n])) \otimes_{\mathbb{Z}_p} O_{K_p}(\lambda^{-1}), O_{K_p}/p^n), \]
where the penultimate arrow is induced by the inclusion $1 + p_nO_{K_p}(E[p^n]) \hookrightarrow \text{Gal}(K_p/K_p(\hat{E}[p^n]))$ given by local class field theory. In particular, for $n = \infty$ we get
\[ E(K_p(\hat{E}[p^\infty])) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to H^1(K_p(\hat{E}[p^\infty]), E[p^\infty]) \xrightarrow{\sim} \text{Hom}(\text{Gal}(K_p/K_p(\hat{E}[p^\infty])), E[p^\infty]) \]
\[ \to \text{Hom}(U^1, E[p^\infty]) \xrightarrow{\sim} \text{Hom}_{O_{K_p}}((U^1 \otimes_{\mathbb{Z}_p} O_{K_p})(\lambda^{-1}), K_p/O_{K_p}), \]
where the penultimate arrow is induced by the inclusion \( U^1 := \lim_{\leftarrow \text{Nm}} (1 + p_n \mathcal{O}_{K_p}(E[p^n])) \hookrightarrow \text{Gal}(\bar{K}_p/K_p(E[p^\infty])) \) given by local class field theory. Hence we get an \( \mathcal{O}_{K_p} \)-linear Kummer pairing

\[
(E(K_p(\bar{E}[p^\infty])) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \times (U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1}) \to K_p/\mathcal{O}_{K_p}.
\]

Let \( \chi_E : \text{Gal}(K(E[p^\infty]/K) = \text{Gal}(K(E[p^\infty])/L) \to \mathcal{O}_{K_p}^\times \) be the reduction \( \text{mod } p^\infty \) (where \( \lambda \) is viewed as a Galois character \( \text{Gal}(K(E[p^\infty]/K) \to \mathcal{O}_{K_p}^\times) \), where \( \epsilon \) is defined as in \([54]\). Recall \( K_{p,\infty} = \bigcup_{n \in \mathbb{Z}_{>0}} K_{n,p} \), where \( K \subset K_n = K_{p^n} \subset K(W(\psi)[p^n]) \) is such that \([K_n : K] = p^{2n}\).

**Proposition 7.27.** The pairing \([164]\) induces a nondegenerate \( \mathcal{O}_{K_p} \)-linear and Galois equivariant pairing

\[
(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \times ((U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1}))_{\chi_E} \to K_p/\mathcal{O}_{K_p}.
\]

Equivalently, we have a natural Galois-equivariant isomorphism

\[
((U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1}))_{\chi_E} \cong \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, K_p/\mathcal{O}_{K_p}).
\]

Moreover, we have

\[
E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = H^1(K_{p,\infty}, E[p^\infty]).
\]

**Proof.** This follows from the same argument as in \([42]\, Proposition 5.2\), except a few steps need to be extended since \( E/\mathcal{O}_{K_p} \) is of bad reduction in our case. Note that since \( \lambda_A \) and \( \lambda \) are both Hecke characters over \( K \) of infinity type \((1,0)\), \( \lambda_A = \lambda \chi \), where \( \chi \) is a finite-order character. By \([42]\, Proposition 5.2\), \([165]\) is a nondegenerate pairing with \( A \) and \( \lambda_A \) in place of \( E \) and \( \lambda \). Let \( K_p \subset K_{p,A} \subset K_{p,A}(p^\infty) \) be such that \([K_{n,p} : K_p] = p^{2n}\) and \( K_{p,\infty} = \bigcup_{n \in \mathbb{Z}_{>0}} K_{n,p}^\prime \). Note that \( K_{p,\infty} = K_{n,p=0}\) is the unique \( \mathbb{Z}_p\)-extension of \( K_p \) contained in \( K_{p,A}(p^\infty) \), and so \( K_{n,p} = K_{n,p}^\prime \) and \( K_{p,\infty} = \bigcup_{n \in \mathbb{Z}_{>0}} K_{n,p}^\prime =: K_{p,\infty}^\prime \). Moreover, viewing \( \lambda \) and \( \lambda_A \) as local Galois characters \( \text{Gal}(\bar{K}_p/K_p) \to \mathcal{O}_{K_p}^\times \), since \( \chi = \lambda_A/\lambda \) is of finite order, we have \( H^1(K_{n,p}, \bar{E}) = H^1(K_{n,p}, \hat{A}) \) for all \( n \geq 0 \), and so Lemma 5.1(i) of loc. cit. implies

\[
\lim_{n \to \infty} H^1(K_{n,p}, \bar{E})[p^{n+1}] = \lim_{n \to \infty} H^1(K_{n,p}, \hat{A})[p^{n+1}] = 0.
\]

Taking the limit of the exact sequence (6) of loc. cit. for \( E \) and using the above limit computation then implies \([167]\).

Note that since \( E \) is isomorphic to \( A \) over \( \mathbb{Z} \), we have that \( E(K_{p,\infty}) \cong A(K_{p,\infty}^\prime) \), and so from \([164]\) we get \([165]\). \( \square \)

**Definition 7.28.** For all \( 0 \leq n \leq \infty \), we have a natural identification

\[
\pi_1(H^1(K_{\pi^n,p}, B[p^\infty] \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})) = H^1(K_{\pi^n,p}, \pi_1) \cong E(K_{\pi^n,p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p.
\]

We then define

\[
E^+(K_{\pi^n,p}) \subset E(K_{\pi^n,p})
\]

to be the subgroup such that

\[
E^+(K_{\pi^n,p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = \pi_1(B^+(K_{\pi^n,p}) \otimes_{\mathbb{Z}_p} K_p/\mathcal{O}_{K_p}) \subset E(K_{\pi^n,p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p
\]

under the above identification. Let \( \bar{E}^+(K_{\pi^n,p}) = E^+(K_{\pi^n,p}) \cap \bar{E}(p^\infty) \).

**Corollary 7.29.** For all \( 0 \leq n \leq \infty \), we have natural Galois-equivariant isomorphisms

\[
H^1_+(K_{\pi^n,p}, T(\lambda)) \cong \text{Hom}_{\mathcal{O}_{K_p}}((E(K_{\pi^n,p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)/(E^+(K_{\pi^n,p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p), K_p/\mathcal{O}_{K_p})
\]
and
\[
\left( \left( U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \right) (\lambda^{-1}) \right)_{\lambda E} / H^1_+ (K_{\pi^m, p}, T(\lambda)) \cong \text{Hom}_{\mathcal{O}_{K_p}} \left( E^+ (K_{\pi^m, p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p, K_p / \mathcal{O}_{K_p} \right)
\]
(169)\[= \text{Hom}_{\mathcal{O}_{K_p}} \left( \hat{E}^+ (K_{\pi^m, p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p, K_p / \mathcal{O}_{K_p} \right)
\]
(170)\[= \text{Hom}_{\mathcal{O}_{K_p}} \left( \hat{E}^+ (K_{\pi^m, p}), \mathcal{O}_{K_p} \right).
\]

Proof. This follows immediately from the definitions and (160). Note that the second and third equality of (160) are easy exercises. \(\square\)

Definition 7.30. By self-duality of \(\lambda\), Tate local duality gives a non-degenerate pairing
\[
H^1 (K_{p, \infty}, W(\lambda)) \times H^1 (K_{p, \infty}, T(\lambda)) \to K_p / \mathcal{O}_{K_p}
\]
(170) obtained by putting together the alternating pairings at each finite level \(n \geq 0\)
\[
H^1 (K_{\pi^n, p}, \mathcal{O}_{K_p} / p^n(\lambda)) \times H^1 (K_{\pi^n, p}, \mathcal{O}_{K_p} / p^n(\lambda)) \to p^{-n} \mathcal{O}_{K_p} / \mathcal{O}_{K_p}.
\]
(171) Hence (165), (171) and (170) induce natural identifications
\[
\left( \left( U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \right) (\lambda^{-1}) \right)_{\lambda E} \cong H^1 (K_{p, \infty}, T(\lambda)).
\]
(172) Note that \(K_{p, \infty} = \bigcup_{n \in \mathbb{Z}_p^+} K_{\pi^n, p}\). Recall that for \(\chi\) as in Proposition 7.2, we have \(F_g = F = K\). From (172), we have a natural inclusion
\[
H^1_+ (K_{p, \infty}, T(\lambda)) \subset \left( \left( U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \right) (\lambda^{-1}) \right)_{\lambda E}.
\]

7.5. The two-variable Coleman and big logarithm maps in the height 2 CM case. In this section, we introduce a construction of the \(+\) big logarithm map for a supersingular CM abelian variety of \(GL_2\)-type. Our construction is perhaps simpler than that of the height 1 case recalled in \([8\ Section 3]\) in the sense that the two-variable big logarithm map arises more naturally from the height 2 formal group law and the \(\mathcal{O}_{K_p}\)-structure of the relevant Lubin-Tate groups. Here, by “height 1 case”, we mean that it is assumed in loc. cit. that \(p\) splits in \(K\), and so the \(\mathbb{Z}_p^\otimes 2\)-extension of \(K\) localizes to a subextension of the height 1 Lubin-Tate \(\mathbb{Z}_p\)-extension of the completion of the maximal unramified extension of \(\mathbb{Q}_p\).

Lemma 7.31 (cf. Lemma 4.1 of \([11]\)). Suppose that \(X\) is a free \(\Lambda_{\mathcal{O}_{K_p}}\)-module of rank 2, and that \(Y \subset X\) is a \(\Lambda_{\mathcal{O}_{K_p}}\)-module of rank 1 such that \(X / Y\) is a torsion-free \(\Lambda_{\mathcal{O}_{K_p}}\)-module. Then \(Y\) is free of \(\Lambda_{\mathcal{O}_{K_p}}\)-rank 1.

Proof. The proof of \([11\ Lemma 4.1]\) applies verbatim, since \(\Lambda_{\mathcal{O}_{K_p}} \cong \mathcal{O}_{K_p} [T_1, T_2]\) is a power series ring over a PID, and so is a UFD. \(\square\)

Lemma 7.32 (cf. Lemma 4.3 of \([11]\)). Suppose that \(Z\) is a \(\Lambda_{\mathcal{O}_{K_p}}\)-module of rank 1 or \(\Lambda_{\mathcal{O}_{K_p}}\)-torsion free and nonzero, and that
\[
\text{Hom}_{\mathcal{O}_{K_p}} (Z, K_p / \mathcal{O}_{K_p})_{\text{Gal}(K_{\infty}/K)} \cong K_p / \mathcal{O}_{K_p}.
\]
Then \(Z\) is free of rank 1.

Proof. This is essentially the same proof as in loc. cit. Let \(\gamma_+, \gamma_-\) be topological generators of \(\text{Gal}(K_{\infty}/K) \cong \mathbb{Z}_p^\otimes 2\). We have by assumption
\[
\text{Hom}_{\mathcal{O}_{K_p}} (Z, K_p / \mathcal{O}_{K_p})_{\text{Gal}(K_{\infty}/K)} = \text{Hom}_{\mathcal{O}_{K_p}} (Z / (\gamma_+ - 1, \gamma_- - 1)Z, \mathcal{O}_{K_p}) \cong K_p / \mathcal{O}_{K_p},
\]
and hence \(Z / (\gamma_+ - 1, \gamma_- - 1)Z \cong \mathcal{O}_{K_p}\). Hence, by Nakayama’s lemma, \(Z\) is generated over \(\Lambda_{\mathcal{O}_{K_p}}\) by a single element, and so is free since \(Z\) is either rank 1 or \(\Lambda_{\mathcal{O}_{K_p}}\)-torsion free and nonzero. \(\square\)
Con**vention 7.33. For Proposition 7.34, Proposition 7.35, Lemma 7.38 and Proposition 7.39, let $X = ((U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1}))_{\lambda A}$ and $Y = H^1_+(Kp, \infty, T(\lambda A))$.

**Proposition 7.34.** $X/Y$ is a nonzero torsion-free $\Lambda_{\mathcal{O}_{K_p}}$-module.

**Proof.** Note that by (169), we have

$$X/Y = \lim_{\rightarrow_n} \text{Hom}_{\mathcal{O}_{K_p}}(E^+(Kp, \infty), \mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E^+(m_{\infty}), \mathcal{O}_{K_p}) = \lim_{\rightarrow_n} \text{Hom}_{\mathcal{O}_{K_p}}(E^+(m_n), \mathcal{O}_{K_p})$$

where $m_n$ is the maximal ideal of $K_{n,p}$. From this, it is clear that one can construct compatible $f_n \neq 0$ for every $n \in \mathbb{Z}_{\geq 0}$, and hence $X/Y \neq 0$. Suppose $f \in X/Y$ satisfies $\lambda f = 0$ for some $\lambda \in \Lambda_{\mathcal{O}_{K_p}} \setminus \{0\}$. Then by the above, $f$ induces a map $f_n : E^+(m_n)/((\lambda E^+(m_n))) \rightarrow \mathcal{O}_{K_p}$ for every $n \in \mathbb{Z}_{\geq 0}$. Specializing the $\Lambda_{\mathcal{O}_{K_p}}$-module $X/Y$ to any character $\phi : \text{Gal}(K_{\infty}/K) \rightarrow \mathbb{Q}_p^\times$ with $\phi(\lambda) \neq 0$, we then get a map $\phi(f_n) : E^+(m_n)/((\phi(\lambda) E^+(m_n))) \rightarrow \mathcal{O}_{K_p}$, where $E^+(m_n)/((\phi(\lambda) E^+(m_n)))$ is a torsion $\mathcal{O}_{K_p}$-module since $E^+(m_n)$ is profinite. However, since $\mathcal{O}_{K_p}$ is torsion-free, this means $\phi(f_n) = 0$. Since $n$ and $\phi$ were arbitrary, this means $f = 0$. Hence $X/Y$ is $\Lambda_{\mathcal{O}_{K_p}}$-torsion free.

**Proposition 7.35.** We have that $X/Y$ is a free $\Lambda_{\mathcal{O}_{K_p}}$-module of rank 1.

**Proof.** By definition, we have $X/Y = \text{Hom}_{\mathcal{O}_{K_p}}(E^+(Kp, \infty), \mathcal{O}_{K_p})$. Let $\gamma_+, \gamma_-$ be topological generators of $\text{Gal}(K_{\infty}/K) \cong \mathbb{Z}_p^\times$. Note that $E^+(Kp) = E(Kp)$, and so

$$\text{Hom}_{\mathcal{O}_{K_p}}(E^+(Kp), \mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E(Kp), \mathcal{O}_{K_p}) \cong \mathcal{O}_{K_p},$$

where the last isomorphism follows by the existence of a finite-index subgroup of $E(K_p)$ isomorphic to $\mathcal{O}_{K_p}$ (using the $p$-adic formal logarithm). Hence

$$\text{Hom}_{\mathcal{O}_{K_p}}(X/Y, Kp/\mathcal{O}_{K_p})^\text{Gal}(K_{\infty}/K) = \text{Hom}_{\mathcal{O}_{K_p}}((X/Y)/(\gamma_+ - 1, \gamma_- - 1)(X/Y), Kp/\mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E^+(Kp), \mathcal{O}_{K_p}) \cong \mathcal{O}_{K_p}.$$

Now Lemma 7.32 applied to $Z = X/Y$ and Proposition 7.34 give the Proposition.

**Definition 7.36.** Let $\phi \in \pi_g$ be as in Definition 7.7 and let $\omega_g \in \Omega^1_{X/\mathbb{Q}}$ be the 1-form associated with the weight 2 newform $g$. Let $\omega_{A_g} \in \Omega^1_{\Lambda_g/\mathbb{Q}}$ be a differential such that $\phi^* \omega_{A_g} = \omega_g$. Let $\omega_B = \omega_{A_g} \otimes \chi \in \Omega^1_{B/K}$. For $0 \leq n \leq \infty$, to $\omega_{A_g}$ we have an associated $p$-adic logarithm map $\log_B = \log_{\omega_B} : H^1(K_{p,n}, Tg, \chi) \rightarrow K_{n,p}$ associated to $\omega_B$. Let $\omega_E \in \Omega^1_{E/\mathbb{Q}}$ be a fixed Néron differential, and let $c_B \in \mathbb{Q}^\times$ so that $\pi_1^* \omega_E = c_B \omega_B$. Let $\log_E = \log_{\omega_E} : H^1(K_{p,n}, TpE \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}) \rightarrow K_{n,p}$ be the $p$-adic logarithm associated to $\omega_E$. We have the following proposition on the non-torsionness of $P_1^+$, which we later refine in Theorem 7.50.

**Proposition 7.37.** We have that $P_1^+$ is not $\Lambda_{\mathcal{O}_{K_p}}^{-1/p}$-torsion.

**Proof.** Suppose $P_1^+ = \lim_{\leftarrow n \in \mathbb{Z}_{\geq 0}} P_{1,n}^+$ were $\Lambda_{\mathcal{O}_{K_p}}^{-1/p}$-torsion. Then for all but finitely many characters $\psi : \text{Gal}(K_{n}/K) \rightarrow \mathbb{Q}_p^\times$, we would have

$$\sum_{\tau \in \text{Gal}(K_{n}/K)} \psi(\tau) \log_E((P_{1,n})^\tau) = \bar{\omega}_{2n}(\psi) \sum_{\tau \in \text{Gal}(K_{n}/K)} \psi(\tau) \log_E((P_{1,n})^\tau) = 0.$$

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But the left-hand side of the above equation is equal to $c_B \mathcal{L}_p(g \times \chi)(\psi^{-1}) = c_B L_p(g \times \chi)(\psi^{-1})$ by [30 Theorem 9.10], Theorem 6.4 and Definition 6.5. Since $L_p(g \times \chi) \neq 0$ as an element of $\Lambda_{\overline{\mathbb{Q}}}$, then $L_p(g \times \chi)(\psi^{-1}) \neq 0$ for all but finitely many $\psi$, a contradiction. □

**Lemma 7.38.** $X$ is free of $\Lambda_{\mathcal{O}_{K_p}}$-rank 2, $Y \subset X$ is a $\Lambda_{\mathcal{O}_{K_p}}$-module of rank 1.

Proof. The first assertion that $X$ is free of rank 2 is [30 Lemma 11.8(ii)].

By Proposition 7.35, we have that $X/Y$ is free and hence torsion-free. Moreover, since $Y \neq 0$ (because $P_1^+ \in \text{cores}_{K_{p,\infty}/K_{p,\infty}}(Y) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$ and $P_1^+ \neq 0$ by Proposition 7.37) we have that $Y$ has $\Lambda_{\mathcal{O}_{K_p}}$-rank at least 1. We now show that it has rank exactly 1. If $Y$ had rank 2, then $X/Y$ would be a torsion-free rank 0 module, meaning $X/Y = 0$, i.e. $X = Y$. However, by the definitions this would imply that $\text{Hom}(E(K_{p,\infty})_{\text{free}}, \text{Hom}(E(K_{p,\infty}), \mathcal{O}_{K_p})) = \text{Hom}(E^+(K_{p,\infty}), \mathcal{O}_{K_p}) = \text{Hom}(E^+(K_{p,\infty})_{\text{free}}, \mathcal{O}_{K_p})$, but clearly $E^+(K_{p,\infty})_{\text{free}}$ is a proper subgroup of $E(K_{p,\infty})$ (for example, this can be seen using the $p$-adic logarithm $\log : E(K_{p,\infty})_{\text{free}} \hookrightarrow K_{p,\infty}$ and the fact that trace commutes with the logarithm), giving a contradiction. □

**Proposition 7.39** (cf. Proposition 4.4 of [36]). $Y$ is a free $\Lambda_{\mathcal{O}_{K_p}}$-module of rank 1. In particular, there is a splitting

\[(173) \quad X \cong Y \oplus X/Y.\]

Proof. From Lemma 7.38 we have that $X/Y$ is $\Lambda_{\mathcal{O}_{K_p}}$-free and so torsion-free, and so by Lemma 7.31 we have that $Y$ is free of $\Lambda_{\mathcal{O}_{K_p}}$-rank 1. Now the exact sequence $0 \to Y \to X \to X/Y \to 0$ splits since $X/Y$ is free, which gives (173). □

**Definition 7.40.** Recall the map from [89] (with $f = 1$)

\[\mu_{\text{glob}} : \mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} \to \Lambda(\tilde{G}, \mathcal{O}_{K_p})[1/p],\]

where we have used that $\mathcal{U}^1 = \mathcal{U}^1$ since $f = 1$. Twisting by $\lambda^{-1}$ and descending to the $\chi_A$-isotypic component (using the natural decomposition $\tilde{G} = \text{Gal}(K_p(E[p^\infty])/K_{p,\infty}) \times \text{Gal}(K_{p,\infty}/K_p)$, where as before $K_{p,\infty} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} K_{n,p}$), we get a map

\[(\mu_{\text{glob}})_E : (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1})_E \to \left(\Lambda(\tilde{G}, \mathcal{O}_{K_p})[1/p] \otimes_{\mathcal{O}_{K_p}} T(\lambda^{-1})\right)_E = \Lambda(\text{Gal}(K_{p,\infty}/K_{p,\infty}), \mathcal{O}_{K_p})[1/p] \otimes_{\mathcal{O}_{K_p}} T(\lambda^{-1})_E\]

(where the target is viewed as a $\Lambda(\tilde{G}, \mathcal{O}_{K_p})$-module). Twisting by $\lambda/\chi_E$, we get a map

\[(\lambda/\chi_E)^*(\mu_{\text{glob}})_E : (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1})_E \to \Lambda(\text{Gal}(K_{p,\infty}/K_{p,\infty}), \mathcal{O}_{K_p})[1/p] \otimes_{\mathcal{O}_{K_p}} T(\lambda^{-1})_E \otimes_{\mathcal{O}_{K_p}} T(\lambda/\chi_E)\]

\[(174) \quad \to \Lambda(\text{Gal}(K_{p,\infty}/K_{p,\infty}), \mathcal{O}_{K_p})[1/p] \otimes_{\mathcal{O}_{K_p}} T(\lambda/\chi_E) \to \Lambda(\text{Gal}(K_{p,\infty}/K_{p,\infty}), \mathcal{O}_{K_p})[1/p] = \Lambda_{\mathcal{O}_{K_p}}[1/p].\]

**Proposition 7.41.** We have a decomposition

\[(175) \quad (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1})_E = \mathcal{U}_1 \oplus \mathcal{U}_2\]

where $\mathcal{U}_2 = \ker ((\lambda/\chi_E)^*(\mu_{\text{glob}})_E)_E$. 

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Proof. Recall that \((U^1)_{\chi_A}\) is a free \(\Lambda_{\mathcal{O}_{K_p}}\)-module of rank 2. Let \(U' \subset U^1\) be as in Proposition 3.61, which is a free \(\Lambda_{\mathcal{O}_{K_p}}\)-module of rank 1, so that \(U'_{K_p} := ((U' \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})/(\lambda_{-1}^{-1}))\) is a free \(\Lambda_{\mathcal{O}_{K_p}}\)-module of rank 2, with a \(\Lambda_{\mathcal{O}_{K_p}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}\)-module structure. The decomposition (176) induces a direct sum decompositions \(U'_{K_p} = U'_1 \oplus U'_2\) where, denoting the \(\Lambda_{\mathcal{O}_{K_p}}\)-action on \(U'\) by \(\cdot\),
\[
U'_1 = \{ u \in U'_{K_p} : \lambda \cdot u = u\lambda \in \Lambda_{\mathcal{O}_{K_p}} \otimes 1 \}, \quad U'_2 = \{ u \in U'_{K_p} : \lambda \cdot u = u\lambda \in \Lambda_{\mathcal{O}_{K_p}} \otimes 1 \}.
\]
By the freeness of \((U^1)_{\chi_A}\), letting \(U'_{K_p} := ((U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})/(\lambda^{-1}))_{\chi_E}\), we also get a decomposition (176)
\[
U'_{K_p} = U_1 \oplus U_2.
\]
Let \(\tau \in \text{Gal}(K_p/\mathbb{Q}_p)\) be the nontrivial element, so that there is a natural decomposition (177)
\[
\Lambda_{\mathcal{O}_{K_p}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p} = \Lambda_{\mathcal{O}_{K_p}} \times \Lambda_{\mathcal{O}_{K_p}}^\tau,
\]
where \(\Lambda_{\mathcal{O}_{K_p}}^\tau\) denotes the abelian group \(\Lambda_{\mathcal{O}_{K_p}}\) with \(\Lambda_{\mathcal{O}_{K_p}}\)-module structure \(\lambda(x) = \tau(x)\). Then (177) preserves the decomposition (176). By (80) and we have
\[
U_2 = \ker ((\lambda/\chi_E)^*(\mu_{\text{glob}})_{\chi_E}).
\]

\[\square\]

Proposition 7.42. The map \((7.40)\) is surjective, and so gives an isomorphism
\[
(\lambda/\chi_E)^*(\mu_{\text{glob}})_{\chi_E} : U_1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathcal{O}_{K_p}[1/p].
\]
Proof. Recall \(A/L\) chosen as in Definition 4.1 and with \(g_0|f\) as in Choice 5.7. By the theory of CM, we have that \(\lambda_A : \text{Gal}(L_p(A[p^\infty])/L_p) \to \mathcal{O}_{K_p}^\times\) is the local reciprocity map \(\kappa\). By Proposition 3.67 the cokernel of \((\lambda_A/\chi_A)^*\mu_{\text{glob}}^{1L_p}\) is
\[
\left(\mathcal{O}_{K_p}^{\chi_A} \otimes \mathcal{O}_{K_p}^{(\kappa^{-1})}\right)_{\chi_A} = (\mathcal{O}_{K_p})_{\chi_A} = 0,
\]
where the last equality follows because \(\chi_A\) is not the trivial character. Hence \((\lambda_A/\chi_A)^*\mu_{\text{glob}}^{1L_p})_{\chi_A}\) is surjective, and thus so is \((\lambda_A/\chi_A)^*(\mu_{\text{glob}})_{\chi_A}\). Now the surjectivity for \((\lambda/\chi_E)^*(\mu_{\text{glob}})_{\chi_E}\) follows from (90).

Recall that \(K_{\pi_2n} = K_n\).

Proposition 7.43. \(\text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi_2n,p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, K_p/\mathcal{O}_{K_p})\) is a free \(\Lambda_{\mathcal{O}_{K_p,n}}\)-module of rank 1, and there is a compatible sequence of isomorphisms
\[
\text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi_2n,p}), \mathcal{O}_{K_p}) \xrightarrow{\sim} \mathcal{O}_{2n-2}\Lambda_{\mathcal{O}_{K_p,n}}
\]
(178)
\[
\text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi_2n,p}), \mathcal{O}_{K_p}) \xrightarrow{\sim} \mathcal{O}_{2n}\Lambda_{\mathcal{O}_{K_p,n}}.
\]
Moreover, letting \(E(K_{\pi_2n,p})_{\text{free}} = E(K_{\pi_2n,p})/E(K_{\pi_2n,p})_{\text{tors}}\) denote the \(\mathcal{O}_{K_p}\)-free part of \(E(K_{\pi_2n,p})\) (which is a direct summand of \(E(K_{\pi_2n,p})\)), and letting \(E^+(K_{\pi_2n,p})_{\text{free}} = E^+(K_{\pi_2n,p}) \cap E(K_{\pi_2n,p})_{\text{free}}\) denote the \(\mathcal{O}_{K_p}\)-free part of \(E^+(K_{\pi_2n,p})\), there are natural \(\Lambda_{\mathcal{O}_{K_p}}\)-equivariant injections
\[
E^+(K_{\pi_2n,p})_{\text{free}} \hookrightarrow \Lambda_{\mathcal{O}_{K_p,n}}
\]
such that for every \( n \) we have the following commutative compatibility diagram

\[
\begin{align*}
E^+(K_{\pi^{2n-2},p})_{\text{free}} & \hookrightarrow \Lambda_{\mathcal{O}_{K_p},n-1} \\
& \downarrow \quad \downarrow \text{Gal}(K_{n-1}/K) \cong \mathcal{O}_{K_p}/p^{n-1} \\
E^+(K_{\pi^{2n},p})_{\text{free}} & \rightarrow \Lambda_{\mathcal{O}_{K_p},n}.
\end{align*}
\]

(179)

**Proof.** By Proposition 7.35 we have that \( \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{p,\infty}), \mathcal{O}_{K_p}) \) is a free \( \Lambda_{\mathcal{O}_{K_p}} \)-module of rank 1, and so

\[
\text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi^{2n},p}), \mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{p,\infty}), \mathcal{O}_{K_p}) \otimes \Lambda_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p},n}
\]

is free of \( \Lambda_{\mathcal{O}_{K_p},n} \)-rank 1. Note that by the freeness of \( \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{p,\infty}), \mathcal{O}_{K_p}) \), the isomorphisms \( \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi^{2n},p}), \mathcal{O}_{K_p}) \cong \Lambda_{\mathcal{O}_{K_p},n} \) can be chosen to be compatible for varying \( n \). Now we have a map

\[
E^+(K_{\pi^{2n},p}) \rightarrow \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi^{2n},p}), \mathcal{O}_{K_p}) \cong \text{Hom}_{\mathcal{O}_{K_p}}(\Lambda_{\mathcal{O}_{K_p},n}, \mathcal{O}_{K_p})
\]

(181)

\[
\xrightarrow{\sim} \Lambda_{\mathcal{O}_{K_p},n}
\]

where the first arrow is given by

\[
E^+(K_{\pi^{2n},p}) \ni c \mapsto \left( \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{\pi^{2n},p}), \mathcal{O}_{K_p}) \ni f \mapsto f(c) \in \mathcal{O}_{K_p} \right),
\]

and the last arrow is given by \( f \mapsto \sum_{\sigma \in \text{Gal}(K_n/K)} \sigma f(\sigma) \), which is clearly a bijection (since \( \text{Gal}(K_n/K) \) forms a \( \mathcal{O}_{K_p} \)-basis of \( \Lambda_{\mathcal{O}_{K_p},n} \)). We now show that the kernel of the first arrow is \( E^+(K_{\pi^{2n},p})_{\text{tors}} \). Given \( c \in E^+(K_{\pi^{2n},p}) \setminus E^+(K_{\pi^{2n},p})_{\text{tors}} \), one can always construct \( f : E^+(K_{\pi^{2n},p}) = E^+(K_{\pi^{2n},p})_{\text{free}} \oplus E^+(K_{\pi^{2n},p})_{\text{tors}} \rightarrow \mathcal{O}_{K_p} \) with \( f|_{\mathcal{O}_{K_p},c} \neq 0 \) and hence \( f \neq 0 \). Thus \( c \) is not in the kernel of the first arrow, and so the kernel belongs to \( E^+(K_{\pi^{2n},p})_{\text{tors}} \); however, \( f(E^+(K_{\pi^{2n},p})_{\text{tors}}) = 0 \) since \( \mathcal{O}_{K_p} \) is torsion-free, and so this inclusion is in fact an equality. Hence (181) gives an injection of \( \Lambda_{\mathcal{O}_{K_p},n} \)-modules

\[
E^+(K_{\pi^{2n},p})_{\text{free}} \hookrightarrow \Lambda_{\mathcal{O}_{K_p},n}.
\]

The compatibilities (178) and (179) follow from the obvious compatibility of the isomorphisms (180) for varying \( n \) and the definition of (181).

\[\square\]

**Proposition 7.44.** For every \( n \in \mathbb{Z}_{\geq 0} \), and (179) induces compatible isomorphisms

\[
E^+(K_{\pi^{2n-2},p})_{\text{free}} \xrightarrow{\sim} \tilde{\omega}_{2n-2} \Lambda_{\mathcal{O}_{K_p},n-1}
\]

(182)

\[
E^+(K_{\pi^{2n},p})_{\text{free}} \xrightarrow{\sim} \tilde{\omega}_{2n} \Lambda_{\mathcal{O}_{K_p},n}.
\]

**Proof.** From the injection \( E^+(K_{\pi^{2n},p})_{\text{free}} \hookrightarrow \Lambda_{\mathcal{O}_{K_p},n} \) from (179), there is some \( c_n[\pi]^{2n} \) such that \( E^+(K_{\pi^{2n},p})_{\text{free}} \) gets mapped isomorphically onto \( c_n \Lambda_{\mathcal{O}_{K_p},n} \). We will show that \( c_n \) generates the same \( \Lambda_{\mathcal{O}_{K_p},n} \)-ideal as \( \tilde{\omega}_{2n} \).

Note that since \( E^+(K_{\pi^{2n},p})_{\text{free}} \) is torsion-free, the \( p \)-adic logarithm gives a \( \text{Gal}(\overline{K}/K_p) \)-equivariant injection

\[
\log : E^+(K_{\pi^{2n},p})_{\text{free}} \subset E(K_{\pi^{2n},p})_{\text{free}} \hookrightarrow K_{\pi^{2n},p} \cong K_p[\text{Gal}(K_n/K)] = \Lambda_{\mathcal{O}_{K_p},n}[1/p].
\]

(67)
using the normal basis theorem and fact that \( \text{Gal}(K_n/K) = \text{Gal}(K_{n,p}/K_p) = \text{Gal}(K_{2n,p}/K_p) \) (since \( p \) is totally ramified in \( K_n/K \)). Note that the above injection induces an isomorphism of \( \Lambda_{\mathcal{O}_{K_p,n}}[1/p] \)-modules

\[
\log : E(K_{2n,p}, p)_{\text{free}} \otimes \mathbb{Z}_p \mathbb{Q}_p \xrightarrow{\sim} \Lambda_{\mathcal{O}_{K_p,n}}[1/p].
\]

By the definition of \( E^+(K_{2n,p}) \), we then see that \( E^+(K_{2n,p})_{\text{free}} \otimes \mathbb{Z}_p \mathbb{Q}_p \) gets mapped isomorphically onto \( \tilde{\omega}_n^+ \Lambda_{\mathcal{O}_{K_p,n}}[1/p] \) under this isomorphism. Hence the \( \Lambda_{\mathcal{O}_{K_p,n}} \)-annihilator of \( E^+(K_{2n,p})_{\text{free}} \) is \([\pi^{2n}_2]_g / \tilde{\omega}_n^- = \omega_2^+ \), and so the annihilator of \( c_n \) is \( \omega_2^+ \), which implies that \( c_n \) generates the same \( \Lambda_{\mathcal{O}_{K_p,n}} \)-ideal as \( \tilde{\omega}_n^- \).

\[
\text{Definition 7.45.} \text{ Let } c_{n,+} \in E^+(K_{2n,p})_{\text{free}} \text{ be a } \tilde{\omega}_n^- \Lambda_{\mathcal{O}_{K_p,n}} \cong \Lambda_{\mathcal{O}_{K_p,n}}/\omega_2^+ \text{-basis (which exists by (182)).}
\]

\[
\text{Definition 7.46 (cf. Proposition 8.19, 8.21 and 8.23 of [26]).} \text{ Recall that by (169), we have an isomorphism}
\]

\[
\left( \left( (U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})_E(\lambda^{-1}) \right)_{\chi_E} / H^1_+(K_{p,\infty}, T(\lambda)) \right) \otimes_{\Lambda_{\mathcal{O}_{K_p}}} \Lambda_{\mathcal{O}_{K_p,n}} \cong \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{2n,p}), \mathcal{O}_{K_p}).
\]

For each \( n \), let

\[
\text{Col}_{n,+} : \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{2n,p}), \mathcal{O}_{K_p}) \xrightarrow{\sim} \tilde{\omega}_n^- \Lambda_{\mathcal{O}_{K_p,n}}
\]

denote the isomorphism given by

\[
\text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{2n,p}), \mathcal{O}_{K_p}) \ni f \mapsto \sum_{\sigma \in \text{Gal}(K_n/K)} \sigma f(c_{n,+}^\sigma) \in \Lambda_{\mathcal{O}_{K_p,n}},
\]

which is indeed an isomorphism by Proposition 7.43 and (182). Again by Proposition 7.43, the \( \text{Col}_{n,+} \) are compatible for varying \( n \in \mathbb{Z}_{\geq 0} \) with respect to the natural restriction morphisms, and so fit into an isomorphism

\[
\text{Col}_+ : \left( (U^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})_E(\lambda^{-1}) \right)_{\chi_E} / H^1_+(K_{p,\infty}, T(\lambda)) \cong \text{Hom}_{\mathcal{O}_{K_p}}(E^+(K_{p,\infty}), \mathcal{O}_{K_p}) \xrightarrow{\sim} \lim_n \tilde{\omega}_n^- \Lambda_{\mathcal{O}_{K_p,n}} = \Lambda_{\mathcal{O}_{K_p}}.
\]

\[
\text{Definition 7.47 (2-variable big logarithm in height 2 CM case).} \text{ Since } \omega_2^+ c_{n,+} = 0, \text{ there exists } \beta_{n,+} \in \Lambda_{\mathcal{O}_{K_p,n}} \text{ with } \tilde{\omega}_n^- \beta_{n,+} = c_{n,+}. \text{ As in [55] Lemma 6.9], we can choose the } \beta_{n,+} \text{ to form a compatible system}
\]

\[
\beta_+ = \lim_{\text{Tate local duality}} \beta_{n,+} \in \lim_{\text{Definition 7.28}} E^+(K_{2n,p})
\]

By Proposition 7.43, \( \beta_+ \) is a \( \Lambda_{\mathcal{O}_{K_p}} \)-basis of \( H^1_+(K_{p,\infty}, T(\lambda)) \). We define

\[
\text{Log}_+ : H^1_+(K_{p,\infty}, T(\lambda)) \rightarrow \Lambda_{\mathcal{O}_{K_p}}, \quad x = \text{Log}_+(x) \cdot \beta_+.
\]
7.6. Explicit reciprocity law. We now relate the images of the + Heegner classes under the maps \( \log_+ \) to the \( p \)-adic \( L \)-function recalled in Definition 6.2 via an explicit reciprocity law. The constructions of the previous section, along with Wiles’s “\( GL_1 \) explicit reciprocity law” will be used with the Kummer theory results of the previous section to prove the “\( GL_2 \) explicit reciprocity law” in the CM case. We note that we do not avoid all \( GL_2 \) input with this approach, as the proof of the \( p \)-adic Waldspurger formula in [30, Chapter 9] uses \( GL_2 \) methods involving \( p \)-adic differential operators acting on families of \( p \)-adic modular forms on Shimura curves.

**Definition 7.48.** Henceforth, fix a \( \Lambda_{O_K} \)-basis

\[
\gamma_+ = (\gamma_+, n)_{n \in \mathbb{Z}_{\geq 0}} \in \left( (\mathcal{U}^1 \otimes \mathbb{Z}_p \mathcal{O}_K)(\lambda^{-1}) \right)_{\chi E} / H^1_+(K_{p, \infty}, T(\lambda)),
\]
as in [184].

Recall that \( H^1_+(K_{p, \infty}, T(\lambda)) \) and \( (\mathcal{U}^1 \otimes \mathbb{Z}_p \mathcal{O}_K)(\lambda^{-1}) \) are free \( \Lambda_{O_K} \)-modules of rank 1, and by (173) we have a splitting

\[
(186) \quad \left( (\mathcal{U}^1 \otimes \mathbb{Z}_p \mathcal{O}_K)(\lambda^{-1}) \right)_{\chi E} \cong H^1_+(K_{p, \infty}, T(\lambda)) \oplus \left( (\mathcal{U}^1 \otimes \mathbb{Z}_p \mathcal{O}_K)(\lambda^{-1}) \right)_{\chi E} / H^1_+(K_{p, \infty}, T(\lambda)).
\]

**Definition 7.49.** Let \( \iota : \text{Gal}(K_{\infty}/K) \to \text{Gal}(K_{\infty}/K) \) be the involution \( \gamma \mapsto \gamma^{-1} \), and let \( \iota : \Lambda_{O_{K_p}} \to \Lambda_{O_{K_p}} \) be the induced map.

**Theorem 7.50.** Recall \( L_p(g \times \chi) \) from Definition 6.5, and let \( \Omega_p \in \mathcal{O}_{L_p}^\times = \mathcal{O}_{K_p}^\times \) be as in Definition 3.27, and \( c_B \in \mathbb{Q}^\times \) as in Definition 7.36. Then we have the identity

\[
\text{Log}_+(P_1^+) \cdot \iota \left( \text{Col}_+(\gamma_+)^{-1} \right) = \sigma_{-1,p} \cdot c_B \Omega_p^{-1} \cdot L_p(g \times \chi) \cdot (\lambda/\chi_E)^*(\mu_{\text{glob}})_{\chi E}(\gamma_+)^{-1}
\]
in \( \Lambda_{O_{K_p}}^{-}[1/p] \). Here, \( \sigma_{-1,p} := \text{rec}_p(-1)|_{K_{\infty}^-} \in \Gamma^- \) where \( \text{rec}_p \) is the local reciprocity symbol, and \( (\lambda/\chi_E)^*(\mu_{\text{glob}})_{\chi E}(\gamma_+), \text{Log}_+ \) and \( \text{Col}_+(\gamma_+)^{-1} \) denote the specializations of \( (\lambda/\chi_E)^*(\mu_{\text{glob}})_{\chi E}(\gamma_+) \), \( \text{Log}_+ \) and \( \text{Col}_+(\gamma_+)^{-1} \) under the projection \( \Lambda_{O_{K_p}} \to \Lambda_{O_{K_p}} \), respectively.

**Proof.** To show (187), it suffices to show it is true when evaluated at a dense set anticyclotomic characters. Suppose \( \psi : \text{Gal}(K_n^-/K) \to \mathbb{Q}_p^\times \) is any nontrivial character. Note that \( \psi \) is trivial on \( \text{Gal}(K(E[p])^\times)/K) \).

Then viewing

\[
P_1^+ \in H^1(K_{\pi n,p}, (\mathcal{O}_{K_p}/p^n \mathcal{O}_{K_p})(\lambda)) \otimes_{\Lambda_{O_{K_p}}} \Lambda_{O_{K_p}}^{-}[1/p],
\]
using the above identifications, and letting \( \alpha_{n}, \beta_{n,+}, \text{Log}_+ \) and \( c_{n,+} = \text{Tr}_{K_{\pi n,p}/K_{\pi n,p}}(c_{n,+}) \) denote the anticyclotomic specializations of the \( n \)-th layer \( \alpha_n \) of the \( p \)-level structure \( \alpha, \beta_{n,+} \), \( \text{Log}_+ \) and \( c_{n,+} \), respectively, we have

\[
\text{Log}_+(P_1^+) \left( c_{n,+}, \gamma_{n,+}^{-} \right)_{2n} = \tilde{\omega}_{2n} \text{Log}_+(P_1^+) \left( \beta_{n,+}, \gamma_{n,+}^{-} \right)_{2n} = \tilde{\omega}_{2n} \left( P_{1,n}(\phi), \gamma_{n,+}^{-} \right)_{2n} = \tilde{\omega}_{2n} \left( P_{1,n}(\phi), \gamma_{n,+}^{-} \right)_{2n} = \frac{1}{qp} \text{Tr}_{K_{\pi n,p}/K_p}(E[p]) \left( \log_E \left( \pi_1 \left( P_{\chi,n}(\phi) \right) \right) \cdot d \log(\gamma_{n,+}^{-}) \right),
\]
where the last equality follows from Wiles’s explicit local reciprocity law [45, Chapter 1.4] applied to the good reduction twist \( A/L_p \) and (140). The above displayed equation implies (cf. [30] Theorem
As in the proof of (93) for the Lubin-Tate formal group \( \hat{A} \).

\[ \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi(\tau) \left( \Log^+_{\pi_1}(P^+_{1,n}) (\gamma_{n,+}^- \gamma_{n,+}^-)_{2n} \right) \]

(190)

\[ = \psi^{-1}\left( \frac{d \log(\gamma_{n,+}^-)}{\log'_E(\alpha_{2n})} \right) \frac{1}{qp^n} \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi(\tau) \log_E (\pi_1 (P_{X,n}(\phi))^\gamma) \]

\[ = c_B \psi^{-1}\left( \frac{d \log(\gamma_{n,+}^-)}{\log'_E(\alpha_{2n})} \right) \frac{1}{qp^n} \psi^{-1}(p^n)g(\psi^{-1}) \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi(\tau) \log_B (P_{X,n}(\phi))^\gamma \]

\[ = c_B \psi^{-1}\left( \frac{d \log(\gamma_{n,+}^-)}{\log'_E(\alpha_{2n})} \right) \frac{1}{qp^n} \psi^{-1}(p^n)g(\psi^{-1}) \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi(\tau) \log_B (P_{X,n}(\phi))^\gamma \]

where

\[ \psi^{-1}\left( \frac{d \log(\gamma_{n,+}^-)}{\log'_E(\alpha_{2n})} \right) := \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi^{-1}(\tau) \left( \frac{d \log(\gamma_{n,+}^-)}{\log'_E(\alpha_{2n})} \right)^\gamma \]

and \( g(\psi) = \sum_{j \in (Z/qp^n)^* \times (Z/q)^*} \psi(j) \zeta_{q}^{j} \) is the usual Gauss sum, the penultimate equality follows from the \( \nu \)–adic Waldspurger formula \([30] \) Theorem 9.10, and the last equality follows from Theorem 6.4 for Hodge-Tate weight \((0,0)\) and from Definition 6.5. Now using the same calculation as in the proof of (93) for the Lubin-Tate formal group \( \hat{A} / \mathcal{O}_L^\gamma \), along with (90), we have

\[ \psi(p^n)g(\psi) \psi^{-1}\left( \frac{d \log(\gamma_{n,+}^-)}{\log'_E(\alpha_{2n})} \right) \cdot L_p(g \times \chi)(\psi^{-1}) \]

\[ = \psi(-1) \Omega_{\nu}^{-1} \cdot (\lambda / \chi_E)^{x}(\mu_{\text{glob}}) \chi_E(\gamma_{+})^{-}(\psi^{-1}) \cdot p^{\nu - 1}(p - 1) L_p(g \times \chi)(\psi^{-1}). \]

The left-hand side of (189) can similarly be evaluated as

\[ \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi(\tau) \left( \Log^{+\gamma}_{\pi_1}(P^+_{1,n}) (\gamma_{n,+}^- \gamma_{n,+}^-)_{2n} \right) \]

(190)

\[ = \Log^{+\gamma}_{\pi_1}(P^+_{1,n}) (\psi^{-1}) \cdot \sum_{\tau \in \Gal(K_n^\gamma / K) \times \Gal(K(E[p])) / K)} \psi(\tau) (\gamma_{n,+}^- \gamma_{n,+}^-)_{2n} \]

\[ = p^{\nu - 1}(p - 1) \Log^{+\gamma}_{\pi_1}(P^+_{1,n}) (\psi^{-1}) \cdot \text{Col}_{\pi_1}(\gamma_{n,+}^-)(\psi), \]

where \( \Log^{+\gamma}_{\pi_1}(P^+_{1,n}) (\psi^{-1}) \) denotes the evaluation of \( \Log^{+\gamma}_{\pi_1}(P^+_{1,n}) \in \Lambda_{\mathcal{O}_{K_n^\gamma}} \) at \( \psi^{-1} \) (i.e., \( \gamma_{n,+}^- \) be a generator of \( \Gamma_n^\gamma \), the image of \( \Log^{+\gamma}_{\pi_1}(P^+_{1,n}) \) under \( \gamma_{n,+}^- \rightarrow \psi^{-1}(\gamma_{n,+}^-) \)).

Hence, combining (189) and (190), we get

\[ \Log^{+\gamma}_{\pi_1}(P^+_{1,n})(\psi^{-1}) \cdot \text{Col}_{\pi_1}(\gamma_{n,+}^-)(\psi) \]

\[ = \psi(-1) c_B \Omega_{\nu}^{-1} \cdot (\lambda / \chi_E)^{x}(\mu_{\text{glob}}) \chi_E(\gamma_{n,+}^-)(\psi^{-1}) \cdot L_p(g \times \chi)(\psi^{-1}). \]

Varying \( n \) and \( \psi \) now completes the proof of (187). \( \square \)
Proposition 7.51. We have

\[ H_+^1(K_{p,\infty}, T(\lambda)) = \mathcal{U}_2. \]

Proof. Recall that \( H_+^1(K_{p,\infty}, T(\lambda)) \cong \text{Hom}_{\mathcal{O}_K}(E(K_{p,\infty})/E^+(K_{p,\infty}), \mathcal{O}_{K_p}) \), and is free of rank 1 with basis \( \beta_+ \). In particular, \( \langle \beta_+, \beta_+, \alpha_{n,2n} \rangle = 0 \). By Wiles’s explicit reciprocity law, we hence get

\[ 0 = \langle \beta_+, \beta_+, \alpha_{n,2n} \rangle = q_p^{n-2n} \text{Tr}_{K_{p,p}(A[p^n])}/K_p \left( \log_{E}(\pi_1(\beta_+)) \frac{d \log(\beta_+)}{d \log(\alpha_{n,2n})} \right). \]

Dividing the calculation in (189), with \( \gamma_{n,\pm} \) replaced by \( \beta_+ \) and without the superscripts “−”, by \( \text{Log}_+(P_1^+) \), from the above expression we get for any character \( \psi \in \text{Gal}(K_n/K) \),

\[ 0 = (\lambda/\chi_E)^* (\mu_{\text{glob}})_{\chi_E}(\beta_+) \cdot \frac{L_p(g \times \chi)(\psi^{-1})}{\text{Log}_+(P_1^+)(\psi^{-1})}. \]

Since this identity holds for any \( \psi, L_p(g \times \chi) \neq 0 \) and \( \text{Log}_+(P_1^+) \neq 0 \), we thus get \( (\lambda/\chi_E)^* (\mu_{\text{glob}})_{\chi_E}(\beta_+) = 0 \), i.e. \( \beta_+ \in \mathcal{U}_2 \) by Proposition 7.51. Hence \( H_+^1(K_{p,\infty}, T(\lambda)) \subset \mathcal{U}_2 \). Now in light of (186) and Proposition 7.51, we get the desired equality.

\[ \square \]

Proposition 7.52. We have

\[ (\lambda/\chi_E)^* (\mu_{\text{glob}})_{\chi_E}(\gamma_+) \cdot \iota(\text{Col}_+(\gamma_+)) \in \Lambda_{\mathcal{O}_{K_p}}^\times. \]

Proof. By Proposition 7.51 we have that the image of the \( \Lambda_{\mathcal{O}_{K_p}} \)-basis

\[ \gamma_+ \in \left( (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})/(\lambda^{-1}) \right)_{\chi_E} / H_+^1(K_{p,\infty}, T(\lambda)) \]

under the projection \( (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})/(\lambda^{-1}) \rightarrow \mathcal{U}_1 \) is a basis of \( \mathcal{U}_1 \). Hence since \( (\lambda/\chi_E)^* (\mu_{\text{glob}})_{\chi_E} : \mathcal{U}_1 \rightarrow \Lambda_{\mathcal{O}_{K_p}} \), we have that \( (\lambda/\chi_E)^* (\mu_{\text{glob}})_{\chi_E}(\gamma_+) \in \Lambda_{\mathcal{O}_{K_p}}^\times \). By (184), we have \( \iota(\text{Col}_+(\gamma_+)) \in \Lambda_{\mathcal{O}_{K_p}}^\times. \)

\[ \square \]

Definition 7.53. Define \( \Psi : \Gamma^- \hookrightarrow (\Lambda_{\mathcal{O}_{K_p}})^\times \) be the canonical character (i.e. the character given by the inclusion \( \Gamma^- \hookrightarrow \Lambda_{\mathcal{O}_{K_p}} \)).

Definition 7.54. Note that we have a localization map

\[ \text{loc}_p : H^1(K, T_{g,\chi} \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \rightarrow H^1(K_p, T_{g,\chi} \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) = H^1(K_p, T(\lambda) \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \oplus H^1(K_p, T(\psi^* \chi) \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})). \]

Note this map also extends \( \otimes_{\mathbb{Z}_p} \otimes_{\mathbb{F}_p} \)-linearly.

We have the following immediate corollary of Proposition 7.52 and Theorem 7.50.

Corollary 7.55. We have

\[ L_p(g \times \chi) \in \Lambda_{\mathcal{O}_{K_p}}^- [1/p], \]

and moreover the map \( \text{Log}_+ \) induces a natural isomorphism

\[ \text{Log}_+ : \frac{H_+^1(K_{p,\infty}, V(\lambda))}{\pi_1 \circ \text{loc}_p(\mathcal{Z}_+)} \cdot \Lambda_{\mathcal{O}_{K_p}}^- [1/p] \cong \frac{\Lambda_{\mathcal{O}_{K_p}}^- [1/p]}{L_p(g \times \chi) \cdot \Lambda_{\mathcal{O}_{K_p}}^- [1/p]}. \]
7.7. + Selmer groups.

**Definition 7.56.** Recall the Definitions of the Selmer groups from Definition 6.7

\[ \mathcal{S}^{\text{rel}}(W) = \ker \left( \prod_v \text{loc}_v : H^1(K, W \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \to \prod_{v \mid p} H^1(K, W \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \right), \]

and similarly define \( \mathcal{S}^{\text{rel}}(V), \mathcal{S}^{\text{rel}}(T), \mathcal{X}^{\text{rel}}(W), \mathcal{X}^{\text{rel}}(V) \) and \( \alpha^{\text{rel}}(T) \). Now suppose that \( W = W_{g,\chi}, T = T_{g,\chi}, V = V_{g,\chi} \). Let \( \mathcal{L}_1, \mathcal{L}_2 \in \{\text{rel, +, str}\} \), and for \( M = W, T, V \),

\[ M_1 = W(\lambda), T(\lambda), V(\lambda), \quad M_2 = W(\psi^*\chi), T(\psi^*\chi), V(\psi^*\chi), \]

and let

\[ H^1_{\mathcal{L}_1, \mathcal{L}_2}(K_p, M \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) := H^1_{\mathcal{L}_1}(K_p, M_1 \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \oplus H^1_{\mathcal{L}_2}(K_p, M_2 \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \]

where

\[ H^1_{\mathcal{L}_1}(K_p, M_1 \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) = \begin{cases} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1}) & \mathcal{L}_1 = \text{rel} \\ \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1}) & \mathcal{L}_1 = + \\ 0 & \mathcal{L}_1 = \text{str} \end{cases} \]

and

\[ H^1_{\mathcal{L}_2}(K_p, M_2 \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) = \begin{cases} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1}) & \mathcal{L}_2 = \text{rel} \\ \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1}) & \mathcal{L}_2 = + \\ 0 & \mathcal{L}_2 = \text{str} \end{cases} \]

We finally define the + Selmer groups:

\[ \mathcal{S}^{\mathcal{L}_1, \mathcal{L}_2}(M) = \ker \left( \mathcal{S}^{\text{rel}}(M) \to H^1_{\mathcal{L}_1}(K_p, M_1 \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \oplus H^1_{\mathcal{L}_2}(K_p, M_2 \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \right). \]

We similarly define \( \mathcal{X}^{\mathcal{L}_1, \mathcal{L}_2}(M) \).

**Remark 7.57.** Note that the factorization \cite{148} gives a natural splitting of the local cohomology \( H^1(K_p, M \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \), which allows one to “split” the Selmer structure at \( p \) into a direct sum \( \mathcal{L}_1 \oplus \mathcal{L}_2 \). This splitting seems to play the analogous role, in some sense, to the splitting induced by the splitting of the local cohomology at \( p \) into a direct sum \( H^1(K_p, M \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \oplus H^1(K_p, M \otimes \mathcal{O}_{K_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})) \) in the split case \( p\mathcal{O}_K = p\mathfrak{p} \). Some arguments in \cite{8} Section 5 go through using this analogy.

7.8. A quotient of \( \mathcal{X}^{\text{rel}}(W_{g,\chi}) \). It turns out that a certain natural quotient of \( \mathcal{X}^{\text{rel}}(W_{g,\chi}) \) will arise in our proof of certain cases of the Rational Heegner point Main Conjecture (Theorem 7.72). It seems to play an analogous role to “\( \mathcal{X}^{\text{rel, str}}(W_{g,\chi}) \)” in \cite{8}, in that it can readily be related to a \( GL_1 \) Rubin-type main conjecture.

**Definition 7.58.** Recall that \( M_\infty/K_\infty \) is the maximal pro-\( p \) abelian extension of \( K_\infty \) unramified outside the places above \( p \). Recall our notation (for \( j = 1 \))

\[ \mathcal{X}_{\mathcal{L}_E} = (\mathcal{X} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}) \otimes_{\Lambda(\mathcal{O}_{K_p}, \chi_E)} \Lambda_{\mathcal{O}_{K_p}}, \]

which is a \( \Lambda_{\mathcal{O}_{K_p}} \)-module of rank 1. Let

\[ \mathcal{X}(\lambda^{-1})_{\mathcal{L}_E} := \mathcal{X}_{\mathcal{L}_E} \otimes_{\mathcal{O}_{K_p}} T(\lambda/\chi_E) = (\mathcal{X} \otimes_{\mathbb{Z}_p} T(\lambda^{-1}))_{\chi_E}. \]

These are \( \Lambda_{\mathcal{O}_{K_p}} \)-modules of rank 1 which are closely related to the \( \Lambda_{\mathcal{O}_{K_p}} \)-modules \( \mathcal{X}^{\text{rel}}(W(\lambda)) \) and \( \mathcal{X}^{\text{rel}}(W(\psi^*\chi)) \), as the next proposition shows.
Proposition 7.59 (cf. Proposition 2.4.12 of [1]). The natural map
\[ \mathcal{X}'(\lambda^{-1})_{\chi_E} := \mathcal{X}'(\lambda^{-1})_{\chi E} \otimes_{\Lambda_{O_{Kp}}} \Lambda_{O_{Kp}}^{-} \to \mathcal{X}'_{\text{rel}}(W(\lambda)) \]
is an isomorphism of \(\Lambda_{O_{Kp}}^{-}\)-modules, and similarly with \(\lambda\) replaced by \(\psi^{*}\chi\). In particular, \(\mathcal{X}'_{\text{rel}}(W(\lambda))\) and \(\mathcal{X}'_{\text{rel}}(W(\psi^{*}\chi))\) have \(\Lambda_{O_{Kp}}^{-}\)-rank 1.

Proof. The first assertion follows from Proposition 5.8 and the same argument as for the statement for the relaxed Selmer group in [1 Proposition 2.4.12], except with \(p^{*} = p\). Note that the analogous statement in the \(p^{*} = p\) case to Lemma 2.4.11 in loc. cit. follows from the argument given in loc. cit., or in [15 Lemma IV.3.5]. The second assertion follows because \(\mathcal{X}'(\lambda^{-1})_{\chi E}\) has \(\Lambda_{O_{Kp}}^{-}\)-rank 1, as recalled above.

Similarly, recall \(\mathcal{X}'\), which is a quotient \(\mathcal{X}' \to \mathcal{X}'\), from Definition 4.26 and let
\[ \mathcal{X}'(\lambda^{-1})_{\chi E} = (\mathcal{X}' \otimes_{\mathbb{Z}_p} T(\lambda^{-1}))_{\chi_E}. \]

Recall the map from Definition 7.60
\[ (\lambda/\chi_E^{*})(\mu_{\text{glob}})_{\chi E} : ((U^{1} \otimes_{\mathbb{Z}_p} O_{K_p})(\lambda^{-1}))_{\chi_E} \to \Lambda_{O_{K_p}}[1/p]. \]

Definition 7.60. Recall the twisted reciprocity map
\[ \text{rec}(\lambda^{-1})_{\chi E} : ((U^{1} \otimes_{\mathbb{Z}_p} O_{K_p})(\lambda^{-1}))_{\chi_E} \to \mathcal{X}(\lambda^{-1})_{\chi E} \]
induced by the global reciprocity map \(\text{rec} : U^{1} \to \mathcal{X}\) (recalling that in our case \(f = 1\) and \(K\) has class number 1, and hence \(L = K(1) = K, U^{1} = U^{1}\)), we have
\[ \mathcal{X}'(\lambda^{-1})_{\chi E} = \mathcal{X}(\lambda^{-1})_{\chi E}/\text{rec}(\lambda^{-1})_{\chi E} (\ker((\lambda/\chi_E^{*})(\mu_{\text{glob}})_{\chi E})). \]

In particular, \(\text{rec}(\lambda^{-1})_{\chi E}\) has \(\text{rank } 1\), and hence
\[ \text{rec}(\lambda^{-1})_{\chi E} \] is a torsion \(\Lambda_{O_{K_p}}^{-}\)-module.

Definition 7.61. Let \(\gamma_{+}\) be a topological generator of \(\text{Gal}(K_{\infty}/K_{\infty}^{\infty})\), and let \(I_{+} = (\gamma_{+} - 1)\Lambda_{O_{K_p}}^{-}\), so that \(\Lambda_{O_{K_p}}^{-} \cong \Lambda_{O_{K_p}}^{-}/I_{+}\). As in [1], we define the descent defect by
\[ \mathcal{D} = \text{char}_{\Lambda_{O_{K_p}}}^{-} (\mathcal{X}'(\lambda^{-1})_{\chi E}[I_{+}]). \]

Proposition 7.62. We have that \(\text{char}(\mathcal{X}'(\lambda^{-1})_{\chi E})\) is coprime with \(I_{+}\), and hence \(\mathcal{X}'(\lambda^{-1})_{\chi E}^{-}\) is \(\Lambda_{O_{K_p}}^{-}\)-torsion, \(\mathcal{D}\) is nonzero and
\[ \text{char}_{\Lambda_{O_{K_p}}^{-}} (\mathcal{X}'(\lambda^{-1})_{\chi E}^{-}) = \text{char}_{\Lambda_{O_{K_p}}^{-}} (\mathcal{X}'(\lambda^{-1})_{\chi E})^{-} \mathcal{D}. \]

Proof. Let \(g \subset O_{K}\) be the conductor of \(\lambda\). Recall that by Theorem 4.70 we have \(\text{char}_{\Lambda_{O_{K_p}}} (\mathcal{X}'(\lambda^{-1})_{\chi E})\) is \((\lambda/\chi_E^{*})(\mu_{\text{glob}})(g; \chi_E)\). By (95) for \(k = 1\), since the root number of \(E\) is \(-1\), by the main result of [39], \((\lambda/\chi_E^{*})(\mu_{\text{glob}})(g; \chi_E)\) does not vanish for all but finitely many values in the anticyclotomic interpolation range, and hence
\[ (\lambda/\chi_E^{*})(\mu_{\text{glob}})(g; \chi_E)^{-} \neq 0. \]

In particular, \(\text{char}(\mathcal{X}'(\lambda^{-1})_{\chi E})\) is coprime with \(I_{+}\). The remaining assertions now follow from [40 Lemma 6.2].

Convention 7.63. Henceforth, given a \(\Lambda_{O_{K_p}}, \Lambda_{O_{K_p}}^{-}\) or \(\Lambda_{O_{K_p}}^{-}[1/p]\)-module \(M\), we let \(M_{\text{tors}} \subset M\) the \(\Lambda_{O_{K_p}}, \Lambda_{O_{K_p}}^{-}\) or \(\Lambda_{O_{K_p}}^{-}[1/p]^{-}\)-torsion submodule, respectively. We will sometimes explicitly describe which module structure we are referring to to avoid ambiguity.
Our next proposition relates the characteristic ideal of $\mathcal{X}'(\lambda^{-1})_{\chi E}$ with that of the $\Lambda_{\mathcal{O}_{K_p}}$-torsion submodule $\mathcal{X}(\lambda^{-1})_{\chi E}$.

**Theorem 7.64** (cf. Lemma 3.8 of [8]). Suppose that $p$ is ramified in $K$. There is an isomorphism of $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-modules

$$
\mathcal{X}'(\lambda^{-1})_{\chi E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,
$$

where the subscript “tors” denotes $\Lambda_{\mathcal{O}_{K_p}}$-torsion, and the superscript “−” denotes tensoring with $\otimes \Lambda_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}$. In particular, we have

$$
\text{char}_{\Lambda_{\mathcal{O}_{K_p}}[1/p]}(\mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \mathcal{L}_p(\lambda)^{-1} \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p].
$$

The key to the proof is the following result on the image of (192).

**Lemma 7.65.** We have a decomposition

$$
\mathcal{X}(\lambda^{-1})_{\chi E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong (\mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \oplus X
$$

where $X$ is a free $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-module of rank 1, and as before the subscript of “tors” denotes $\Lambda_{\mathcal{O}_{K_p}}$-torsion. Moreover, the map induced by (192)

$$
\text{rec}(\lambda^{-1})_{\chi E} : ([U_1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}](\lambda^{-1}))_{\chi E} \xrightarrow{(192)} ([U_1^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p] \oplus [U_2^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p] \to \mathcal{X}(\lambda^{-1})_{\chi E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
$$

maps $U_2^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ isomorphically onto $X$.

**Proof.** We have a tautological exact sequence of $\Lambda_{\mathcal{O}_{K_p}}$-modules

$$
0 \to \mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}} \to \mathcal{X}(\lambda^{-1})_{\chi E} \to \mathcal{X}(\lambda^{-1})_{\chi E}/\mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}} \to 0.
$$

Then $\mathcal{X}(\lambda^{-1})_{\chi E}/\mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}}$ is $\Lambda_{\mathcal{O}_{K_p}}$-torsion free. Since $\Lambda_{\mathcal{O}_{K_p}} \cong \mathcal{O}_{K_p}[Y][1/p]$ is a Dedekind domain, then $\mathcal{X}(\lambda^{-1})_{\chi E}/\mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}}$ is a projective $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-module. Hence the above exact sequence splits after tensoring with $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, so letting $X = (\mathcal{X}(\lambda^{-1})_{\chi E}/\mathcal{X}(\lambda^{-1})_{\chi E,\text{tors}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ we have (196).

We now prove the assertion on (197). Note that we have a global $\mathcal{O}_{K_p}$-bilinear Kummer pairing

$$
E(K_{\infty}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \times \text{Gal}(\overline{K}/L(A[p^\infty]))(\lambda^{-1})_{\chi E} \to K_p/\mathcal{O}_{K_p}.
$$

By the theory of CM and the assumption that $K$ has class number 1, we have $L(A[p^\infty]) \subset K(p^{\infty})$, and hence the above pairing factors through

$$
E(K_{\infty}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \times \mathcal{X}(\lambda^{-1})_{\chi E} \to K_p/\mathcal{O}_{K_p}
$$

whence we get a map

$$
\mathcal{X}(\lambda^{-1})_{\chi E} \to \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{\infty}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p, K_p/\mathcal{O}_{K_p}).
$$

Let $g$ denote the conductor of $\lambda$. Recall that $\mathcal{C}(g) \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}$ denotes the free $\Lambda_{\mathcal{O}_{K_p}}$-module of rank 1. By Remark 4.27, we have $([U_1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}](\lambda^{-1}))_{\chi E} \cap U_2 = 0$. By the $\Lambda_{\mathcal{O}_{K_p}}$-freeness of $([U_1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p}](\lambda^{-1}))_{\chi E}$, and the fact that $((E/\mathcal{C}(g)) \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_p})(\lambda^{-1})_{\chi E}$ is $\Lambda_{\mathcal{O}_{K_p}}$-torsion, we have
that $(\mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_{K_p})(\lambda^{-1})_{\chi_E} \cap U_2 = 0$. Let $\mathcal{E}(\lambda^{-1})_{\chi_E} := (\mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_{K_p})(\lambda^{-1})_{\chi_E}$ to shorten notation. We hence have a commutative diagram relating (165) and (198)

$$
\begin{array}{ccc}
U_1 \oplus U_2 & \xrightarrow{(200)} & \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, K_p/\mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}), \mathcal{O}_{K_p}) \\
\downarrow & & \downarrow \sim \\
U_1/\mathcal{E}(\lambda^{-1})_{\chi_E} \oplus U_2 & \xrightarrow{(199)} & \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, K_p/\mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, \mathcal{O}_{K_p}) \\
\downarrow \text{rec}(\lambda^{-1})_{\chi_E} & & \downarrow \\
\chi(\lambda^{-1})_{\chi_E} & \xrightarrow{(199)} & \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, K_p/\mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, \mathcal{O}_{K_p})
\end{array}
$$

where the top horizontal arrow is an isomorphism, and the left non-labelled vertical arrow is (clearly) a surjection.

**Claim 7.66.** The right non-labelled vertical arrow in (200) is a surjection of $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-modules after tensoring with $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

$$\text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, K_p) \to \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, K_p).$$

**Proof of Claim 7.66.** One sees this as follows. Any $\mathcal{O}_{K_p}$-linear homomorphism $f : E(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$ extends to $f : \bar{E}(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$, where $\bar{E}(K_{\infty}) = \lim_{\leftarrow n} E(K_{\infty})/p^n E(K_{\infty})$ is the $p$-adic completion of $E(K_{\infty})$ in $E(K_{p,\infty})$. Note that since $K_p$ is torsion free, $f : \bar{E}(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$ factors through $f : \bar{E}(K_{\infty})_{\text{free}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$, where $\bar{E}(K_{\infty})_{\text{free}} = \bar{E}(K_{\infty})/\bar{E}(K_{\infty})_{\text{tors}} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \bar{E}(K_n)/\bar{E}(K_n)_{\text{tors}}$ is the $\mathcal{O}_{K_p}$-free part (recall that finitely generated torsion-free over $\mathcal{O}_{K_p}$, a PID, implies freeness, so each $\bar{E}(K_n)/\bar{E}(K_n)_{\text{tors}}$ is free). Finally, this uniquely extends linearly to $f : \text{Sat}_{E(K_{p,\infty})}(\bar{E}(K_{\infty})_{\text{free}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$ where $E(K_{p,\infty})_{\text{free}} = E(K_{p,\infty})/E(K_{p,\infty})_{\text{tors}}$, and

$$\text{Sat}_{E(K_{p,\infty})}(\bar{E}(K_{\infty})_{\text{free}}) = \{ x \in E(K_{p,\infty})_{\text{free}} : \exists r \in \mathbb{Z}_{\geq 0}, \ p^r x \in \bar{E}(K_{\infty})_{\text{free}} \}.$$

Note that using the $p$-adic logarithm, there is a finite-index subgroup $E_n \subset E(K_{n,p})$ isomorphic to $\mathcal{O}_{K_{n,p}}$. It is clear that $\text{Sat}_{E(K_{p,\infty})_{\text{free}},(\bar{E}(K_{\infty})_{\text{free}})} \cap E_n \subset E_n$ is a direct summand of $E_n$ as a $\mathcal{O}_{K_p}$-module, and hence we can extend the restrictions $f : \text{Sat}_{E(K_{p,\infty})_{\text{free}}} \cap E_n \to K_p$ to $f : E_n \to K_p$ and then extend linearly to $f : E(K_{n,p})_{\text{free}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$. By construction, these extensions are compatible for varying $n$, and so we get an extension $f : E(K_{p,\infty})_{\text{free}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$. Finally, using $E(K_{p,\infty}) = E(K_{p,\infty})_{\text{free}} \oplus E(K_{p,\infty})_{\text{tors}}$, to $f : E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to K_p$. Note that all Galois actions continuous extend using $\text{Gal}(K_{\infty}/K) = \text{Gal}(K_{p,\infty}/K_p).$}

**Claim 7.67.** $\text{Hom}_{\mathcal{O}_{K_p}}(E(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, \mathcal{O}_{K_p})$ is $\Lambda_{\mathcal{O}_{K_p}}$-torsion free.

**Proof of Claim 7.67.** We have

$$\text{Hom}_{\mathcal{O}_{K_p}}(E(K_{\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, \mathcal{O}_{K_p}) = \text{Hom}_{\mathcal{O}_{K_p}}(E(K_{p,\infty}), \mathcal{O}_{K_p}) = \bigcup_{n} \text{Hom}_{\mathcal{O}_{K_p}}(E(K_n) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p, \mathcal{O}_{K_p}).$$

Using the above identification, write any element $f$ as $f = (f_n)_{n \in \mathbb{Z}_{\geq 0}}$. Suppose that a nonzero $\lambda \in \Lambda_{\mathcal{O}_{K_p}}$ satisfies $(\lambda \cdot f_n)_{n \in \mathbb{Z}_{\geq 0}} = \lambda \cdot f = 0$, i.e. $\lambda \cdot f_n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Each $f_n$ factors as

$$f_n : (E(K_n) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p)/\lambda(E(K_n) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p) \to \mathcal{O}_{K_p},$$

and by continuity, factors through

$$f_n : \bar{E}(K_{n,\infty})/\lambda \bar{E}(K_{n,\infty}) \to \mathcal{O}_{K_p}. \quad \Box$$
However, since $E(K_n)$ is profinite, and $\lambda E(K_n) \neq 0$, then the source of the above map is $p$-torsion, and hence $f_n = 0$. Since $n$ was arbitrary, this implies $f = 0$. □

**Claim 7.68.** We have $\text{Hom}_{\mathcal{O}_p}(E(K_\infty) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p}) \neq 0$.

**Proof of Claim 7.68.** This follows from the argument of Proposition 7.37, which shows (from the fact that $L_p(g \times X) \neq 0$) that $\pi_1(P_{X,n}(\phi)) \neq 0$ for all $n \gg 0$, and so $\text{Hom}_{\mathcal{O}_p}(E(K_n) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p}) \neq 0$ for all $n \gg 0$, which gives the claim.

Tensoring (200) with $\otimes_{\mathcal{O}_p} \mathcal{Q}_p$ gives

$$
\begin{align*}
(U_1 \otimes_{\mathcal{O}_p} \mathcal{Q}_p) \oplus (U_2 \otimes_{\mathcal{O}_p} \mathcal{Q}_p) & \longrightarrow \text{Hom}_{\mathcal{O}_p}(E(K_{p,\infty}), K_p) \\
(U_1/E(\lambda^{-1})_{\chi_E} \otimes_{\mathcal{O}_p} \mathcal{Q}_p) \oplus (U_2 \otimes_{\mathcal{O}_p} \mathcal{Q}_p) & \longrightarrow \text{Hom}_{\mathcal{O}_p}(E(K_{p,\infty}) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p}) \\
X(\lambda^{-1})_{\chi_E} \otimes_{\mathcal{O}_p} \mathcal{Q}_p & \longrightarrow \text{Hom}_{\mathcal{O}_p}(E(K_\infty) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p})
\end{align*}
$$

where the top horizontal arrow is an isomorphism and, by Claim 7.66, the unlabelled vertical arrows are surjections. Hence, the bottom horizontal arrow is a surjection. Tensoring (201) with $\otimes_{\mathcal{O}_p} \Lambda_{\mathcal{O}_{K_p}}^-$ and using (196) we get

$$
\begin{align*}
(U_1^- \otimes_{\mathcal{O}_p} \mathcal{Q}_p) \oplus (U_2^- \otimes_{\mathcal{O}_p} \mathcal{Q}_p) & \longrightarrow \text{Hom}_{\mathcal{O}_p}(E(K_{p,\infty}), K_p^-) \\
(U_1^-/E(\lambda^{-1})_{\chi_E} \otimes_{\mathcal{O}_p} \mathcal{Q}_p) \oplus (U_2^- \otimes_{\mathcal{O}_p} \mathcal{Q}_p) & \longrightarrow \text{Hom}_{\mathcal{O}_p}(E(K_{p,\infty}) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p})^- \\
X(\lambda^{-1})_{\chi_E} \otimes_{\mathcal{O}_p} \mathcal{Q}_p \oplus X & \longrightarrow \text{Hom}_{\mathcal{O}_p}(E(K_\infty) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p})^-
\end{align*}
$$

where the top horizontal arrow is an isomorphism and the unlabelled vertical arrows and the bottom horizontal arrow are surjections. Since the target of the bottom horizontal arrow is torsion-free by Claim 7.67 and nonzero by Claim 7.68, we have that the bottom horizontal arrow maps $X$ injectively into $\text{Hom}_{\mathcal{O}_{K_p}}(E(K_\infty) \otimes_{\mathbb{Z}} \mathcal{O}_{K_p}, K_p^-)$, and maps $(X(\lambda^{-1})_{\chi_E} \otimes_{\mathcal{O}_p} \mathcal{Q}_p)$ to 0. Hence by the commutativity of (202), we have that $U_2$ maps surjectively onto $X$ via the composition of the left vertical arrows. This completes the proof of the Lemma. □

**Lemma 7.69.** We have $X(\lambda^{-1})_{\chi_E}[I_+] = 0$ and $\mathcal{D} = 1$.

**Proof.** Since $U_2$ is $\Lambda_{\mathcal{O}_{K_p}}$-free, we have $X(\lambda^{-1})_{\chi_E, \text{tors}} \cap \text{rec}(\lambda^{-1})_{\chi_E}(U_2) = 0$. Hence $X(\lambda^{-1})_{\chi_E}[I_+] \hookrightarrow X'(\lambda^{-1})_{\chi_E}$ (using (193)). Since the latter is $\Lambda_{\mathcal{O}_{K_p}}$-torsion by Proposition 7.62, we have that $X(\lambda^{-1})_{\chi_E}[I_+]$ is $\Lambda_{\mathcal{O}_{K_p}}$-torsion, and hence $X(\lambda^{-1})_{\chi_E}[I_+]$ is $\Lambda_{\mathcal{O}_{K_p}}$-pseudonull by [10, Lemma 6.2(i)]. The first assertion now follows because $X(\lambda^{-1})_{\chi_E}$ has no nontrivial pseudonull $\Lambda_{\mathcal{O}_{K_p}}$-submodules.

We now prove that $X'(\lambda^{-1})_{\chi_E}[I_+] = 0$, which gives the second assertion. Note that we have a pseudoisomorphism $X(\lambda^{-1})_{\chi_E}/X(\lambda^{-1})_{\chi_E, \text{tors}} \rightarrow \Lambda_{\mathcal{O}_{K_p}}$ (which is even an injection since it is
apparent that \( \mathcal{X}(\lambda^{-1})_{\chi_E}/\mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}} \) is torsion-free, and by the above we have \( \text{rec}(\lambda^{-1})_{\chi_E}: U_2 \hookrightarrow \mathcal{X}(\lambda^{-1})_{\chi_E}/\mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}} \). Under the composition of these maps, \( U_2 \) maps isomorphically onto \( f \cdot \Lambda_{\mathcal{O}_{K_p}} \) for some \( f \in \Lambda_{\mathcal{O}_{K_p}} \). Hence we have an exact sequence

\[
0 \to (\mathcal{X}(\lambda^{-1})_{\chi_E}/\mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}}) / \text{rec}(\lambda^{-1})_{\chi_E}(U_2) \to \Lambda_{\mathcal{O}_{K_p}} / f \cdot \Lambda_{\mathcal{O}_{K_p}} \to Z \to 0
\]

where \( Z \) is \( \Lambda_{\mathcal{O}_{K_p}} \)-pseudonull. We have a trivial surjection

\[
(\mathcal{X}(\lambda^{-1})_{\chi_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to (\mathcal{X}(\lambda^{-1})_{\chi_E} / \mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}})^{-} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,
\]

which induces a surjection

\[
(\mathcal{X}(\lambda^{-1})_{\chi_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\text{rec}(\lambda^{-1})_{\chi_E}(U_2))^{-} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
\]

\[
\to ((\mathcal{X}(\lambda^{-1})_{\chi_E} / \mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}})^{-} / \text{rec}(\lambda^{-1})_{\chi_E}(U_2)^{-}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\]

However, the source of this surjection is 0 by Lemma 7.65, and hence so is the target. From (203), we now see that \( (f, I_+) = 1 \), or else \( f \equiv 0 \mod (I_+)^+ \) and from the previously remarked vanishing we would have a map

\[
0 \to \Lambda_{\mathcal{O}_{K_p}} / f \cdot \Lambda_{\mathcal{O}_{K_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \Lambda_{\mathcal{O}_{K_p}}[1/p] \to Z \otimes \Lambda_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}[1/p] \to 0,
\]

where \( Z \otimes \Lambda_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}[1/p] \) is \( \Lambda_{\mathcal{O}_{K_p}}[1/p] \)-torsion since \( Z \) is \( \Lambda_{\mathcal{O}_{K_p}} \)-pseudonull. However, this is clearly impossible. From \( (f, I_+) = 1 \), and the second arrow of (203), we then see that

\[
(\mathcal{X}(\lambda^{-1})_{\chi_E} / \mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}}) / \text{rec}(\lambda^{-1})_{\chi_E}(U_2)[I_+] = 0.
\]

Now from the trivial exact sequence

\[
0 \to \mathcal{X}(\lambda^{-1})_{\chi_E}[I_+] \to (\mathcal{X}'(\lambda^{-1})_{\chi_E})[I_+] \to (\mathcal{X}(\lambda^{-1})_{\chi_E} / \mathcal{X}(\lambda^{-1})_{\chi_E,\text{tors}}) / \text{rec}(\lambda^{-1})_{\chi_E}(U_2)[I_+] = 0
\]

and the first assertion of the Proposition that \( \mathcal{X}(\lambda^{-1})_{\chi_E}[I_+] = 0 \), we get that \( (\mathcal{X}'(\lambda^{-1})_{\chi_E})[I_+] = 0 \). As remarked previously, this gives the second assertion of the Proposition.

\[\square\]

**Proof of Theorem 7.64.** By (193) and Lemma 7.65 we get (194). Recall that \( U_2 \cap \mathcal{E}(\lambda^{-1})_{\chi_E} \equiv 0 \) by the above, and so \( \text{rec}(\lambda^{-1})_{\chi_E}(U_2) \subset \mathcal{X}(\lambda^{-1})_{\chi_E} \equiv \Lambda_{\mathcal{O}_{K_p}} \)-free.

Now follows from (194) and Corollary 4.70, Proposition 7.59, Lemma 7.69 and [40, Lemma 6.2(ii)].

\[\square\]

7.9. **Rational Heegner point Main Conjecture.** We make the following Perrin-Riou type “Heegner point Main Conjecture”, which we view as a \( \Lambda \)-adic version of an equivalent form of the Birch and Swinnerton-Dyer conjecture over \( K \) in the rank 1 setting. Our formulation closely follows the non-CM supersingular version formulated in [8, Conjecture 1.2].

**Conjecture 7.70.** We have that \( S^{++}(T_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = S^{++}(V_{g,\chi}), \mathcal{X}^{++}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) and \( z_+ \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p] \) all have \( \Lambda_{\mathcal{O}_{K_p}}[1/p] \)-rank 1. Moreover, we have the following equality of characteristic ideals in \( \Lambda_{\mathcal{O}_{K_p}}[1/p] \):

\[
\text{char}_{\Lambda_{\mathcal{O}_{K_p}}[1/p]}(\mathcal{X}^{++}(W_{g,\chi})_{\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \text{char}_{\Lambda_{\mathcal{O}_{K_p}}[1/p]}(S^{++}(V_{g,\chi}) / (z_+ \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p]))^2,
\]

where the subscript “tors” denotes \( \Lambda_{\mathcal{O}_{K_p}} \)-torsion.

**Remark 7.71.** Note that the fact that \( z_+ \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p] \) is of \( \Lambda_{\mathcal{O}_{K_p}}[1/p] \)-rank 1 follows immediately from the explicit reciprocity law given in Theorem 7.50 and the fact that \( L_p(g \times \chi) \neq 0 \) by Theorem 6.4. It can also be deduced from the results of [12].
7.10. Proof of Rational Heegner point Main Conjecture. The aim of this section is to prove that the rational Heegner point Main Conjecture holds for elliptic curves $E/\mathbb{Q}$ with complex multiplication by an imaginary quadratic field $K$ of class number 1 and in which $p$ is ramified. This will suffice for the purposes of applications to Sylvester’s conjecture and the congruent number problem. Our strategy is to reduce Conjecture 7.70, using the results of Section 6 and the explicit reciprocity law Theorem 7.50, to the $GL_1$ Rubin-type main conjecture (proven in certain settings in Theorem 4.69 and Corollary 4.70).

Theorem 7.72. Suppose $K$ has class number 1 and $p$ is ramified in $K$. Then Conjecture 7.70 is true.

We first need some preliminary results.

Proposition 7.73. We have that $\mathcal{X}^{\text{str}}(W(\lambda))$ and $\mathcal{X}^{\text{str}}(W(\psi^*\chi))$ are torsion $\Lambda_{\mathcal{O}_K^p}$-modules.

Proof. By the definitions, (193) and Proposition 7.59, we have the inclusion

$$ S^{\text{str}}(W(\lambda)) \subset \text{Hom}_{\mathcal{O}_K^p}(\mathcal{X}'(\lambda^{-1}), K_p/\mathcal{O}_K^p). $$

Dualizing, this gives

$$ \mathcal{X}'(\lambda^{-1})_{\mathcal{O}_K^p} \rightarrow \mathcal{X}^{\text{str}}(W(\lambda)). $$

Now by Proposition 7.62, the source of the above surjection is $\Lambda_{\mathcal{O}_K^p}$-torsion, and hence so is the target. The argument for $\psi^*\chi$ is exactly the same. \qed

Proposition 7.74 (cf. Lemma 1.1.9 of [1]). We have a natural isomorphism of $\Lambda_{\mathcal{O}_K^p}$-modules

$$ S^{\text{rel}}(T(\lambda)) \cong \text{Hom}_{\Lambda_{\mathcal{O}_K^p}^-}(\mathcal{X}^{\text{rel}}(W(\lambda)), \Lambda_{\mathcal{O}_K^p}^-). $$

In particular, $\text{rank}_{\Lambda_{\mathcal{O}_K^p}^-}(S^{\text{rel}}(T(\lambda))) = \text{rank}_{\Lambda_{\mathcal{O}_K^p}^-}(\mathcal{X}^{\text{rel}}(W(\lambda)), \Lambda_{\mathcal{O}_K^p}^-) = 1$.

Proof. The isomorphism of $\Lambda_{\mathcal{O}_K^p}$-modules follows by the same argument as in loc. cit., mutatis mutandis, using the fact that $[K_p(E[p]) : K_p]/\#(\mathcal{O}_K^p/p) = p - 1$ by the theory of complex multiplication and our assumption that $K$ has class number 1, and so is in particular prime to $p$. The final rank equality now follows from the isomorphism and Proposition 7.59. \qed

Lemma 7.75. Let $\psi = \lambda\chi^*/(\lambda^*\chi)$, so that $\lambda\psi^{-1} = \psi^*\chi$. Then there are natural isomorphisms induced by the Kummer pairing (163)

$$ (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_K^p)(\mathcal{O}_K^p)(\lambda^{-1})_{\mathcal{O}_K^p} \cong H^1(K_{p,\infty}, T(\lambda)), $$

$$ U_2 \cong H^1_+(K_{p,\infty}, T(\lambda)), $$

$$ (\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_K^p)(\lambda^{-1}\psi)_{\mathcal{O}_K^p} \cong H^1(K_{p,\infty}, T(\psi^*\chi)), $$

$$ U_2(\psi|_{\infty}) \cong H^1(K_{p,\infty}, T(\psi^*\chi)), $$

where the superscript “$-$” denotes tensoring with $\otimes_{\Lambda_{\mathcal{O}_K^p}} \Lambda_{\mathcal{O}_K^p}^-$, and $U_2(\psi|_{\infty}) = U_2 \otimes_{\mathcal{O}_K^p} \mathcal{O}_K^p(\psi|_{\infty})$.

Proof. By [42] Lemma 5.1(ii)], and essentially the argument as in Proposition 7.27, we have $E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q} \otimes \mathbb{Z}_p \cong H^1(K_{p,\infty}, E[p]\mathbb{Q})$ via the Kummer map, and the pairing

$$ (E(K_{p,\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q} \otimes \mathbb{Z}_p) \times ((\mathcal{U}^1 \otimes_{\mathbb{Z}_p} \mathcal{O}_K^p)(\lambda^{-1}))_{\mathcal{O}_K^p} \rightarrow K_p/\mathcal{O}_K^p $
is perfect. Hence, by local duality (as in (172)) we get the first isomorphism in the statement. By Proposition 7.51, we get
\[ U_{2}^{-} \cong H_{+}^{1}(K_{p,\infty}, T(\lambda)) = \ker_{K_{p,\infty}/K_{p,\infty}}(H_{+}^{1}(K_{p,\infty}, T(\lambda))) =: H_{+}^{1}(K_{p,\infty}, T(\lambda)) \]
which is the second isomorphism.

Now consider the isomorphism \( \text{Tw}_{\psi^{-1}} : \Lambda_{\mathcal{O}_{K_{p}}} \xrightarrow{\sim} \Lambda_{\mathcal{O}_{K_{p}}} \) given by \( \gamma \mapsto \psi^{-1}(\gamma)\gamma \) on group-like elements. We then have an isomorphism
\[ H^{1}(K_{p,\infty}, T(\lambda)) \otimes_{\Lambda_{\mathcal{O}_{K_{p}}}} \Lambda_{\mathcal{O}_{K_{p}}} \cong H^{1}(K_{p,\infty}, T(\lambda\psi^{-1})) \]
of \( \Lambda_{\mathcal{O}_{K_{p}}} \)-modules. Hence the third and fourth equalities of the Proposition follow from this isomorphism and the first and second isomorphisms of the Proposition.

\[ \square \]

**Proposition 7.76.** We have
\[ \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(S^{+}(T(\lambda))) = \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(X^{+}(W(\lambda))) = 1, \]
\[ \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(S^{+}(T(\psi^{*}\chi))) = \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(X^{+}(W(\psi^{*}\chi))) = 0, \]
\[ \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(S^{+}(T_{g,\chi})) = \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(X^{+}(W_{g,\chi} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_{p}})) = 1. \]

In fact, \( S^{+}(T(\psi^{*}\chi)) = 0 \) and so \( S^{+}(T_{g,\chi}) = S^{+}(T(\lambda)) \).

**Proof.** By global duality, we have an exact sequence
\[ 0 \to S^{+}(T(\lambda)) \to S^{+}(T(\lambda)) \to H^{1}(K_{p,\infty}, T(\lambda))/H_{+}^{1}(K_{p,\infty}, T(\lambda)) \to X^{+}(W(\lambda)) \to X^{+}(W(\lambda)) \to 0. \]

Now counting ranks, using the fact that \( H^{1}(K_{p,\infty}, T(\lambda)) \) has \( \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}} \)-rank 2 by Lemma 7.75, Proposition 7.35 which shows that \( H_{+}^{1}(K_{p,\infty}, T(\lambda)) := \ker_{K_{p,\infty}/K_{p,\infty}}(H_{+}^{1}(K_{p,\infty}, T(\lambda))) \) has \( \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}} \)-rank 1, Proposition 7.74 gives the first equality of ranks. Moreover, we know that this rank is 1 because \( \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(S^{+}(T(\lambda))) \leq 1 \) by the exactness of the first nonzero term of the above exact sequence and the previously established equality \( \text{rank}_{\Lambda_{\mathcal{O}_{K_{p}}}}(S^{+}(T(\lambda))) = 1 \), and has rank at least 1 because the submodule \( z_{+} \cdot \Lambda_{\mathcal{O}_{K_{p}}}([1/p]) \subset S^{+}(V(\lambda)) \) has rank 1 by Theorem 7.50.

For the second chain of rank equalities, by Lemma 7.75, we have the localization map
\[ \text{loc}_{p} : S^{+}(T(\psi^{*}\chi)) \to H^{1}(K_{p,\infty}, T(\psi^{*}\chi)) \cong ((\mathcal{U}^{1} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_{p}})(\lambda^{-1}))[\chi_{E}] \subset X^{+} \circledast \ker_{\Lambda_{\mathcal{O}_{K_{p}}}}(\mathcal{O}_{K_{p}})(\lambda^{-1}[\chi_{E}]). \]

Now tensoring (174) by \( \otimes_{\Lambda_{\mathcal{O}_{K_{p}}}} \Lambda_{\mathcal{O}_{K_{p}}}([\psi^{-1}_{1}]) \), we have a map
\[ \langle \lambda / \chi E \psi^{-1}_{1} \rangle^{*}(\mu_{\text{glob}})_{X_{E}} : ((\mathcal{U}^{1} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_{p}})(\lambda^{-1}\psi))_{X_{E}} \to \Lambda_{\mathcal{O}_{K_{p}}}^{-} \]
which factors though \( (\mathcal{U}^{1} \otimes_{\mathbb{Z}_p} \mathcal{O}_{K_{p}})(\lambda^{-1}\psi))_{X_{E}} / \mathcal{U}_{2}^{-}(\psi^{-1}) \). By the fact that the root number of \( L(\psi^{*}\chi, s) \) is +1, and the main theorem 39 and (95) for \( k = 1 \), we have
\[ (\lambda / \chi E \psi^{-1}_{1})^{*}(\mu_{\text{glob}})_{X_{E}}(\text{loc}_{p}(\overline{\psi}(1)(\lambda^{-1}\psi))_{X_{E}}) \neq 0, \]
and hence \( \text{loc}_p(\mathcal{C}(1)(\lambda^{-1}\psi)_{\chi_E}) \neq 0 \). Since \( \mathcal{S}^{\text{rel}}(T(\psi^*\chi)) \) is \( \Lambda_{\mathcal{O}_{K_p}}^- \) -torsion free, we hence have that \( \text{loc}_p \) is injective. Now by global duality, we have

\[
0 \to \mathcal{S}^+(T(\psi^*\chi)) \to \mathcal{S}^{\text{rel}}(T(\psi^*\chi)) \xrightarrow{\text{loc}_p} H^1(K_{p,\infty}^-, T(\psi^*\chi))/H^1_+(K_{p,\infty}^-, T(\psi^*\chi)) \to \mathcal{X}^+(W(\psi^*\chi)) \to \lambda^{\text{str}}(W(\psi^*\chi)) \to 0.
\]

Hence \( \mathcal{S}^+(T(\psi^*\chi)) = 0 \). Counting ranks again and using Proposition 7.73 and Lemma 7.75 we see that \( \text{rank}_{\Lambda_{\mathcal{O}_{K_p}}^-}(\lambda^{\text{str}}(W(\psi^*\chi))) = 0 \).

Finally, recall that by definition, \( \mathcal{S}^{+,+}(T_{g,\chi}) = \mathcal{S}^+(T(\lambda)) \oplus \mathcal{S}^+(T(\psi^*\chi)) \) and also \( \lambda^{+,+}(W_{g,\chi}) = \mathcal{X}^+(W(\lambda)) \oplus \mathcal{X}^+(W(\psi^*\chi)) \), which immediately gives the remaining assertions of the Proposition. □

**Proposition 7.77** (cf. Lemma 3.8 of [S]). We have

\[
\text{char}_{\Lambda_{\mathcal{O}_{K_p}}^-[1/p]}(\mathcal{X}^+(W(\lambda)))_{\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \text{char}_{\Lambda_{\mathcal{O}_{K_p}}^-[1/p]}(\mathcal{X}^+(W(\psi^*\chi)))_{\text{tors}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p),
\]

where the subscript “tors” denotes \( \Lambda_{\mathcal{O}_{K_p}}^- \) -torsion.

**Proof.** This follows from essentially the same argument as in [11 Section 1.2]. We note that, by definition, \( H^1_+(K_{p,\infty}^-, T(\lambda)) \) is the orthogonal complement to \( H^1_+(K_{p,\infty}^-, W(\lambda)) \). Let

\[
H^1_+(K, (K_p/\mathcal{O}_{K_p})(\chi)) \subset H^1(K, (K_p/\mathcal{O}_{K_p})(\chi))
\]

denote the subgroup defined by the local conditions that are trivial outside of \( p \), and by the local condition \( H^1_+(K_p, W(\lambda)) \) at \( p \). By [33, Theorem 4.1.13] (which does not use the assumption H.4b of loc. cit., as its proof does not use the Cebotarev results of loc. cit.), for every character \( \rho : \Gamma^- \to \mathcal{O}_{L_0}^\times \) (where \( L/\mathbb{Q}_p \) is a finite extension), there is a non-canonical isomorphism

\[
H^1_+(K, W(\lambda\rho))[p^i] \cong (L/\mathcal{O}_L)^r[p^i] \oplus H^1_+(K, W(\lambda^p \rho^{-1}))[p^i]
\]

for some \( r \in \mathbb{Z}_{\geq 0} \) for all \( i \in \mathbb{Z}_{\geq 0} \). (One can in fact compute this core rank \( r \), as in [S, Lemma 3.8], but we will not need the precise \( r \) for our purposes.) Letting \( i \to \infty \), we hence get a non-canonical isomorphism

\[
H^1_+(K, W(\lambda\rho)) \cong (L/\mathcal{O}_L)^r \oplus H^1_+(K, W(\lambda^p \rho^{-1})).
\]

Following the same arguments as in [11 Lemma 1.2.4], one can show that the natural restriction maps

\[
H^1_+(K, W(\lambda\rho)) \to \mathcal{S}(W(\lambda))(\rho)^{\Gamma^-},
\]

\[
H^1_+(K, W(\lambda^p \rho^{-1})) \to \mathcal{S}(W(\lambda^p))(\rho^{-1})^{\Gamma^-}
\]

are injective with uniformly bounded cokernel as \( \rho \) varies (for fixed \( L/\mathbb{Q}_p \)). Now the equality of characteristic ideals of torsion submodules follows from the same argument as in [11 Lemma 1.2.6]. □

**Proposition 7.78.** Recall the isomorphism \( Tw_{\psi^{-1}} : \Lambda_{\mathcal{O}_{K_p}}^- \xrightarrow{\sim} \Lambda_{\mathcal{O}_{K_p}}^- \) given by \( \gamma \mapsto \psi^{-1}(\gamma)\gamma \) on group-like elements \( \gamma \). We have

\[
Tw_{\psi^{-1}}(L_p(\lambda)^-) = L_p(\lambda\psi^{-1})^- = L_p(\psi^*\chi)^-.
\]

**Proof.** This follows immediately from [95]. □
Proof of Theorem 7.72. It suffices to prove that

$$\text{length}_\mathfrak{p}(\mathcal{X}^{\rel, +}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \text{length}_\mathfrak{p} \left( \frac{S^{+, +}(V_{g,\chi})}{z_+ \cdot \Lambda^-_{\mathcal{O}_{K_p}} [1/p]} \right)$$

for every height 1 prime $\mathfrak{p}$ of $\Lambda_{\mathcal{O}_{K_p}}[1/p]$. We divide this into several steps. By Shapiro's lemma, we have $H^1(K_{p, \infty}, T(\lambda)) = H^1(K_p, \mathcal{O}_{K_p} \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1}))$, and similarly for $H_+^1(K_{p, \infty}, T(\lambda))$ and $\lambda$ replaced by $\psi^*\chi$. For $i = 1, 2$, let

$$\text{loc}_p^i = \pi_i \circ \text{loc}_p : S^{+, +}(V_{g,\chi}) \xrightarrow{\text{loc}_p^i} H_+^1(K_p, T_{g,\chi} \otimes_{\mathbb{Z}_p} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p])$$

(205)

$$\pi_1 \begin{cases} H_+^1(K_p, T(\lambda) \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p]) \quad i = 1 \\ H_+^1(K_p, T(\psi^*\chi) \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p]) \quad i = 2 \end{cases}$$

denote the localization map composed with projection onto the $i$th factor with respect to the decomposition induced by $\mathfrak{p}$. We claim that we have

$$\text{length}_\mathfrak{p}(L_p(g \times \chi)) = \text{length}_\mathfrak{p} \left( \text{coker}(\text{loc}_p^1) \right) + \text{length}_\mathfrak{p} \left( \frac{S^{+, +}(V_{g,\chi})}{z_+ \cdot \Lambda^-_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p]} \right).$$

This follows from an argument analogous to that in Proof of Lemma 5.10. We have a tautological exact sequence

$$0 \to S^{\text{str, +}}(V_{g,\chi}) \to S^{+, +}(V_{g,\chi}) \to H_+^1(K_p, T(\lambda) \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p]) \to \text{coker}(\text{loc}_p^1) \to 0.$$ 

By Theorem 7.50, we have that $z_+ \in S^{+, +}(V_{g,\chi})$ is not $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-torsion. From Proposition 7.76, we have that $S^{+, +}(V_{g,\chi})$ has $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-rank 1, so $S^{\text{str, +}}(V_{g,\chi})$ is $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-torsion and hence 0 since $H^1(K, V_{g,\chi} \otimes_{\mathcal{O}_{K_p}} \Lambda_{\mathcal{O}_{K_p}}[1/p])$ is $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-torsion free by Proposition 7.14. Hence we have an exact sequence

$$0 \to \frac{S^{+, +}(V_{g,\chi})}{z_+ \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p]} \to H_+^1(K_p, T_{g,\chi} \otimes_{\mathbb{Z}_p} \Lambda^-_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p]) \to \text{coker}(\text{loc}_p^1) \to 0,$$

as well as an isomorphism induced by the big logarithm map (Corollary 7.55)

$$\text{Log}_+ : \frac{H_+^1(K_p, T_{g,\chi} \otimes_{\mathbb{Z}_p} \Lambda^-_{\mathcal{O}_{K_p}}(\Psi^{-1})[1/p])}{\Lambda_{\mathcal{O}_{K_p}}[1/p] \cdot \text{loc}_p^1(z_+)} \simeq \frac{\Lambda^-_{\mathcal{O}_{K_p}}[1/p]}{\Lambda_{\mathcal{O}_{K_p}}[1/p] \cdot L_p(g \times \chi)}.$$ 

The equality of lengths now immediately follows.

Step 2: We claim that $\mathcal{X}^{\text{rel, str}}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is $\Lambda_{\mathcal{O}_{K_p}}[1/p]$-torsion and that for any height 1 prime $\mathfrak{p}$ we have

$$\text{length}_\mathfrak{p}(\mathcal{L}_p(\lambda)^- \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p]) = \text{length}_\mathfrak{p}(\mathcal{X}^+(W(\lambda)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) + \text{length}_\mathfrak{p}(\text{coker}(\text{loc}_p^1)).$$

and

$$\text{length}_\mathfrak{p}(\mathcal{L}_p(\psi^*\chi)^- \cdot \Lambda_{\mathcal{O}_{K_p}}[1/p]) = \text{length}_\mathfrak{p}(\mathcal{X}^+(W(\psi^*\chi)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

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This is essentially the same argument as that in the proof of [8, Lemma 5.11]. From global duality, we get an exact sequence

\[(206) \quad 0 \to \text{coker}(\text{loc}_p^1) \to \mathcal{X}_\text{rel}(W(\lambda)) \otimes \mathbb{Z}_p \mathbb{Q}_p \to \mathcal{X}^+(W(\lambda)) \otimes \mathbb{Z}_p \mathbb{Q}_p \to 0.\]

From the previous step, we have that \(\text{coker}(\text{loc}_p^1)\) is \(\Lambda^{-}_{O_{K_p}}[1/p]\)-torsion. By Proposition 7.76, we have that \(\mathcal{X}^+(W(\lambda)) \otimes \mathbb{Z}_p \mathbb{Q}_p\) has \(\Lambda^{-}_{O_{K_p}}[1/p]\)-rank 1, and \(\mathcal{X}_\text{rel}(W(\lambda)) \otimes \mathbb{Z}_p \mathbb{Q}_p\) also has \(\Lambda^{-}_{O_{K_p}}[1/p]\)-rank 1 by Proposition 7.74. Hence Proposition 7.73 implies that \(\mathcal{X}^+(W(\lambda))\) is \(\Lambda^{-}_{O_{K_p}}[1/p]\)-torsion. Taking the \(\Lambda^{-}_{O_{K_p}}[1/p]\)-torsion of (206) and invoking (195), we get the first desired equality of lengths.

For the second equality, by global duality we have an exact sequence

\[0 \to \text{coker}(\text{loc}_p^2) \to \mathcal{X}_\text{rel}(\psi^*\chi) \to \mathcal{X}^+(\psi^*\chi) \to 0.\]

By Proposition 7.76, we have that \(S^+(T(\psi^*\chi)) = 0\), and so \(\text{coker}(\text{loc}_p^2) = H_1^1(K_{P,\infty}, T(\psi^*\chi))\), and since

\[H_1^1(K_{P,\infty}, T(\psi^*\chi)) \cong \mathcal{U}_2(\psi|_{\Gamma}^{-1})\]

via the Kummer map, by Proposition 7.51 and Lemma 7.75, and since the global reciprocity map corresponds to the second arrow of the above exact sequence via local duality pairing and the Kummer map, we get an exact sequence

\[0 \to \mathcal{U}_2(\psi|_{\Gamma}^{-1}) \xrightarrow{\text{rec}(\lambda^{-1}) \chi_E(\psi|_{\Gamma}^-)} \mathcal{X}_\text{rel}(\psi^*\chi) \to \mathcal{X}^+(\psi^*\chi) \to 0.\]

Hence by (193) (twisted by \(\psi^{-1}\)), and recalling that \(\psi^*\chi = \lambda \psi^{-1}\), from the above exact sequence we get an isomorphism

\[\mathcal{X}'(\psi^*\chi) \cong \mathcal{X}^+(\psi^*\chi).\]

By Corollary 4.70, Proposition 7.59, Lemma 7.69 and [10, Lemma 6.2(ii)] (twisted by \(\psi^{-1}\)), we see that \(\text{char}_{\Lambda^{-}_{O_{K_p}}} (\mathcal{X}'(\psi^*\chi)) = \mathcal{L}_p(\psi^*\chi)^- \cdot \Lambda^{-}_{O_{K_p}}\). This, along with the above isomorphism, gives the second equality.

**Step 3:** We have

\[\text{length}_{\mathfrak{p}}(\text{coker}(\text{loc}_p^1)) = 0.\]

To show this, note that by Proposition 7.77 we have

\[\text{length}_{\mathfrak{p}}(\mathcal{X}^+(W(\lambda)_{\text{tors}})) = \text{length}_{\mathfrak{p}}(\mathcal{X}^+(W(\lambda^{-1}))) = \text{length}_{\mathfrak{p}}(\mathcal{X}^+(W(\lambda^*)))\]

\[= \text{length}_{\mathfrak{p}}(\mathcal{X}^+(W(\psi^*\chi))) \overset{\text{Step 2}}{=} \text{length}_{\mathfrak{p}}(\mathcal{L}_p(\psi^*\chi)^-\, \mathcal{L}_p(\lambda)^-)\]

where \(\text{Tw}_{(\lambda/\chi^*)} : \Lambda^{-}_{O_{K_p}} \xrightarrow{\sim} \Lambda^{-}_{O_{K_p}}\) is given on group-like elements by \(\gamma \mapsto (\chi/\chi^*)(\gamma)\). However, again by Step 2 we see that the left-hand side of the above chain of equalities is equal to

\[\text{length}_{\mathfrak{p}}(\mathcal{L}(\lambda)^-) + \text{length}_{\mathfrak{p}}(\text{coker}(\text{loc}_p^1)).\]

Putting these two equalities together completes this Step.

**Step 4:** The first two steps imply that the equality (204) is equivalent to the equality

\[(207) \quad \text{length}_{\mathfrak{p}}(\mathcal{L}(\lambda)^-) + \text{length}_{\mathfrak{p}}(\mathcal{L}_p(\psi^*\chi)^-) = 2\text{length}_{\mathfrak{p}}(\mathcal{L}_p(g \times \chi))\]

for every height 1 prime \(\mathfrak{p}\) of \(\Lambda^{-}_{O_{K_p}}[1/p]\). By (151) we have the equality

\[L_p(g \times \chi)^2 \cdot \Lambda^{-}_{O_{K_p}}[1/p] = (\mathcal{L}(\lambda)^- \mathcal{L}_p(\psi^*\chi)^-) \cdot \Lambda^{-}_{O_{K_p}}[1/p]\]
in $\Lambda_{\mathcal{O}_{K_p}}[1/p]$. This gives \eqref{207}, and hence we are done.

\[\Box\]

8. Descent, Rank 1 $p$-converse, Sylvester’s Conjecture and Goldfeld’s conjecture for the Congruent Number Family

In this section, we give the rank-1 converse theorem implied by the Heegner Point Main Conjecture proved in the previous section. We first need a control theorem analogous to that proven in \cite{26}.

8.1. $p$-converse theorem.

**Definition 8.1.** Fix a topological generator $\gamma^- \in \Gamma^-$, and let $I^- := (\gamma^- - 1)\Lambda_{\mathcal{O}_{K_p}}^-$ be the associated augmentation ideal. We have the following control theorem, analogous to Kobayashi’s control theorem \cite[Theorem 9.3]{26}. Define the finitely generated $\mathcal{O}_{K_p}$-module

$$\text{Sel}_{p,\infty}(B) = \lim_{n} \text{Sel}_{p^n}(B),$$

where

$$\text{Sel}_{p}(B) = \ker \left( \prod_v \text{loc}_0 : H^1(K, B[p^n]) \to H^1(K_v, B[p^n]) \right).$$

**Proposition 8.2.** The natural map

$$(\chi^{+,+}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(I^- (\chi^{+,+}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)) \to \text{Hom}_{\mathcal{O}_{K_p}}(\text{Sel}_{p,\infty}(B), K_p/\mathcal{O}_{K_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is surjective with finite kernel.

We first need the following lemma controlling the local condition at $\mathfrak{p}$.

**Lemma 8.3** (cf. Lemma 5.1 of \cite{2}). The natural map

\[\text{ (208) } H^1_+(K_p, W_{g,\chi}) \to H^1_+(K_{p,\infty}, W_{g,\chi})[I^-] \]

is an isomorphism.

**Proof.** First, note that since the local characters $\lambda_{\mathfrak{p}} : \text{Gal}(K_p(\hat{E}[p^\infty])/K_p) \to \mathcal{O}_{K_p}^\times$ and $\psi_{p,\mathfrak{p}} : \text{Gal}(K_p(\hat{E}[p^\infty])/K_p) \to \mathcal{O}_{K_p}^\times$ are not trivial on the subgroup $\text{Gal}(K_p(\hat{E}[p^\infty])/K^-_{p,\infty})$, we have

$$W_{g,\chi} \text{Gal}_{K_p}^{ab}/K_{p,\infty} = (W(\lambda) \oplus W(\psi_{p,\mathfrak{p}})) \text{Gal}_{K_p}^{ab}/K_{p,\infty} = 0,$$

and so in particular $H^i(K_{p,\infty}/K_p, W_{g,\chi}) \text{Gal}_{K_p}^{ab}/K_{p,\infty} = 0$ for all $i \in \mathbb{Z}_{\geq 0}$. Hence by the inflation-restriction exact sequence we have

$$0 \to H^1(K_{p,\infty}/K_p, W_{g,\chi}) \text{Gal}_{K_p}^{ab}/K_{p,\infty}) = 0 \to H^1(K_p, W_{g,\chi}) \to H^1(K_{p,\infty}, W_{g,\chi}) \text{Gal}_{K_p}^{ab}/K_{p,\infty} = 0$$

$$\to H^2(K_{p,\infty}/K_p, W_{g,\chi}) \text{Gal}_{K_p}^{ab}/K_{p,\infty}) = 0.$$

Hence the restriction map

$$H^1(K_p, W_{g,\chi}) \to H^1(K_{p,\infty}, W_{g,\chi}) \text{Gal}_{K_p}^{ab}/K_{p,\infty} = H^1(K_{p,\infty}, W_{g,\chi})[I_-]$$

is an isomorphism. Since $H^1(K_p, W_{g,\chi}) = H^1_+(K_p, W_{g,\chi})$, the above induces an isomorphism $H^1_+(K_p, W_{g,\chi}) \to H^1_+(K_{p,\infty}, W_{g,\chi})[I_-]$. \[\Box\]
Proof of Proposition \[8.2\] We follow the proof of \[2\] Theorem 5.2. Let 
\[ L_+ = H^1(K_p^-, W_{g,\chi})[I_-]/H^1_+(K_p^-, W_{g,\chi}) \quad \text{ and } \quad L_{\infty,+} = H^1(K_{p,\infty}, W_{g,\chi})[I_-]/H^1_+(K_{p,\infty}, W_{g,\chi})[I_-]. \]
Recall that \( g \in \mathcal{O}_K \) is the conductor of \( \lambda \) (and hence also that of \( \psi^* \chi \) by Proposition \[7.2\]), and let \( K^{p,g} \) denote the maximal extension of \( K \) unramified outside of \( pg \). Let \( g_0 \) denote the prime-to-\( p \) part of \( g \), and let 
\[ H^1(K_{g_0}, W_{g,\chi}) := \bigoplus_{v \mid g_0} H^1(K_v^-, W_{g,\chi}), \quad H^1(K_{\infty,g_0}, W_{g,\chi}) := \bigoplus_{v \mid g_0} H^1(K_{\infty,v}^-, W_{g,\chi}). \]
We then have a diagram with exact rows
\[
\begin{array}{cccccc}
0 & \to & \text{Sel}_+(K, W_{g,\chi})[I_-] & \to & H^1(K^{p,g}/K, W_{g,\chi})[I_-] & \to & H^1(K_{g_0}, W_{g,\chi}) \oplus L_+ \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Sel}_+(K, W_{g,\chi})[I_-] & \to & H^1(K^{p,g}/K, W_{g,\chi})[I_-] & \to & H^1(K_{\infty,g_0}, W_{g,\chi}) \oplus L_{\infty,+}
\end{array}
\tag{209}
\]
\[
H^1(K_{p,\infty}/K_p, B(K_{\infty,p}^-[p^{\infty}])) = H^1(K_{p,\infty}/K_p, E(K_{\infty,p}^-[p^{\infty}])) \oplus H^1(K_{p,\infty}/K_p, E_\chi(K_{\infty,p}^-[p^{\infty}])),
\]
where \( E_{\chi} \) is the CM abelian variety associated with the type \((1,0)\) algebraic Hecke character \( \psi^* \chi \), and where \( K_{\infty,p}^- = \hat{K}_{p,\infty}^- \) is the \( p \)-adic completion. The factors of the above decomposition are in turn isomorphic to a quotient of \( E(K_{\infty,p}^-[p^{\infty}]) \) and a quotient of \( E_\chi(K_{\infty,p}^-[p^{\infty}]) \), respectively. Since \( W_{g,\chi} \) is ramified at the primes dividing \( g_0 \), \( E(K_{\infty,p}^-[p^{\infty}]) \) is a proper \( \mathcal{O}_{K_p} \)-submodule of \( E[p^{\infty}] \) and \( E_\chi(K_{\infty,p}^-[p^{\infty}]) \) is a proper \( \mathcal{O}_{K_p} \)-submodule of \( E_\chi[p^{\infty}] \). Hence since \( E[p^{\infty}] \) and \( E_\chi[p^{\infty}] \) are cofree of corank \( \mathcal{O}_{K_p} \), any proper \( \mathcal{O}_{K_p} \)-submodule of \( E[p^{\infty}] \) and \( E_\chi[p^{\infty}] \) is finite, and thus by the above the kernel of \( H^1(K_p, W_{g,\chi}) \to H^1(K_{p,\infty}, W_{g,\chi}) \) is finite.
To control the kernel of \( L_+ \to L_{\infty,+} \), consider the diagram
\[
\begin{array}{cccccc}
0 & \to & H^1_+(K_p, W_{g,\chi}) & \to & H^1_+(K_p, W_{g,\chi})[I_-] & \to & L_+ \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1_+(K_{p,\infty}, W_{g,\chi})[I_-] & \to & H^1(K_{p,\infty}, W_{g,\chi})[I_-] & \to & L_{\infty,+}
\end{array}
\tag{210}
\]
By Lemma \[8.3\] the left vertical arrow is an isomorphism. As in the proof of Lemma \[8.3\] the inflation-restriction exact sequence shows that the middle vertical arrow is an isomorphism, and hence so is the right vertical arrow.
Hence we have shown that the kernel of the right vertical arrow of (209) is finite, and so by the above we are done.\[\square\]

As usual, let
\[\text{Sel}_{p^\infty}(E/\mathbb{Q}) = \ker \left( \prod_{\ell} \text{loc}_{\ell} : H^1(\mathbb{Q}, E[p^\infty]) \to \prod_{\ell} H^1(\mathbb{Q}_\ell, E) \right) \subset H^1(\mathbb{Q}, E[p^\infty])\]
denote the $p^\infty$-Selmer group of $E/\mathbb{Q}$. For an algebraic Hecke character $\rho : \text{Gal}(K^{ab}/K) \to \mathcal{O}_K^{\times}$, we let
\[
H^1_f(K, W(\rho)) = \ker \left( \prod_v \text{loc}_v : H^1(K, W(\rho)) \to \prod_v H^1(K_v, W(\lambda))/H^1_f(K_v, W(\rho)) \right)
\]
where, as usual, we define the Bloch-Kato conditions by
\[
H^1_f(K_v, K_p(\rho)) = \begin{cases} 
\ker (H^1(K_v, K_p(\rho)) \to H^1(I_v, K_p(\rho))) & v \neq p \\
\ker (H^1(K_v, K_p(\rho)) \to H^1(K_v, K_p(\rho) \otimes_{K_p} \mathcal{O}_{\text{cris}})) & v = p,
\end{cases}
\]
and $H^1_f(K_v, W(\rho))$ is the image of $H^1_f(K_v, K_p(\rho))$ under the natural map $H^1_f(K_v, K_p(\rho)) \to H^1_f(K_v, W(\rho))$. It is a standard fact that
\[
H^1_f(K, W(\lambda)) = \text{Sel}_{p^\infty}(E/\mathbb{Q}).
\]

**Proposition 8.4.** Suppose $\chi$ is as in Proposition 7.2 so that in particular $L(\psi^* \chi, 1) \neq 0$. Then letting $E_\chi$ denote the CM abelian variety associated with the type $(1, 0)$ algebraic Hecke character $\psi^* \chi$, we have $\# H^1_f(K, W(\psi^* \chi)) < \infty$.

**Proof.** Note that the conjugate character $\psi \chi^*$ (a type $(1, 0)$ algebraic Hecke character) is associated with a CM abelian variety $E_\chi/K$. Repeating the argument of Theorem 5.13 with $E_\chi$ in place of $E$, applying the Rubin-type Main Conjecture Corollary 4.70 and using the fact that $H^1_f(K_p, W(\psi^* \chi))$ is the Kummer local condition, one sees that $S_0(\psi^* \chi^*) = H^1_f(K, W(\psi^* \chi))$ and
\[
\# H^1_f(K, W(\psi^* \chi)) = \# S_0(\psi^* \chi) \sim \# (O_{K_p}/(L(\psi^* \chi, 1)/\Omega_{E_\chi} \mathcal{O}_{K_p}))
= (O_{K_p}/((L(\psi^* \chi, 1)/\Omega_{E_\chi} \mathcal{O}_{K_p}))
\]
where the penultimate equality follows from (95) and $\Omega_{E_\chi}$ is a Néron period attached to $E_\chi$. (Recall that “$\sim$” denotes equality up to finite power of $p$.)

Now note that there is an involution $i : G_{\infty} \sim\sim G_{\infty}$ given on group-like elements by complex conjugation $\gamma \mapsto \tau \gamma \tau^{-1}$ (where $\tau \in \text{Gal}(K/\mathbb{Q})$ is the non-trivial element), and that $i$ induces an isomorphism $H^1_f(K, W(\psi^* \chi)) \sim H^1_f(K, W(\psi^* \chi))$. Hence we have
\[
\# H^1_f(K, W(\psi^* \chi)) = \# H^1_f(K, W(\psi^* \chi)) \sim (O_{K_p}/((L(\psi^* \chi, 1)/\Omega_{E_\chi} \mathcal{O}_{K_p}))
\]
which gives the Proposition. \[\square\]

**Theorem 8.5.** Suppose that $K$ is an imaginary quadratic field with class number 1 in which $p$ is ramified, and that $E/\mathbb{Q}$ is an elliptic curve with CM by $\mathcal{O}_K$. Then
\[
\text{corank}_{\mathbb{Z}} \text{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \implies 0 \neq P_{\chi, 0}(\phi) \in B(K) \otimes_{\mathbb{Z}} \mathbb{Q}, \text{ ord}_{s=1} L(E/\mathbb{Q}, s) = \text{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1.
\]
Proof. Note that by (148) we have
\[ \text{Sel}_p\infty(B) = \text{Sel}_p\infty(E/Q) \oplus H^1_f(K, W(\psi^*\chi)). \]
By Proposition 8.2, the natural map
\[ (\mathcal{X}^{+,+}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(I^- (\mathcal{X}^{+,+}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)) \rightarrow \text{Hom}(\text{Sel}_p\infty(B), K_p/\mathcal{O}_{K_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \]
is surjective with finite kernel. Hence corank 1 assumption corank_{\mathbb{Z}_p} \text{Sel}_p\infty(E/Q) = 1 and Proposition 8.4 imply that (\mathcal{X}^{+,+}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(I^- (\mathcal{X}^{+,+}(W_{g,\chi}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)) is not K_p-torsion. By Theorem 7.72, this means that \( z_+ \) has non-torsion image in \( S^{+,+}((V_{g,\chi})/I^- S^{+,+}(V_{g,\chi}) \). Now from the natural injection
\[ (S^{+,+}(V_{g,\chi}))/I^- (S^{+,+}(V_{g,\chi})) \rightarrow \text{Sel}_p\infty(B) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \]
we see that \( 1(z_+) \) is non-zero (under the Kummer map) in \( \text{Sel}_p\infty(V_{g,\chi}) \). But by construction, \( 1(z_+) = P_{\chi,0}(\phi) \), and so the Heegner point \( P_{\chi,0}(\phi) \in (B(K^{ab}) \otimes_{\mathbb{Z}} F)^{\text{Gal}(K^{ab}/K)} \) is non-torsion. Now by the Gross-Zagier formula on Shimura curves [57], we have that
\[ P_{\chi,0}(\phi) \neq 0 \implies \text{ord}_{s=1} L(\lambda, s) + \text{ord}_{s=1} L(\psi^*\chi, s) = 1 \]
and so by Proposition 7.2 (which ensures \( L(\psi^*\chi, 1) \neq 0 \)), we have
\[ \text{ord}_{s=1} L(E/Q, s) = \text{ord}_{s=1} L(\lambda, s) = 1. \]
Now the remaining assertions of the Theorem follow from the Gross-Zagier formula [21] (or Yuan-Zhang-Zhang [57]) and Kolyvagin [27]. \( \square \)

8.2. Sylvester’s conjecture. Recall the cubic twist family of elliptic curves
\[ E_d : x^3 + y^3 = d. \]
Sylvester conjectured in 1879 (50) that for \( d = p \) prime, if \( d \equiv 4, 7, 8 \pmod{9} \) then \( E_d \) has a rational solution.

Corollary 8.6. Sylvester’s conjecture is true. That is, for any prime \( p \), if \( p \equiv 4, 7, 8 \pmod{9} \) then there exist \( x, y \in \mathbb{Q} \) such that \( x^3 + y^3 = p \).

Remark 8.7. Previously, the case \( p \equiv 4, 7 \pmod{9} \) was announced by Elkies [14], though the full proof remains unpublished. See also the article of Dasgupta-Voight [13], which gives another proof of Elkies’s result under additional assumptions.

Proof of Corollary 8.6. By standard 3-descent (see [13]), we have \( \text{corank}_{\mathbb{Z}_3}(\text{Sel}_3\infty(E_d)) = 1 \) for \( d \equiv 4, 7, 8 \pmod{9} \). Now the Corollary follows immediately from Theorem 8.5 with \( p = 3 \) and \( K = \mathbb{Q}(\sqrt{-3}) \). \( \square \)

8.3. Goldfeld’s conjecture for the congruent number family. For a general elliptic curve \( E : y^2 = x^3 + ax + b \), let \( r_m(E/Q) = \text{ord}_{s=1} L(E/Q, s) \) denote the analytic rank. Let \( E^d : y^2 = x^3 + ad^2x + bd^3 \) be the quadratic twist by \( \mathbb{Q}(\sqrt{d}) \). The celebrated conjecture of Goldfeld states that:

Conjecture 8.8 (Goldfeld’s conjecture [19]). For \( r = 0, 1, \]
\[ \lim_{X \to \infty} \frac{\# \{ 0 < |d| < X : d \text{ squarefree}, r_m(E^d/Q) = r \}}{\# \{ 0 < |d| < X : d \text{ squarefree} \}} = \frac{1}{2}. \]
The best known unconditional results towards Goldfeld’s conjecture in general are [28], [31] and [32]. For the congruent number family \( E_d : y^2 = x^3 - d^2 x \), the previously best known result follows from the main result of [49]. Using the results of loc. cit., we can establish Goldfeld’s conjecture for certain elliptic curves among those considered in loc. cit. including the congruent number family.

**Corollary 8.9.** Suppose \( E/\mathbb{Q} \) is an elliptic curve with \( E(\mathbb{Q})[2] \cong (\mathbb{Z}/2)^{\oplus 2} \) and no cyclic 4-isogeny defined over \( \mathbb{Q} \). Suppose further that \( E \) has CM by \( K \) (so that \( K \) necessarily has class number 1), and that 2 is ramified in \( K \). Then Goldfeld’s conjecture (Conjecture 8.8) is true for \( E \).

In particular, 100% of squarefree \( d \equiv 1, 2, 3 \pmod{8} \) are not congruent numbers and 100% of squarefree \( d \equiv 5, 6, 7 \pmod{8} \) are congruent numbers, and Goldfeld’s conjecture is true for the congruent number family \( E_d : y^2 = x^3 - d^2 x \).

**Proof.** This follows from the Selmer distribution results of Smith [49], Corollary 5.16 and Theorem 8.5 for \( p = 2 \) ramified in \( K \) of class number 1 (so that, in particular, \( K \) satisfies the assumptions of Theorem 8.5).

\[ \square \]

**References**
