Remarks on Kuratowski's Planarity Theorem

A planar graph is a graph that can be drawn in a plane without edges crossing one another.

This theorem says that a graph is planar if and only if it does not contain, lurking in it somewhere, a K_5 or a $K_{3,3}$, which means a complete graph on 5 vertices or a complete bipartite graph with each part consisting of 3 vertices.

What does "lurking in it" mean?

If you take an edge of a graph and put a vertex or several vertices on it, so that, for example, the edge (v,w) is replaced by the path (v,x), (x,y),(y,z)(z,w) the change does not affect the planarity of the graph. Also, adding extra edges to a non-planar graph will not make it planar. So we mean that G is non-planar if and only if we can throw away edges from G and replace paths in G by single edges to obtain a K_5 or a $K_{3,3}$.

This statement is usually shortened to "G is non planar if and only if G has K_5 or $K_{3,3}$ as a minor". (A graph H is a minor of G if we can obtain H from G by contracting or deleting edges. Contracting an edge means merging its vertices into one vertex, Thus a path can be contracted to an edge.)

There is a slightly different version of the theorem, which states: a graph is not planar if and only if it contains disjoint paths that, considered as edges containing its endpoints, is a K_5 or a $K_{3,3}$. This mean essentially the same thing as the statement about minors..
We can prove that a K_5 or a $K_{3,3}$ is not planar by drawing one in the plane with one pair of edges that cross one another, and proving that any two drawings of either one in the plane must have the same parity of the number of crossings between edges that have no vertex in common; (that is both drawings must have an even number of crossings or both must have an odd number of crossings, between edges that do not share a vertex.)

To prove that any non-planar graph G must contain a K_5 or a $K_{3,3}$ minor, we first throw away (delete) all edges e of G, one at a time if G -{e} is non-planar. When we are done, removing any other edge of G would make it planar.

We then observe that G must have a cycle C such that the graph whose edges are those of G limited to the vertices of G-C has at least two connected components. (Here is the argument. Pick any edge of G, say (v,w). Omit that edge from G and then draw G – {(v,w)}in the plane. If any face in that drawing contains both v and w, then we can connect them without any crossings and G would be planar. Otherwise the outer boundary of the faces that contain v contain a cycle C that separates v from w, and is therefore a separating cycle.

Since G-C has at least two connected components, and G less any edge is planar, C plus any connected component of G-C has to be planar. C has to be drawn as a cycle in the plane. Non-planarity can therefore arise only from the interference in drawing the various connected components of G-C with one another and with C in the plane.

From now on we will call a connected component of G-C a bridge. Every bridge will have vertices that are adjacent to some vertices on the cycle C. which we can call its points of attachment to C. For our purposes the only interesting thing about a bridge are these points of attachment. You can imagine all the interior vertices (if there are any) of a bridge contracted into one vertex, so that the bridge forms either a chord of C, or is a "star" graph, which has one inside vertex and edges connecting that vertex to its points of attachment on C.

When we draw a cycle in the plane it has two sides: an inside and an outside. We now ask: when can we draw two bridges on the same side of C in a planar drawing?

We can draw two bridges A and B on the same side of C unless either: A and B have points of attachment that alternate ABAB around C. or A and B each have three points of attachment to C and they are the same three vertices of C. For example, if A has points of attachment 1 and 3 and B has 2 and 4, then we cannot draw A and B on the same side of C without a crossing. The same is true if A and B both have points of attachment 1 2 and 3. Otherwise the two bridges are "compatible" by which we mean they can both be drawn on the same side of C without any crossing.

Can you prove this please?

Suppose bridge A has a point of attachment x that is not one of B. Then let w and y be the points of attachment of B closest to x on either side of x on C. Then w and y cut C into two paths which we can call the x and not x paths of C.. If A has a point of attachment z on the not-x path of C then w x y z form a BABA set of points of attachment; otherwise, A and B are compatible bridges. A's points of attachment all lying on the x side of y and z and all B's other than y and z. if any lying on the other side. If A and B have identical points of attachment and there are at least three of them, then A and B will also be incompatible.

Remember that C has two sides. Thus G can be drawn in the plane if and only if the bridges of G-C can be partitioned into two parts, such that the bridges in each part are mutually compatible.

So non-planarity means that the "Bridge incompatibility graph" whose vertices are bridges, and with an edge between incompatible bridges, is not two colorable, (which means not partitionable into two independent sets.).

Now remember that the condition for a graph not being two colorable is that it has an odd cycle. So G is non-planar only if G's bridge incompatibility graph has an odd cycle. And G will be minimally incompatible (removal of any edge makes G planar) only if G's bridge incompatibility graph is an odd cycle.

So suppose the bridges of G have an incompatibility graph that is an odd cycle. If that cycle is a 5-cycle (sometimes called a $C_{5)}$, we can contract edges around it to form a K_5. If the bridge incompatibility graph for G is a 7-cycle or larger, we can change the cycle C and omit some edges from G to produce a graph having a bridge incompatibility graph that is a C_5. This means that we can contract and get a K_5 in G and also that G was not minimal.

In either case above, in which the odd cycle bridge incompatibility graph is not a triangle, we can confine our attention to bridges that are chords of C; bridges that have 3 or more points of attachment to C can be replaced by chords without changing the bridge incompatibility graph.

How come?

In these cases the bridge incompatibility graph contains no triangle, so each vertex of it meets two others that themselves do not meet. If bridge B meets bridge A and bridge D which themselves do not meet, this means that the vertices of A and D are confined to intervals on C whose interiors are disjoint. Bridge B must have at least one point of attachment in the interior of each of these intervals, and B will have the same incompatibility if we replace it by the chord joining these two points of attachment.

What happens when the bridge incompatibility graph is C_5?

If so, and the bridges A B D E and F are chords of C, there is only one way to arrange their points of attachment, namely, AFBADBEDFE. If we contract the edges (fb)(ad)(be)(df)(ea), then each resulting vertex is adjacent to its neighbor around the cycle, and to the opposite two through the bridges. Thus fb is adjacent to be and df through bridges b and d and to ad and ea around the cycle.

What happens when the bridge incompatibility graph is a C_7 or is a bigger odd cycle?

If this happens we can switch our separating cycle C with a shorter cycle to obtain a bridge incompatibility graph that is a C_5. Here is how. Suppose the bridges are numbered 1, 2, … 7, and they have points of attachment arranged as 21324354657617 along C. We can omit the edges 76 and 17 of C, consider the separating cycle D which consists of the part of C between the two points of attachment of chord 7 (through the points of attachment of 2,3 etc.) and the chord 7. We can then consider the path consisting of the chords 1 and 6 and the edge (61) of C as a path-chord of this cycle. The bridge incompatibility graph then becomes C_5. A very similar construction works for longer cycles. We can also deduce that G was not minimally non-planar if the bridge incompatibility graph is neither a triangle nor a C_5.

OK what happens when the bridge incompatibility graph is a triangle?

If all three bridges are chords, their points of attachment must be arranged as 1,2,3, 1',2',3' and the vertices 1,3 and 2' and 1',3' and 2 form the vertices of the two parts of a $K_{3,3}$.

We do however have to worry about the possibility of non-chordal bridges. There are several examples of three bridges that lack points of attachments that occur in the order 1,2,3, 1',2',3' (which form a $K_{3,3}$) that nevertheless are mutually incompatible as bridges.

Fortunately, we need only consider attachment sequences for which between each pair of attachments of any non-chordal bridge there is an attachment of another bridge. Otherwise, if say bridge 1 has two points of attachment that are consecutive around C we can replace

C by the cycle that lacks the edge of C between these two and instead goes through bridge 1 between these two points.Each such replacement increases the length of the separating cycle. Thus if we restrict our attention to separating cycles of minimal non-planar G which have maximum length there can be no bridges with adjacent points of attachment on C.

This argument means we don't have to consider for example, the case in which there are three 3-star bridges with identical points of attachment. This configuration is a $K_{3,3}$ and replacements as above make it into the all chord case first considered above.

A K_5 can be described as a C_4 with two chordal bridges each connecting to a pair of opposite vertices of the C_4 and one 4-star bridge, connecting to all of its vertices. Replacement of the C_4 separating cycle by a C_5 cycle as above reduces this to a situation in which the bridge incompatibility graph is a C_5 which we have already considered,.

There are other graphs with G with triangular bridge incompatibility graph that seem to require non-chordal bridges and cannot be eliminated by this replacement argument. All of these are not minimally non-planar and also have longer separating cycles with only chordal bridges, but they ought to be mentioned.

A. 3, 2=1,3,1,2,1 (In this example bridge 1 is a 3-star and one of its points of attachment is also a point of attachment for bridge 2. (see picture). It is not minimal because the graph obtained by eliminating the edge of C between the 2=1 attachment vertex and either 3 attachment vertex is also non-planar.
   It also has a separating C_7 cycle, for which all bridges are chords.
B. 1=2, 3, 1=2, 3, 1=2. 3. By contracting the edges between each of the 1 and 2 bridge attachment vertices and the next adjacent bridge 3 attachment vertex, we reduce this case to the three 3-star bridges with identical points of attachment already discussed. This graph is not minimally non-planar and also has a longer separating cycle.whose bridges can all be chords.
C. All other sequences of 1's 2's and 3's that represent points of attachment of incompatible bridges either contain consecutive vertices with the same integer label, or have one of A B or 123123 (or something the same with labels interchanged) as a cyclic subsequence. You can prove this by showing that when the points of attachments of different bridges are all different then there must be a 123123 sequence, and otherwise, if no bridge can have points of attachment that are adjacent around the separating cycle C, then a sequence of points of attachment equivalent to that of A above (3, 2=1,3,1,2,1) or B above (1=2, 3, 1=2, 3, 1=2, 3) must be present.