29. Parentheses, Catalan Numbers and Ruin

1. Introduction

A sequence of zeroes and ones can represent a message, a sequence of data in a computer or in digital communications, but it also can represent all sorts of mathematical or real world objects.

Thus a sequence of length n can represent **the outcome of n coin tosses, or a binary valued vector of length n, or a polynomial of degree up to n-1 with binary coefficients, or a number.**

We will now discuss three other interpretations for such a sequence.

First, a sequence of length n can represent **a subset of an n-element set, S**.

Each position, j, in the sequence can represent one of the n elements.

A **given subset, T, of S can then be represented by the n-length sequence having 1's in the positions corresponding to the elements of T and 0's elsewhere.**

For example, for n=6, the subset {2,3,5} can be represented as 011010.

The 2^n binary sequences of length n then correspond to the 2^n subsets of an n-element set.

We already know something about subsets of a set. Thus the number of such subsets that have k elements in them is the binomial coefficient C(n,k) (which is $n_{(k)}/k!$). We will deduce some more properties of collections of subsets from the next interpretation.

2. Parentheses

A another curious interpretation of binary sequences is that obtained by replacing **each 1 by a left parenthesis and each 0 by a right one**.

Thus 011010 can be written as)(( )( ).

The gain we get from this silly looking substitution is **that there is a natural definition of “closing” among parentheses. If a left and right parenthesis are adjacent in the proper order, like ( ), we can “close” them with one another, and ignore them in future closings so that those around them become adjacent** (as happens similarly with the pairings in the Hu Tucker algorithm).

Thus in our example above,)(( )( ), the first two parentheses remain open while the last two close.

A famous ancient question in this context is: **how many distinct arrangements of n pairs of left-right parentheses are there all of which close?**

The answer to this question is called **the n-th Catalan number, C(n).**

Here are the first few answers:

C(1)=1	( )
C(2)=2	()() and (())
C(3)=5	()()(), ()(()), (())(), (()()) and ((()))

We can deduce the answer to this and many similar questions in terms of binomial coefficients by noticing **that every sequence of 2n parentheses, or in other terms, every subset of a 2n-element set, has some open part and some closed part when considered as a sequence of parentheses.**

This allows us to define **a natural partition of all subsets of a 2n-element set, (or of an odd sized set as well.)**

Each block of the partition consists of all subsets which have the same closed part in the same locations.

Thus, in our original example,)(( )( ), the closed part consists of the last 4 parentheses, and the first two represent its open part.

The magical fact that makes this information extraordinarily useful, is that **open parts are extremely simple**. No open part can contain a left parenthesis, (, followed by a right parenthesis,).

Thus, open parts, wherever they occur in a sequence, **all either consist of all parentheses of the same kind, or some right ones followed by some left ones.**

If, for example, the open part of a sequence has four elements, the only possibilities are ((((,)(((,))((,)(((and)))), wherever they are located.

The binary sequences that correspond to these possibilities are 1111, 0111, 0011, 0001 and 0000.

The sets corresponding to them are: none of them, the last element, the last two elements, the last three elements, and all four elements.

Here and in general these sets form what we can call a **full chain.**

Each set in a full chain contains or is contained in all the others, and there are no gaps among them: every set in the middle of one has another set in the chain with one more element and another with one less element.

What other sequences have the same closed part as our example)(()()?

We can replace **the open part by any other open part of the same size and location**: here we can replace)(at the beginning, by)) and also by ((.

The results here, expressed as binary sequences, are that 001010, 011010 and 111010 **all have the same closed part and hence lie in the same block in this partition**. Considered as subsets these are {3,5}, {2,3,5} and {1,2,3,5}.

The subsets of a set, S, that have the same closed parts all have open parts in the same locations, and therefore inherit the wonderful properties of open parts.

Recall that the set S corresponding to n pairs of parentheses must have 2n elements all together, one for each parenthesis.

We will quickly be able to read off what the Catalan number C(n) is from these properties.

Once again, these properties are

1. Each block in this partition is a full chain, and is completely ordered by inclusion.

2. Each block is symmetric about the middle size (|S|/2). If, as here, S has cardinality 2n, their sizes are symmetric about n. This statement follows from the fact that the closed parts all have the same number of left parentheses as right ones and so the corresponding sets have one element for each closed parenthesis. The open parts are also symmetric about the middle size.

3. Full chains have no gaps in the them. They have an element of every size from some minimum, 2n/2 - k to 2n/2 + k.

In short, each **block has some smallest set in it, and then consists of what you get by adding the elements in the open part one at a time (highest index first) to that set until all are added**.

We can draw the following conclusions from these facts:

1. Every set of size 2n/2+j is in one full chain of this kind that stretches from at most size 2n/2-j to at least 2n/2+j.

2. Any such chain contains at least 2j+1 member subsets.

3. Every one of these chains of length at least 2j+1 contains a set of size n +j.

We may deduce from the first and third fact that **the number of chains of length 2j+1 or more is the number of subsets of S of size n+j.**

This latter is the binomial coefficient C(2n, n+j).

We can also answer the question:

How many of our chains in this partition have length <u>exactly</u> 2j+1?

These are chains of length **at least 2j+1 <u>but no longer</u>**.

They must all **have a member having n+j elements, but no member having n+j+1 elements.**

The number of these must therefore **be the difference in binomial coefficients:**

C(2n, 2n/2 +j) - C(2n,2n/2+j+1).

(The first term counts how many of our chains have a member of size m+j while the second tells us how many of these have larger elements as well and so are longer.)

Now we have found the Catalan number and much more!

For, **parentheses that close completely**, which the Catalan numbers count count, are exactly those that **have no open part** and therefore lie **in chains having exactly one member. This is the j=0 answer here, which is:**

C(n) = C(2n,n) - C(2n,n+1).

3. Gambling Sequences

It turns out that the open and closed parts of parentheses have meaning in terms of gambling sequences.

For, if we make a left parenthesis correspond to a win, and a right one to a loss, in every closed part and in every portion thereof, there are always at least as many wins as losses counting from the left-hand beginning to any point.

In other words, **you never fall behind from being in a closed part**.

In fact the first time you fall behind is at the first) of the open part of your sequence, and the maximum you are ever behind **is the number of left parentheses with which your open part begins.**

Suppose you gamble by betting on the outcome of a series of coin tosses or roulette wheel spins, and you bet the same amount, on each toss or turn of the wheel, and either win or lose one unit per event.

Suppose further that you start with a stake of X units, and can stay for N events unless you lose all your money.

You will be wiped out in any sequence in which you ever lose X events more than you have won.

This means that among the gambling sequences corresponding to sets in one of our full chains, if the chain has length X+q then you will be wiped out in exactly q of its sequences. (You will not be wiped out in those sequences having 0 up to X-1 starting left parentheses and will be wiped out otherwise.)

Thus you get at most X sequences in which you survive to play N events from each chain (and fewer if the chain has fewer members than X).

But the number of sequences obeying this condition, at most X from any chain, is exactly the number of sets of the X middle sizes. These also have X members in each chain of length at least X and contain all elements of all shorter chains.

The probability of not being wiped out after N events, when odds of winning are even, is therefore the sum of the X middle (i.e, largest) binomial coefficients divided by 2^N.

We can also ask:

What is the probability of winning k times more than you lose given that you must drop out and lose your stake if you ever fall X behind?

There are C(N,k/2+N/2) gambling sequences of length N for which you end up winning k more times than you lose. But we must throw out those that start with X or more right parentheses in their open parts. (In them you are wiped out.)

Suppose an open part has Q parentheses in it. If you win k more times than you lose, there must be Q/2+k/2 left parentheses and Q/2-k/2 right parentheses in the open part, since the closed part balances between wins and losses. You are wiped out if the latter number is at least X since these will be sequences in which you fall Q/2-k/2 behind before you start winning.

Thus the number of win k non wipe out sequences is C(N,k/2+N/2) less the number of chains of length Q+1 with Q=X+k/2 or more, (each of which will contain a wipeout "net k win" sequence).

We have counted such chains before and the latter number is C(N,N/2+X+k/2), so that the number of non-wipeout k win sequences is:

C(N,(k+N)/2) - C(N,(k+N)/2+X)

unless of course k is -X or less in which case you are definitely wiped out, while the expression here can become negative.

This information allows you to compute your probability of winning whatever given an initial stake with any constant probability of winning, p, by summing **the number above** multiplied by $kp^{(k+N)/2}(1-p)^{(N-k)/2}$ over k above -X and subtracting X $p^{(k+N)/2}(1-p)^{(N-k)/2}$C(N,(k+N)/2-X) in this range and $Xp^{(k+N)/2}(1-p)^{(N-k)/2}$C(N,(k+N)/2) for k at most -X, to account for your losses when you are wiped out.

Exercise: Use a spreadsheet to evaluate this for p=.49, N=100, and X =5 and 10. Do the same for p=.51. What do you find?

4. Applications to Extremal Set Theory

The partition of the subsets into blocks defined by closed parts has other implications that are worth noting, in the area of "extremal set theory."

In particular we will deduce an upper bound for the size of an antichain of subsets (defined below) from it, as well as an upper bound to the number of antichains of subsets.

We define an **antichain** of subsets of a set S to be a collection of its subsets having the property that **no two of its members are ordered by inclusion**.

In short **no member contains another**.

We ask then**:**

How many members can an antichain have?

The answer to this question, (which is called “Sperner’s Theorem”), is **the largest binomial coefficient C(N, [N/2])** when we are considering subsets of S and S has N elements.

This answer follows immediately from the following two facts.

1. Since the chain-blocks of our partition are completely ordered by inclusion, an antichain can have at most one member from each block.

2. Since each of our chain blocks contains a member of size [N/2], the size of an antichain, (which is at most the number of blocks as follows from the previous observation), is at most the number of [N/2] size subsets of S, which is C(N,[N/2]).

Since the sets of size [N/2] form an antichain of this size, this bound cannot be improved.

Another curious question (first posed by Dedekind in the 19th Century) is:

How many antichains of subsets of an N element set are there?

There are $2^{C(N,[N/2])}$ subsets of the **antichain consisting of sets all of size [N/2] and all these are antichains., so this is a lower bound**.

Hansel found an upper bound of $3^{C(N,[N/2])}$ for this number by the following argument:

1. An antichain A of subsets of S corresponds to a function, f(A) defined on all subsets of S which is 1 on every member of the antichain and any set containing such a member, and 0 on sets not containing any member.

(Such a function is called a monotone Boolean function, monotone because when it is 1 on some set q and r contains q it is 1 on r as well, and Boolean because its values are 0 and 1)

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2. We can define all such monotone Boolean functions by defining them on each chain, and if we do so starting on the short chains first**, we have at most 3 possibilities on each chain.**

Why is this so? a monotone Boolean function can have at most k+1 values on a chain of length k:

Thus, if some member gets value one, everything above it must get value 1, and if it gets 0 everything below it must get 0. It is completely defined by specifying the smallest 1 valued member if any.

Thus the possible monotone Boolean functions on a 3 element chain can be described as 000, 001, 011 and 111.

But if we start defining a function on short chains, the values on them have implication for the larger chains as well.

In particular, suppose we look **at three successive members of a large chain**; these will have the **form)),)(and ((in 2 successive places in their open parts, and be otherwise identical**.

This means that **the set z with ( ) in these same places in their open part and otherwise identical lies in a shorter chain** (it has a bigger closed part) and hence will already be defined when it comes to defining our function on this chain.

But if z has been assigned value 0 then the set with corresponding parenthesese)) in these two places in our chaine must get 0 value 0 as well since it is smaller than z and z has value 0.

Similarly if z has been assigned value 1 then the set with corresponding parentheses ((in these places must get the value 1 as well, since it is bigger than z.

This implies that when it comes to define a monotone Boolean function on each our chains, at least one of every three members of each chain must be predetermined by their values on shorter chains, so there can be at most 2 members whose definition is still open.

This means there are only at most 3 possible definitions per chain (00,01 or 11 on these members), which means we can define all possible monotone Boolean functions with **at most $3^{C(N,[N/2])}$ possibilities all together**.

(By careful argument this upper bound can be reduced to something whose logarithm is asymptotic to the lower bound.)

Exercise: How many members can a collection of subsets of an N element set S have if no three members, A B and C have A contains B contains C?

5. Catalan Numbers in other Contexts

Catalan numbers appear in many other places in mathematics. In particular, suppose you take a polygon and draw it in the plane so that its interior is convex. (the line between any two interior points is entirely in the interior)

We can ask: how many ways are there to introduce chords to your polygon so that all interior faces are triangles?

The answer is a Catalan number.

Exercise: For a polygon having N vertices , which Catalan number is it?

We found above a way to evaluate Catalan numbers that had a certain generalization, based upon lots of talk and little calculation.

There is another way to evaluate Catalan numbers that gives an equally simple formula equivalent to the previous answer of course, but different and with an entirely different generalization, that is also talk with practically no calculation.

And here it is.

Suppose we take a good parenthesization, one that closes completely, with n parentheses, and add to it at the end another extra right parenthesis.

You obviously get 2n+1 parentheses, with one more right one than left ones, and these correspond to subsets of size n of a 2n+1 element set .

We can show, by appropriate talk, that if you take the 01 sequence that represents this set and take all its cyclical permutations, and do this for all good n-parenthesizations, you get sequences that correspond to all n element sets that are chosen from among 2n+1 elements, and get each set exactly once.

This means that each good n parenthesization corresponds to one unique cycle of n element subsets of a 2n+1 element set.

Since there are 2n+1 cyclic permutations of a sequence of length 2n+1, we find: C(n) = C(2n+1,n)/(2n+1).

Exercise: Show that this formula is the same as the previous one.

Suppose we represent our parentheses and sets by sequences not ot 1 and 0 but instead of 1 and -1. Then a good parenthesization has all the partial sums of its sequences non-negative, since every -1 or right parenthesis must have its partner left parenthesis or +1 to its left.

Similarly the partial sums from the right are non positive.

If we start with a good parenthesization and add a right parenthesis -here a -1 on its right, every partial sum but the whole will still be non-negative.

On the other hand if we make any cyclic permutation, then the resulting sequence will begin with a partial sum of the original from the right, then the -1 then the rest. When you hit the -1 the partial sum will definitely be negative.

This means that the cyclic permutations of the sequence obtained from one good parenthesization, cannot be obtained from any other good parenthesization.

On the other hand, if you take partial sums of any 1 -1 sequence of length 2n+1 with 2n ones, they will have some minimum value. If you start right after the corresponding first minimum point, all partial sums up to the last one will be non-negative, so that removing the -1 at that point and starting beyond it will give a good n parethesization.

In short, there is a one to one correspondence between good n parenthesizations and cyclic permutations of n element subsets of a 2n+1 element set and that is the proof of our formula.

Now suppose we consider a different kind of parenthesization, where one left parenthesis closes with 2 (or more generally with k) right parentheses.

Here are examples for n=2 (( )))), ( ) ( ))), ( )) ( ))
in terms of 01 sequences they look like 1 100 00, 10 100 0, 100 100

We can count these by the identical argument previously used. Take a good parenthesization here, add an extra right parenthesis at the right end, and argue that cyclic permutations of these give all sets of size n, now out of 3n+1 elements.

The answer is therefore C(3n+1,n)/(3n+1) for the number of these objects.

Exercises: What is the general formula if each left parenthesis closes with k right ones in the same way as considered here for k=2?
Find the answers for k=2 and n=3 and verify the correctness of the formula above.

There is a generalization of the Catalan numbers as follows. Suppose each left hand parenthesis closes with two right hand ones. (or with k of them but we stick with 2 for the moment.) How many arrangements of n such parentheses are there?

For n=1 there is 1, for n=2 you can have **(**()))**)** or ()())) or ()))()).

To be a legal parenthesisation, there must be some adjacent triple of the form ()) and you can close it, and remove it, and the same must be true on what remains.