Weyl group representations, nilpotent orbits, and the orbit method

David Vogan

Department of Mathematics
Massachusetts Institute of Technology

Lie groups: structure, actions and representations
In honor of Joe Wolf, on his 75th birthday
January 11–14, 2012, Bochum

Outline

What is representation theory about?

Nilpotent orbits from G reps

W reps from G reps

W reps and nilpotent orbits

What it all says about representation theory

The old good fan-fold days

Gelfand's "abstract harmonic analysis"

Say Lie group G acts on manifold M. Can ask about

- topology of M
- solutions of G-invariant differential equations
- special functions on M (automorphic forms, etc.)

Method step 1: LINEARIZE. Replace M by Hilbert space $L^2(M)$. Now G acts by unitary operators.

Method step 2: DIAGONALIZE. Decompose $L^2(M)$ into minimal G-invariant subspaces.

Method step 3: REPRESENTATION THEORY. Study minimal pieces: irreducible unitary repns of G.

What repn theory is about is 2 and 3.

Today: what do irr unitary reps look like?

Short version of the talk

Irr (unitary) rep of G $\leftrightsquigarrow$ (coadjoint) orbit of G on $\mathfrak{g}_0{}^*$.

$\leftrightsquigarrow$ $\leftrightsquigarrow$ fifty years of shattered dreams, broken promises

but I'm fine now, and not bitter

G reductive: coadjt orbit $\leftrightsquigarrow$ conj class of matrices

Questions *re* matrices $\rightsquigarrow$ nilpotent matrices

nilp matrices $\leftrightsquigarrow$ combinatorics: partitions (or. . .)

partitions (or. . .) $\leftrightsquigarrow$ Weyl grp reps

Conclusion:

irr ~~unitary~~ reps $\longleftrightarrow$ Weyl grp reps

$\searrow$ nilp orbits $\nearrow$

Plan of talk: explain the arrows.

When everything is easy 'cause of p

Corresp G rep $\rightsquigarrow$ nilp coadjt orbit hard$/\mathbb{R}$, easier$/\mathbb{Q}_p$.

$G \subset GL(n, \mathbb{Q}_p)$ reductive alg, $\mathfrak{g}_0 \subset \mathfrak{gl}(n, \mathbb{Q}_p)$.

Put $\mathcal{N}_G^* =$ nilp elts of $\mathfrak{g}_0^*$, the *nilpotent cone*.

Orbit $\mathcal{O}$ has natural G-invt msre $\mu_{\mathcal{O}}$, homog deg dim $\mathcal{O}/2$, hence *tempered distribution*; Fourier trans $\widehat{\mu}_{\mathcal{O}} =$ temp gen fn on $\mathfrak{g}_0$.

Theorem (Howe, Harish-Chandra local char expansion)

$\pi \in \widehat{G}$, Θ_π character (generalized function on G).

$\theta_\pi =$ lift by exp to neighborhood of $0 \in \mathfrak{g}_0$.

Then there are unique constants $c_{\mathcal{O}}$ so that

$$\theta_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}} \qquad \pi \xrightarrow{\text{WF}} \{\mathcal{O} | c_{\mathcal{O}} \neq 0\}.$$

on some conj-invt nbhd of $0 \in \mathfrak{g}_0$.

Depends only on π restr to any small (compact) subgp.

$$\frac{\pi \text{ restr to compact open}}{\text{singularity of } \Theta_\pi \text{ at } e} \rightsquigarrow \text{WF}(\pi)$$

Nilp orbits from G reps by analysis: WF(π)

$(\pi, \mathcal{H}_\pi)$ irr rep $\xrightarrow{\text{HC}}$ Θ_π = char of π.

Morally $\Theta_\pi(g) = \operatorname{tr} \pi(g)$; unitary op $\pi(g)$ never trace class, so get not function but "generalized function":

$$\Theta_\pi(\delta) = \operatorname{tr}\left(\int_G \delta(g)\pi(g)\right) \quad (\delta \text{ test density on } G)$$

Singularity of Θ_π (at origin) measures infinite-dimensionality of π.

Howe definition:

WF(π) = wavefront set of Θ_π at $e \subset T_e^*(G) = \mathfrak{g}_0^*$

passage $\mathcal{H}_\pi \to$ WF(π) is analytic classical limit.

WF(π) is G-invt closed cone of nilp elts in $\mathfrak{g}_0^*$, so finite union of nilp coadjt orbit closures.

Nilp orbits from G reps by comm alg: $\mathcal{AV}(\pi)$

G real reductive Lie, K maximal compact subgp.

$(\pi, \mathcal{H}_\pi)$ irr rep $\xrightarrow{\text{HC}} \mathcal{H}_\pi^K$ Harish-Chandra module of K-finite vectors; fin gen over $U(\mathfrak{g})$ (with rep of K).

Choose (arbitrary) fin diml gen subspace $\mathcal{H}_{\pi,0}^K$, define

$$\mathcal{H}_{\pi,n}^K =_{\text{def}} U_n(\mathfrak{g}) \cdot \mathcal{H}_{\pi,0}^K, \qquad \text{gr}\, \mathcal{H}_\pi^K =_{\text{def}} \sum_{n=0}^{\infty} \mathcal{H}_{\pi,n}^K / \mathcal{H}_{\pi,n-1}^K.$$

$\text{gr}\, \mathcal{H}_\pi^K$ is fin gen over poly ring $S(\mathfrak{g})$ (with rep of K).

$$\begin{aligned} \mathcal{AV}(\pi) &=_{\text{def}} \text{support of gr}\, \mathcal{H}_\pi^K \\ &= K\text{-invariant alg cone of nilp elts in } (\mathfrak{g}/\mathfrak{k})^* \\ &\subset \mathfrak{g}^* = \text{Spec}\, S(\mathfrak{g}). \end{aligned}$$

passage $\mathcal{H}_\pi \rightarrow \mathcal{AV}(\pi)$ is algebraic classical limit.

Interlude: real nilpotent cone

G real reductive Lie, K max compact, θ Cartan inv.

$\mathfrak{g}_0$ real Lie alg, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $G(\mathbb{C})$ cplx alg gp.

$K(\mathbb{C})$ = complexification of K: cplx reductive alg.

$\mathcal{N}^* \subset \mathfrak{g}^*$ = nilp cone; finite union of $G(\mathbb{C})$ orbits.

$\mathcal{N}^*_{\mathbb{R}} =_{\text{def}} \mathcal{N}^* \cap \mathfrak{g}_0^*$; finite union of G orbits.

$\mathcal{N}^*_{\theta} =_{\text{def}} \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$; finite union of $K(\mathbb{C})$ orbits.

Theorem (Kostant-Sekiguchi, Schmid-Vilonen).
There's natural bijection

$$\mathcal{N}^*_{\mathbb{R}}/G \leftrightarrow \mathcal{N}^*_{\theta}/K(\mathbb{C}), \qquad \text{WF}(\pi) \leftrightarrow \mathcal{AV}(\pi).$$

Conclusion: two paths reps $\rightsquigarrow$ nilp coadj orbits,
$\pi \mapsto \text{WF}(\pi)$ and $\pi \mapsto \mathcal{AV}(\pi)$ are same!

And now for something completely different...

...to distinguish representations: $\mathfrak{Z}(\mathfrak{g}) =_{\text{def}}$ center of $U(\mathfrak{g})$.

π irr rep $\rightsquigarrow$ infinitesimal character $\xi(\pi)\colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ is homomorphism giving action in π.

Fix Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, $W =$ Weyl group.

HC: there's bijection [infl chars $\xi_\lambda\colon \mathfrak{Z}(\mathfrak{g})] \to \mathbb{C} \leftrightarrow \mathfrak{h}^*/W$.

$$\Pi(G)(\lambda) = \text{(finite) set of irr reps of infl char } \xi_\lambda$$
$$W(\lambda) = \{w \in W \mid w\lambda - \lambda = \text{integer comb of roots}\}$$

Theorem (Lusztig-V). If λ is regular, then $\Pi(G)(\lambda)$ has natural structure of $W(\lambda)$-graph. In particular,

1. $W(\lambda)$ acts on free $\mathbb{Z}$-module with basis $\Pi(G(\mathbb{R}))(\lambda)$.
2. There's preorder $\le_{\text{LR}}$ on $\Pi(G)(\lambda)$:
   $$y \le_{\text{LR}} x \Leftrightarrow x \text{ appears in } w \cdot y \quad (w \in W(\lambda))$$
   $$\Leftrightarrow \pi(x) \text{ subquo of } \pi(y) \otimes F \quad (F \text{ fin diml of } \mathrm{Ad}(G))$$
3. Each double cell ($\sim_{\text{LR}}$ class) in $\Pi(G)(\lambda)$ is $W(\lambda)$-graph, so carries $W(\lambda)$-repn.
4. $\mathrm{WF}(\pi(x))$ constant for x in double cell.

W reps and nilpotent orbits

Nilp cone $\mathcal{N}^*$ = fin union of $G(\mathbb{C})$ orbits $\mathcal{O}$.

Springer corr $\widehat{W} \hookrightarrow \{(\mathcal{O}, \mathcal{S})\}$, ($\mathcal{S}$ loc sys on $\mathcal{O}$).

Write $\sigma \mapsto (\mathcal{O}(\sigma), \mathcal{S}(\sigma)) \quad (\sigma \in \widehat{W})$.

Write $a(\sigma) =$lowest degree with $\sigma \subset S^{a(\sigma)}(\mathfrak{h})$

1. $a(\sigma) \geq [\dim(\mathcal{N}^*) - \dim(\mathcal{O})]/2$. Equality iff $\mathcal{S}(\sigma)$ trivial.
2. For each $\mathcal{O}$ $\exists!$ $\sigma(\mathcal{O}) \in \widehat{W}$ corr to $(\mathcal{O}, \text{trivial})$.
3. $\sigma(\mathcal{O})$ has mult one in $S^{a(\sigma(\mathcal{O}))}(\mathfrak{h})$.
4. Every special rep of W (Lusztig) is of form $\sigma(\mathcal{O})$.

$$\widehat{W} \supset \widehat{W}_{\text{nilpotent}} \supset \widehat{W}_{\text{special}}$$

Type A_{n-1}: all size $p(n)$. E_8: 112 $\widehat{W} \supset 70$ $\widehat{W}_{\text{nilp}} \supset 46$ $\widehat{W}_{\text{special}}$.

$\sigma \in \widehat{W}$ close to trivial $\Leftrightarrow a(\sigma)$ small $\Leftrightarrow \mathcal{O}(\sigma)$ large.

$\sigma \in \widehat{W}$ triv $\Leftrightarrow a(\sigma) = 0 \Leftrightarrow \mathcal{O}(\sigma) =$ princ nilp.

$\sigma \in \widehat{W}$ sgn $\Leftrightarrow a(\sigma) = |\Delta^+| \Leftrightarrow \mathcal{O}(\sigma) =$ zero nilp.

$W(\lambda)$ reps and nilpotent orbits

$\lambda \in \mathfrak{h}^* \rightsquigarrow \Delta(\lambda)$ integral roots $\rightsquigarrow$ endoscopic gp $G(\lambda)(\mathbb{C})$

Weyl group $W(\lambda)$, nilp cone $\mathcal{N}^*(\lambda)$).

$\sigma(\lambda) \in \widehat{W(\lambda)}_{\text{special}} \leftrightarrow \mathcal{O}(\lambda) \subset \mathcal{N}^*(\lambda)$; codim $= 2a(\sigma(\lambda))$.

Proposition $L \subset F$ fin gps, $\widehat{L} \ni \sigma_L \subset X$ (reducible) rep of F. If σ_L has mult one in X then $\exists! \; \sigma \in \widehat{F}$, $\sigma_L \subset \sigma \subset X$.

$W(\lambda) \subset W$, $\sigma(\lambda) \subset S^{a(\sigma(\lambda))}(\mathfrak{h}) \rightsquigarrow \sigma \in \widehat{W}, \; a(\sigma) = a(\sigma(\lambda))$.

Theorem Representation $\sigma \in \widehat{W}$ constructed from special $\sigma(\lambda) \in \widehat{W(\lambda)}_{\text{special}}$ belongs to $\widehat{W}_{\text{nilp}}$. Get endoscopic induction

special nilps for $G(\lambda)(\mathbb{C}) \rightsquigarrow$ nilps for $G(\mathbb{C})$,

preserves codimension in nilpotent cone.

$\mathcal{O}(\lambda)$ principal $\rightsquigarrow \mathcal{O}$ principal

$\mathcal{O}(\lambda) = \{0\} \rightsquigarrow \mathcal{O}$ orbit for maxl prim ideal of infl char λ

$G(\lambda)(\mathbb{C})$ Levi $\Rightarrow \mathcal{O}(\lambda) \rightsquigarrow \mathcal{O}$ Lusztig-Spaltenstein induction

Our story so far…

$\lambda \in \mathfrak{h}^* \rightsquigarrow$ endoscopic gp $G(\lambda)(\mathbb{C})$

Block **B** of reps of $G(\mathbb{R})$ of infl char λ

$\rightsquigarrow$ conn comp **D** of $W(\lambda)$-graph $\Pi(G(\mathbb{R}))(\lambda)$

$\rightsquigarrow$ real form $G(\lambda)(\mathbb{R})$; $\mathbf{D} \simeq \mathbf{D}(\lambda) \subset \Pi(G(\lambda)(\mathbb{R}))$

Conclusion: for each double cell of reps of infl char λ

$$\begin{array}{ccccccc} \mathcal{C} \subset \Pi(G(\mathbb{R}))(\lambda) & & & \sigma \in \widehat{W}_{\text{nilp}} & \leftrightarrow & \mathcal{O} \subset \mathcal{N}^* \\ \updownarrow\simeq & \searrow & & \uparrow & & \uparrow \\ \mathcal{C}(\lambda) \subset \Pi(G(\lambda)(\mathbb{R})) & \leftrightarrow & & \sigma(\lambda) \in \widehat{W(\lambda)}_{\text{spec}} & \leftrightarrow & \mathcal{O}(\lambda) \subset \mathcal{N}^*(\lambda) \end{array}$$

Further conclusion: to understand G reps $\rightsquigarrow$ nilpotent orbs, must understand W graphs $\rightsquigarrow$ special W reps.

Real nilp orb(s) $\mathrm{WF}(\mathcal{C})$ (Howe) refine this correspondence.

Special reps and W-graphs

Want to understand W graphs $\rightsquigarrow$ special W reps.

Silently fix reg int infl char ξ (perhaps for endoscopic gp).

$$x \in W\text{-graph} = \text{HC/Langlands param for irr of } G(\mathbb{R})$$
$$= (H(\mathbb{R})_x, \Delta_x^+, \Lambda_x) \text{ mod } G(\mathbb{R}) \text{ conj}$$
$$= (H_x, \Delta_x^+, \theta_x, (\mathbb{Z}/2\mathbb{Z} \text{ stuff})_x) \text{ mod } G \text{ conj}$$
$$= (H_p, \Delta_p^+, \theta_x, (\mathbb{Z}/2\mathbb{Z} \text{ stuff})_x)$$

Last step: move to (Cartan, pos roots) from pinning.

$\mathbb{Z}/2\mathbb{Z}$ stuff grades θ-fixed roots as cpt/noncpt (real form) and $-\theta$-fixed roots as nonparity/parity (block).

Rep of W on graph is sum (over real Cartans) of induced from stabilizer of $(\theta_x, \mathbb{Z}/2\mathbb{Z} \text{ stuff})$...

...so maps to sum of induced from stabilizer of θ_x.

W-graph for $G(\mathbb{R}) \twoheadrightarrow$ quotient with basis $\{$involutions$\}$.

Kottwitz calculated RHS ten years ago: for classical G it's sum of all special reps of W, each with mult given by Lusztig's canonical quotient (of $\pi_1(\mathcal{O})$).

What is to be done?

W-rep with basis $\{$Langlands params$\}$ $\twoheadrightarrow$ quotient with basis $\{$involutions$\}$.

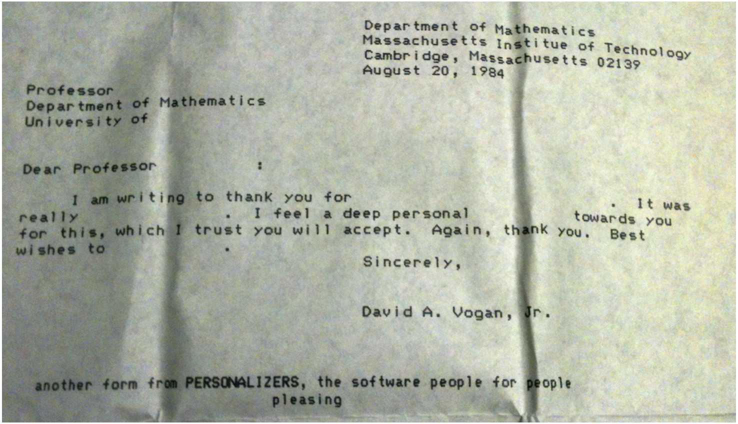
Kottwitz calculation of RHS explains ($G(\mathbb{R})$ reps) $\rightarrow$ (complex special orbits).

Lusztig-V (arxiv 2011): there's a W-graph with vertex set $\{$involutions$\}$.

PROBLEM: Relate LV W-graph structure on involutions to classical one on Langlands params.

PROBLEM: refine Kottwitz calculation to include $\mathbb{Z}/2\mathbb{Z}$ stuff, to explain/calculate Howe's wavefront set map ($G(\mathbb{R})$ reps) $\rightarrow$ (real special orbits).

Trying to outwit an old friend



Department of Mathematics
Massachusetts Institue of Technology
Cambridge, Massachusetts 02139
August 20, 1984

Professor
Department of Mathematics
University of

Dear Professor :

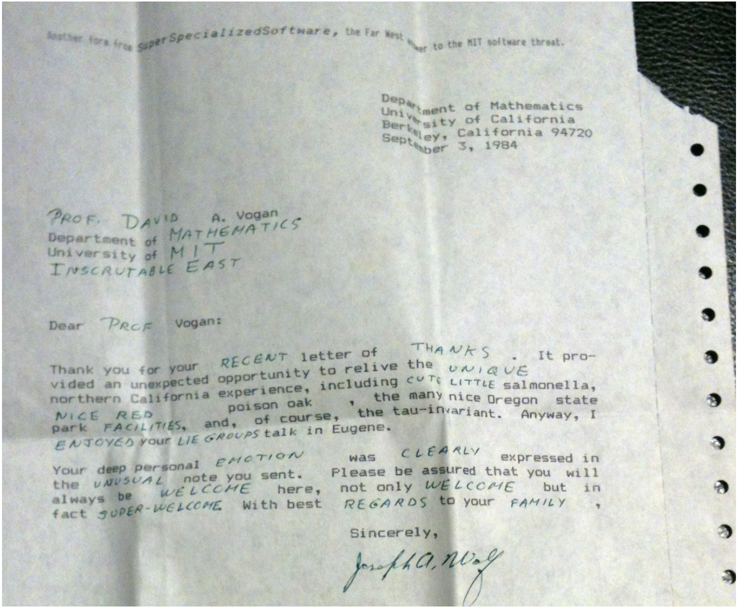
I am writing to thank you for . It was really . I feel a deep personal towards you for this, which I trust you will accept. Again, thank you. Best wishes to .

Sincerely,

David A. Vogan, Jr.

another form from PERSONALIZERS, the software people for people pleasing

Being outwitted by an old friend. . .



Another form from SuperSpecializedSoftware, the Far West answer to the MIT software threat.

Department of Mathematics
University of California
Berkeley, California 94720
September 3, 1984

PROF. DAVID A. Vogan
Department of MATHEMATICS
University of MIT
INSCRUTABLE EAST

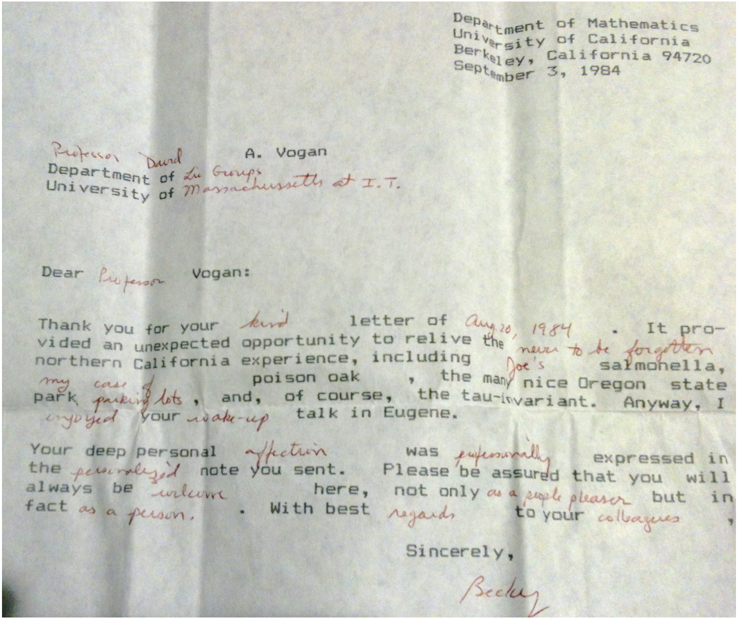
Dear PROF Vogan:

Thank you for your RECENT letter of THANKS . It provided an unexpected opportunity to relive the UNIQUE northern California experience, including CUTE LITTLE salmonella, NICE RED poison oak , the many nice Oregon state park FACILITIES, and, of course, the tau-invariant. Anyway, I ENJOYED your LIE GROUPS talk in Eugene.

Your deep personal EMOTION was CLEARLY expressed in the UNUSUAL note you sent. Please be assured that you will always be WELCOME here, not only WELCOME but in fact SUPER-WELCOME. With best REGARDS to your FAMILY ,

Sincerely,

…who has old friends of his own…



Department of Mathematics
University of California
Berkeley, California 94720
September 3, 1984

Professor David A. Vogan
Department of Lie Groups
University of Massachussetts at I.T.

Dear Professor Vogan:

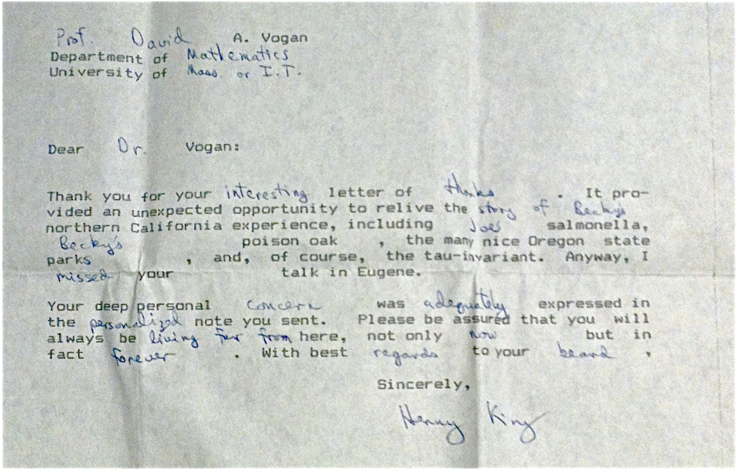
Thank you for your kind letter of Aug 20, 1984. It provided an unexpected opportunity to relive the never to be forgotten northern California experience, including Joe's salmonella, my case of poison oak, the many nice Oregon state park parking lots, and, of course, the tau-invariant. Anyway, I enjoyed your wake-up talk in Eugene.

Your deep personal affection was professionally expressed in the personalized note you sent. Please be assured that you will always be welcome here, not only as a people pleaser but in fact as a person. With best regards to your colleagues,

Sincerely,

Becky

. . . doesn't know when to quit.



Prof. David A. Vogan
Department of Mathematics
University of Mass. or I.T.

Dear Dr. Vogan:

Thank you for your interesting letter of thanks . It provided an unexpected opportunity to relive the story of Becky's northern California experience, including Joe's salmonella, Becky's poison oak , the many nice Oregon state parks , and, of course, the tau-invariant. Anyway, I missed your talk in Eugene.

Your deep personal concern was adequately expressed in the personalized note you sent. Please be assured that you will always be living far from here, not only now but in fact forever . With best regards to your beard .

Sincerely,

Henry King