Kazhdan-Lusztig polynomials for disconnected groups

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Outline

What are KL polynomials for?

How to compute KL polynomials

Twisting by outer automorphisms

Computing twisted KL polynomials

Making repn theory algebraic

G conn reductive alg gp def over $\mathbb{R}$

Natural problem: Describe irr repns $(\pi, \mathcal{H}_\pi)$ of $G(\mathbb{R})$.

Function-analytic: $\mathcal{H}_\pi$ is Hilbert space.

To do algebra: fix $K(\mathbb{R}) \subset G(\mathbb{R})$ max compact.

Analytic reps of $K(\mathbb{R})$ = algebraic reps of K (Weyl's unitarian trick)

$\mathcal{H}_\pi^K = K(\mathbb{R})$-finite vecs in $\mathcal{H}_\pi$ HC module of π

Harish-Chandra: $\mathcal{H}_\pi^K$ is $(\mathfrak{g}, K)$-module (alg rep of $\mathfrak{g} = \mathrm{Lie}(G)$ with compatible alg rep of K).

HC: analytic $G(\mathbb{R})$ reps $=$ algebraic $(\mathfrak{g}, K)$-mods.

$\Pi(G(\mathbb{R})) = \Pi(\mathfrak{g}, K) =$ equiv classes of irr mods.

From $(\mathfrak{g}, K)$-mods to constructible sheaves

$$X = \text{flag var of } G, \quad d = \dim X = \#(\text{pos roots})$$
$$= \text{variety of Borel subalgebras of } \mathfrak{g} = \text{Lie}(G)$$

X is "universal boundary" for G-homog spaces.

First made precise by Helgason conj (Kashiwara-Kowata-Minemura-Okamoto-Oshima-Tanaka).

Beilinson-Bernstein loc thm: any rep (more or less) appears in secs of (more or less) eqvt line bundle on X.

Which bundle $\leftrightsquigarrow$ infl char $=$ action of cent of $U(\mathfrak{g})$.

Example: F fin diml irr rep of $G \rightsquigarrow$ line bdle $\mathcal{L}_F \to X$; fiber at $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is lowest wt space

$$\mathcal{L}_{F,\mathfrak{b}} = F/\mathfrak{n}F = H_0(\mathfrak{n}, F); \qquad F = \text{alg secs of } \mathcal{L}_F$$

V fin length $(\mathfrak{g}, K)$-mod $\rightsquigarrow \mathcal{H}_F(V) \in K$-eqvt derived category of constructible sheaves on X.

Cohom $\mathcal{H}^i_F(V)$ is K-eqvt constr sheaf on X; fiber is

$$\mathcal{H}^i_F(V)_{\mathfrak{b}} = \mathrm{Hom}_{\mathfrak{h}}(F/\mathfrak{n}F, H_{i+d}(\mathfrak{n}, V)).$$

Irreducible modules and perverse sheaves

Given irr fin-diml rep F of G, have a functor $\mathcal{H}_F$ from finite length $(\mathfrak{g}, K)$-mods to K-eqvt derived category of constr sheaves on flag var X.

1970s: for V irr, $\mathcal{H}_F(V) \neq 0$ if (Harish-Chandra, Langlands, Schmid) and only if (Casselman-Osborne) V has same infl char as F.

Theorem (Beilinson-Bernstein, Kashiwara/Mebkhout,...)
Map $V \to \mathcal{H}_F(V)$ is bijection from irr $(\mathfrak{g}, K)$-mods, infl char of F to irr K-eqvt perverse sheaves on X.

$$\begin{aligned} &\text{irr } K\text{-eqvt perverse sheaves on } X \\ \leftrightarrow\ &\text{pairs } (\mathcal{O}, \mathcal{S}) \quad (K\text{-orbit on } X, \text{ irr eqvt local system}) \\ \leftrightarrow\ &\left\{(B, \sigma) \mid B \subset G \text{ Borel}, \sigma \in (B \cap K)/(B \cap K)_0\widehat{\ } \right\}/K \\ \leftrightarrow\ &\left\{(H, \Delta^+, \sigma) \mid H\ \theta\text{-stable CSG}, \sigma \in (H \cap K)/(H \cap K)_0\widehat{\ } \right\}/K \\ \leftrightarrow\ &\left\{(H(\mathbb{R}), \Delta^+, \sigma) \mid H(\mathbb{R}) \text{ real CSG}, \sigma \in H(\mathbb{R})/H(\mathbb{R})_0\widehat{\ } \right\}/G(\mathbb{R}). \end{aligned}$$

What KL polys tell you

$$\Pi_F(G(\mathbb{R})) = \text{params for irr reps, infl char of } F$$
$$= \left\{(H(\mathbb{R}), \Delta^+, \sigma)\right\} / G(\mathbb{R}).$$

$x \in \Pi_F(G(\mathbb{R})) \rightsquigarrow I(x)$ standard rep (like Verma module), $J(x)$ irr quotient.

Recall $d = \dim X$; put $d_x = \dim(K\text{-orbit for } x)$.

$$\mathcal{H}_F(I(x)) = \text{loc sys } \mathcal{S}(x)[d_x] \text{ on one } K\text{-orbit}$$
$$\mathcal{H}_F(J(x)) = \text{perverse extension } \mathbb{P}(x)$$

Coeff of t^i in KL poly $P_{y,x}$ is

$$\text{mult of loc sys } \mathcal{S}(y) \text{ in cohom } \mathbb{P}^{-d_x+2i}(x).$$
$$= \text{mult of } \sigma_y \text{ in } \mathrm{Hom}_{\mathfrak{h}_y}(F/\mathfrak{n}_y F, H_{(d-d_x)+2i}(\mathfrak{n}_y, J(x))).$$

$$J(x) = \sum_{y \le x} P_{y,x}(1)(-1)^{d_y - d_x} I(y).$$

The Hecke algebra

Interested in category $\mathcal{M}(X, K)$ of K-eqvt perverse sheaves on X.

Analogue over finite field $\mathbb{F}_q$: vector space

$$M(X, K)_q = \{K(\mathbb{F}_q)\text{-invt functions on } X(\mathbb{F}_q)\}.$$

This vec space is module for Hecke algebra at q

$$\mathcal{H}_q = \{G(\mathbb{F}_q)\text{-invt functions on } X(\mathbb{F}_q)\}.$$

$\mathcal{M}(X, K)$ ind of field; irrs are ratl $/\mathbb{F}_q$.

Frobenius F is alg aut of G, K, X... Fixed pts = $\mathbb{F}_q$-ratl points.

Alt sum of traces of F on fibers of cohomology sheaves at rational points maps (ratl) objects of $\mathcal{M}(X, K)$ to $M(X, K)_q$.

Lusztig: relate $\mathcal{H}_q$ action to geometry of perverse sheaves $\rightsquigarrow$ compute $P_{y,x}$.

Disconnected groups

Want to consider $G' \supset G$, G'/G finite.

Rep theory for finite groups harder than for Lie groups, because Lie algebra linearizes problems.

So "arbitrary" disconnected reductive $G' \supset G$ too hard.

Easy/useful special case: G'/G finite cyclic.

Action of G' on $G \rightsquigarrow$ candidates corr to

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

Fix $G \supset B_p \supset H_p \rightsquigarrow \Pi_p$ simple roots, root datum

$$(X^*(H_p), \Pi_p, X_*(H_p), \Pi_p^\vee).$$

Pinning p is $B_p \supset H_p$ plus choice of maps

$$\phi_p(\alpha)\colon SL(2) \to G, \qquad \alpha \in \Pi_p.$$

Now automorphism group sequence split by

$$\begin{aligned}\mathrm{Aut}_p(G) &=_{\mathrm{def}} \text{auts permuting } \{\phi_p\}\\ &\simeq \text{root datum automorphisms} \simeq \mathrm{Out}(G).\end{aligned}$$

Pinned disconnected groups

Cplx conn reductive G, pinning p, Cartan inv θ preserves $H_p \subset B_p$ (fundamental), almost permutes $\{\phi_p\}$.

Fix order two root datum aut $\delta \rightsquigarrow \delta_p \in \mathrm{Aut}_p(G_0)$.

Order two not critical, but easier and covers current applications.

$G^\Delta =_{\mathrm{def}} G \rtimes \{1, \delta_p\}$ our model disconn reductive group.

$\delta_p\theta = \theta\delta_p$, so δ_p normalizes K; $K^\Delta =_{\mathrm{def}} K \rtimes \{1, \delta_p\}$.

Ex: δ induced by θ, so $\theta = \mathrm{Ad}(t)\delta_p$ ($t \in H_p \cap K$).

Ex: $G = SO(2n, \mathbb{C})$, $H_p = SO(2, \mathbb{C})^n$,

$$\theta = \mathrm{Ad}(\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1, 1, -1))$$

($2p + 1$ 1s, $2q + 1$ -1s); means $G(\mathbb{R}) \simeq SO(2p + 1, 2q + 1)$. Have

$$\delta_p = \mathrm{Ad}(\mathrm{diag}(1, \ldots, 1, -1));$$

so $G^\Delta \simeq O(2n, \mathbb{C})$.

Reps of pinned disconnected groups

$G^\Delta =_{\text{def}} G \rtimes \{1, \delta_p\}$, $K^\Delta =_{\text{def}} K \rtimes \{1, \delta_p\}$.

V any $(\mathfrak{g}, K)$-module $\rightsquigarrow$ V^{δ_p} same vector space; actions of $\mathfrak{g}$, K twisted by δ_p. Gives action of Δ on $\Pi(\mathfrak{g}, K)$.

First way to make irr $(\mathfrak{g}, K^\Delta)$-module: start with $V \not\simeq V^{\delta_p}$ irr, $V^\Delta = V \oplus V^{\delta_p}$. Gives bijection

$$V^\Delta \in \Pi(\mathfrak{g}, K^\Delta) \text{ reducible on } (\mathfrak{g}, K)$$

$$\leftrightarrow \text{two-elt orbits } \{V, V^{\delta_p}\} \text{ of } \Delta \text{ on } \Pi(\mathfrak{g}, K).$$

Second way to make irr $(\mathfrak{g}, K^\Delta)$-module: start with $V \simeq V^{\delta_p}$ irr; choose intertwining op

$$D_p \colon V \to V^{\delta_p}, \qquad D_p^2 = \text{Id}$$

(two choices diff by sgn). Extend V to V^Δ, making δ_p act by D_p. Gives 2-to-1 map

$$\text{elts } V^\Delta \text{ of } \Pi(\mathfrak{g}, K^\Delta) \text{ irr on } (\mathfrak{g}, K)$$

$$\leftrightarrow \text{fixed pts } V \text{ of } \Delta \text{ on } \Pi(\mathfrak{g}, K).$$

Application of disconnected groups

δ root datum aut induced by Cartan inv θ.

δ is trivial iff rk $G =$ rk K.

Theorem (consquence of Knapp-Zuckerman)

V irr $(\mathfrak{g}, K)$-mod of real infl char. Then V admits invt Hermitian form iff $V \simeq V^\delta$.

Corollary (Adams-van Leeuwen-Trapa-V-Yee)

Every irr $(\mathfrak{g}, K^\Delta)$-mod V^Δ of real infl char admits preferred invt Hermitian form.

Case $V^\Delta = V \oplus V^{\delta_p}$: form "hyperbolic," V isotropic.

Case $V^\Delta \rightsquigarrow V$ irr: V admits invt Herm form. Two exts V^Δ to $(\mathfrak{g}, K^\Delta)$-mod $\leftrightsquigarrow$ two invt forms on V.

Relate invt forms on irrs to invt forms on std modules $\leftrightsquigarrow$ compute char formulas for $(\mathfrak{g}, K^\Delta)$-mods.

Classifying irr mods for pinned disconn gps

F fin-diml irr of G, lowest weight $\lambda_p \in X^*(H_p)$.

Twist $F^{\delta_p} =$ irr of lowest weight $\delta\lambda_p$; isom to F iff λ_p is fixed by root datum aut δ.

Case $F \not\simeq F^{\delta_p}$: no irr of infl char F has invt Herm form; irr $(\mathfrak{g}, K^\Delta)$-mods all induced from irr $(\mathfrak{g}, K)$-mods.

Case $F \simeq F^{\delta_p}$: fix canonical extension F^Δ of F to G^Δ rep, δ_p acts triv on B_p-lowest wt space.

Get G^Δ-eqvt alg line bdle $\mathcal{L}_{F^\Delta}$ on X.

Localization $+$ Riemann-Hilbert gives functor $\mathcal{H}_{F^\Delta}$ from fin length $(\mathfrak{g}, K^\Delta)$-mod to K^Δ-eqvt derived category of constr sheaves on X.

Theorem (Beilinson-Bernstein, Kashiwara/Mebkhout,...)
Map $V^\Delta \to \mathcal{H}_{F^\Delta}(V^\Delta)$ is bijection from irr $(\mathfrak{g}, K^\Delta)$-mods, infl char of F^Δ to irr K^Δ-eqvt perverse sheaves on X.

K^Δ-eqvt irr perverse sheaves

$B \subset G$ Borel $\rightsquigarrow B^\Delta = N_{G^\Delta}(B)$.

K^Δ preserves $K \cdot \mathfrak{b} \Leftrightarrow B \cap K$ index two in $B^\Delta \cap K^\Delta$

$\Leftrightarrow B^\Delta \setminus B$ meets $K^\Delta \setminus K$ in elt d

K-eqvt loc sys $\mathcal{S}_\sigma$ extends to K^Δ-eqvt

$\Leftrightarrow$ automorphism d fixes char σ of $(B \cap K)/(B \cap K)_0$.

First way to make irr K^Δ-eqvt perverse sheaf: start with $\mathbb{P} \not\simeq \mathbb{P}^{\delta_p}$ irr, $\mathbb{P}^\Delta = \mathbb{P} \oplus \mathbb{P}^{\delta_p}$.

Built from local systems on orbits via classical KL polys.

Second way to make irr K^Δ-eqvt perverse sheaf: start with $\mathbb{P} \simeq \mathbb{P}^{\delta_p}$ irr; choose isom

$$D_p \colon \mathbb{P} \to \mathbb{P}^{\delta_p}, \quad D_p^2 = \mathrm{Id}$$

(two choices diff by sgn). Make δ_p act by D_p; $\mathbb{P}$ K^Δ-eqvt.

"First way" local systems in $\mathbb{P} \leftrightsquigarrow$ classical KL polys.

New KL polys $P^\delta_{y,x}$: x, y K^Δ-eqvt loc systems irr for K.

Coeff of t^i in $P^\delta_{y,x}$ is

mult[local sys $\mathcal{S}(y)$] $-$ mult[local sys $\mathcal{S}(-y)$] in $\mathbb{P}^{-d_x+2i}(x)$.

The twisted Hecke algebra

$\mathcal{M}(X, K^\Delta) = K^\Delta$-eqvt perverse sheaves on X.

Over $\mathbb{F}_q$: use Frobenius map F twisted by δ_p to get quasisplit form $G(\mathbb{F}_q) \supset K(\mathbb{F}_q)$

Easy $\mathbb{F}_q$-analogue of perverse sheaves is

$$M(X, K^\Delta)_q = \{K(\mathbb{F}_q)\text{-invt functions on } X(\mathbb{F}_q)\}.$$

This vec space is module for Lusztig's quasisplit Hecke algebra at q

$$\mathcal{H}_q^\delta = \{G(\mathbb{F}_q)\text{-invt functions on } X(\mathbb{F}_q)\}.$$

Basis indexed by $W^\delta =$ fixed subgp of root datum aut on W; gens indexed by Π_p/Δ, aut orbits of simple roots.

Relations involve dimensions of "restricted root" subgps: (very particular) unequal parameter Hecke algebra.

Alt sum of traces of F on fibers of cohom sheaves at ratl pts maps (ratl) objects of $\mathcal{M}(X, K)$ to $M(X, K)_q$.

Lusztig: relate $\mathcal{H}_q^\delta$ action to geom $\rightsquigarrow$ compute $P_{y,x}^\delta$.