The Langlands philosophy and representation theory

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Outline

What Langlands can do for me

Introduction to number theory

Analytic number theory

Automorphic forms

Langlands conjectures, a bit more precisely

What's better than a representation of $\mathsf{Gal}(\mathbb{C}/\mathbb{R})$?

End of this talk is beginning of promised talk.

What happened to “Unitary dual”?

Announced title was, *What's the unitary dual look like?*

Unfortunately, I often prepare talks at the last minute.

On this occasion the last minute coincided with a (highly divisible) wedding anniversary; so burying myself in my computer seemed like a poor idea.

We celebrated our tenth anniversary at the Arbeitstagung in Bonn.
My wife has not forgotten.

There was on my computer a set of slides on a (related!) topic, and so I have elected to use those.

Now the end of the talk can be a surprise to me as well as to you.

I beg your forgiveness for this deviation from plan, and hope that you find something to enjoy.

Here's the punchline

$GL_n(\mathbb{R})$ is everybody's favorite reductive group.

Want to understand $\widehat{GL_n(\mathbb{R})}$ = set of irr repns.

Studied by Gelfand, Harish-Chandra *et alia* 1950s), as part of functional analysis. That was really hard.

Langlands (1960s) studied $\widehat{GL_n(\mathbb{R})}$ for number theory.

Langlands idea: $\widehat{GL_n(\mathbb{R})} \overset{\approx ?}{\leftrightsquigarrow}$ n-diml reps of $\mathrm{Gal}(\overline{\mathbb{R}}/\mathbb{R})$.

That is, $\widehat{GL_n(\mathbb{R})} \overset{\approx ?}{\leftrightsquigarrow} \{n \times n \text{ matrices } J,\ J^2 = I\}/(\text{conjugation})$.

FINALLY we have an algebra-friendly problem:

$$\widehat{GL_n(\mathbb{R})} \overset{\approx ?}{\leftrightsquigarrow} \{\text{decompositions } n = p + q\}.$$

This is a bit too simple to be true. Plan today:

1. look at the origin of Langlands' idea;
2. how Langlands complicated the idea so it can be true.
3. how to complicate it even more so it can be even truer.

A one-minute introduction to number theory

Number theory $\leftrightsquigarrow$ solutions in $\mathbb{Q}$ to polynomial eqns.

Can always find solutions by enlarging the field, so

Number theory $\leftrightsquigarrow$ understanding finite extensions of $\mathbb{Q}$.

$$\begin{aligned} E &= \text{(separable) degree } n \text{ extension of } k \\ &= n\text{-dimensional vector space over } k. \end{aligned}$$

$$GL(E/k) = \{\text{invertible } k\text{-linear } E \to E\} \simeq GL_n(k).$$

Multiplication in $E \rightsquigarrow E^\times \hookrightarrow GL_k(E)$ maximal abelian

Theorem.

$$\begin{matrix} \text{separable extensions } E_j \\ \sum_j [E_j : k] = n \end{matrix} \longleftrightarrow \begin{matrix} \text{nice max abelian} \\ A \subset GL_n(k) \end{matrix}$$

$$A = E_1^\times \times \cdots \times E_m^\times.$$

Number theory $\leftrightsquigarrow$ group theory for $GL_n(\mathbb{Q})$.

Another one-minute intro to number theory

Number theory $\leftrightsquigarrow$ solutions in $\mathbb{Q}$ to polynomial eqns.

What's hard about that is that there's no analysis.

Embed $\mathbb{Q} \hookrightarrow \mathbb{R}$, study real solutions using analysis.

$x^2 + 4y^2 = -3$: no real solutions, so no rational solutions.

Embed $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, study p-adic solutions using analysis.

$x^2 + 4y^2 = 135$: no solutions $\pmod 4$, so no rational solutions.

Adeles of $\mathbb{Q}$ is (restricted) direct product $\mathbb{A}(\mathbb{Q}) = \mathbb{R} \times \prod_p \mathbb{Q}_p$.

$\mathbb{A}(\mathbb{Q})$ = loc compact ring $\supset \mathbb{Q}$ = discrete cocompact subring.

arithmetic on $\mathbb{Q} \leftrightsquigarrow$ analysis on compact $\mathbb{A}(\mathbb{Q})/\mathbb{Q}$.

What's analysis look like on the adeles?

Adeles of $\mathbb{Q}$ $\qquad \mathbb{A}(\mathbb{Q}) = \mathbb{R} \times \prod_p \mathbb{Q}_p$.

$\mathbb{A}(\mathbb{Q}) =$ loc compact ring $\supset \mathbb{Q} =$ discrete cocompact subring.

Like $\mathbb{R} \supset \mathbb{Z}$ but with more number-theoretic content.

Since $(\mathbb{A}(\mathbb{Q}), +)$ loc compact abelian, have a dual group

$$\widehat{\mathbb{A}(\mathbb{Q})} =_{\text{def}} \{\chi \colon \mathbb{A}(\mathbb{Q}) \to U(1) \text{ continuous}, \chi(a+b) = \chi(a)\chi(b)\},$$

Haar measures da on $\mathbb{A}$ and $d\chi$ on $\widehat{\mathbb{A}}$, and Fourier transform

$$\widehat{\ } \colon \mathcal{S}(\mathbb{A}) \xrightarrow{\sim} \mathcal{S}(\widehat{\mathbb{A}}), \quad \widehat{F}(\chi) = \int_{\mathbb{A}} F(a)\chi(a)da.$$

Theorem. Fix nontrivial character $\chi_1 \in \widehat{\mathbb{A}}$ trivial on $\mathbb{Q}$.
For $\xi \in \mathbb{A}$ define $\chi_\xi(a) =_{\text{def}} \chi_1(\xi \cdot a)$.

1. $\xi \mapsto \chi_\xi$ is an isomorphism $\mathbb{A} \simeq \widehat{\mathbb{A}}$.
2. $\{\chi_\xi \mid \xi \in \mathbb{Q}\} \simeq \widehat{\mathbb{A}/\mathbb{Q}}$.

Nice basis of functions on $\mathbb{A}(\mathbb{Q})/\mathbb{Q}$ indexed by $\mathbb{Q}$.

A one-minute intro to automorphic forms

Wasn't there a reductive group here somewhere?

$$\begin{matrix} \text{separable extensions } E_j \\ \sum_j [E_j : k] = n \end{matrix} \longleftrightarrow \begin{matrix} \text{nice max abelian} \\ A \subset GL_n(k) \end{matrix}$$

$$A = E_1^\times \times \cdots \times E_m^\times.$$

Number theory $\leftrightsquigarrow$ group theory for $GL_n(\mathbb{Q})$.

To do analysis in this world, use locally compact group

$$GL_n(\mathbb{A}) = \prod_v GL_n(\mathbb{Q}_v).$$

Diagonal embedding is

$$GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{A}),$$

discrete subgroup that's nearly cocompact.

arithm on $GL_n(\mathbb{Q})$ $\leftrightsquigarrow$ analysis on nearly cpt space $GL_n(\mathbb{A})/GL_n(\mathbb{Q})$.

Automorphic forms = nice fns on $GL_n(\mathbb{A})/GL_n(\mathbb{Q})$.

Automorphic representations

$GL_n(\mathbb{A}) = \prod_v GL_n(\mathbb{Q}_v)$ locally compact group.

Number theory $\leftrightsquigarrow$ $\mathcal{A}(\mathbb{Q})$ = nice fns on $GL_n(\mathbb{A})/GL_n(\mathbb{Q})$.

$\mathcal{A}(\mathbb{Q})$ = vector space where $GL_n(\mathbb{A})$ acts: representation!

Automorphic rep = irr rep of $GL_n(\mathbb{A})$ on $\mathcal{A}(\mathbb{Q})$.

Irr rep of product = tensor product of irr reps.

Any irr rep $\pi \in \widehat{GL_n(\mathbb{A})}$ is $\pi = \otimes_v \pi_v$, $\pi_v \in \widehat{GL_n(\mathbb{Q}_v)}$.

π automorphic $\iff$ $\otimes_v \pi_v$ has $GL_n(\mathbb{Q})$-fixed vector.

Analogous to “matching chars” in reciprocity laws of class field theory.

Langlands philosophy, take one

$\mathrm{Gal}(\mathbb{Q}_v) \hookrightarrow \mathrm{Gal}(\mathbb{Q})$ mod conjugacy.

Langlands' philosophy $\rightsquigarrow$ conjectural maps:

$$(n\text{-diml reps of } \mathrm{Gal}(\mathbb{Q}_v)) \xrightarrow{\text{local}} \widehat{GL_n(\mathbb{Q}_v)}, \quad \sigma_v \mapsto \pi_v(\sigma_v).$$

$$(n\text{-diml reps of } \mathrm{Gal}(\mathbb{Q})) \xrightarrow{\text{global}} (\text{automorphic reps of } GL_n)$$

σ n-diml of $\mathrm{Gal}(\mathbb{Q}) \rightsquigarrow$ automorphic $\pi(\sigma) = \otimes_v \pi_v(\sigma)$.

Local/global compatibility: $\pi_v(\sigma) = \pi_v\left(\sigma|_{\mathrm{Gal}(\mathbb{Q}_v)}\right)$.

Offers indirect "answer" to question of which local Galois group representations can be assembled to global ones...

Set $\{\sigma_v\}$ of n-diml reps of $\mathrm{Gal}(\mathbb{Q}_v)$ assemble to n-diml rep σ of $\mathrm{Gal}(\mathbb{Q})$ $\iff$ tensor product of corresponding $GL_n(\mathbb{Q}_v)$ reps has $GL_n(\mathbb{Q})$ fixed vector.

N.B.: the maps $\xrightarrow{\text{local}}$ and $\xrightarrow{\text{global}}$ aren't surjective!

What makes the Langlands conjectures true?

Number theory $\leftrightsquigarrow \mathcal{A}(\mathbb{Q}) =$ nice fns on $GL_n(\mathbb{A})/GL_n(\mathbb{Q})$

supports conjectural global correspondence

$$\sigma \text{ } n\text{-diml of } \mathrm{Gal}(\mathbb{Q}) \rightsquigarrow \text{automorphic } \pi(\sigma) = \otimes_v \pi_v(\sigma),$$

suggests image includes "most" automorphic reps.

Nature of embeddings $GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{Q}_v)$ supports

$$\{\text{comps } \pi_v \text{ of automorphic } \pi\} \supset \text{"most of" } \widehat{GL_n(\mathbb{Q}_v)}.$$

Now a local correspondence

$$(n\text{-diml reps of } \mathrm{Gal}(\mathbb{Q}_v)) \xrightarrow{\text{local}} \widehat{GL_n(\mathbb{Q}_v)}, \quad \sigma_v \mapsto \pi_v(\sigma_v)$$

needs to be defined, with image including "most" of $\widehat{GL_n(\mathbb{Q}_v)}$, for local/global compatibility to make sense.

What makes the Langlands conjectures false?

Galois grps are compact, so sets of repns are discrete.

Predicted sets of automorphic representations and $GL_n(\mathbb{Q}_v)$ representations are therefore discrete.

$GL_n(\mathbb{Q}_v)$, $GL_n(\mathbb{A})/GL_n(\mathbb{Q})$ are both noncompact (like $\mathbb{R}$)...

...so have continuous spectra (like Fourier transform for $\mathbb{R}$).

Langlands understood this difficulty very well.

Class field theory (case of GL_1 for Langlands' conjectures) already sees this difficulty.

Langlands followed Andre Weil's resolution: replace $\mathrm{Gal}(\mathbb{Q}_v)$ by closely related noncompact Weil group W_v.

Seems to work perfectly for nonarchimedean $\mathbb{Q}_v$...

...but less well for $\mathbb{R}$ and $\mathbb{C}$.

Where do automorphic forms come from?

Automorphic forms tied to L-functions: meromorphic functions, analytic behavior $\leftrightsquigarrow$ interesting number theory.

Fundamental example is Riemann zeta function.

Emil Artin gave a construction (for number fields)

representation of Galois group $\rightarrow$ L-function.

This is part of the basis of Langlands' conjectures:

Galois reps $\leftrightsquigarrow$ L-functions $\leftrightsquigarrow$ automorphic forms.

Another source of L-functions is varieties/number fields.

Connection with Artin L-functions looks like this:

variety $X/F \rightarrow$ cohomology $H^*(X) \rightarrow$ rep of $\mathrm{Gal}(F)$ on $H^*(X)$.

This is a good way to think, but the arrows don't really work...

...Artin uses cplx reps, so want cohom with cplx coeffs.

But $\mathrm{Gal}(F)$ does not act on such cohomology.

What does this suggest about Langlands?

aut form on $GL(n) \longleftrightarrow$ n-diml cohom space of alg variety/$\mathbb{Q}$ $\longleftrightarrow$ n-diml rep of $\mathrm{Gal}(Q)$

Local version at $\mathbb{R}$ is

rep of $GL_n(\mathbb{R}) \longleftrightarrow$ n-diml cohom space of alg variety/$\mathbb{R}$ $\longleftrightarrow$ n-diml rep of $\mathrm{Gal}(\mathbb{R})$

In both settings, first problem is that red arrows don't work.

To address that, Langlands needed a structure on an n-diml cplx vector space V related to cohom of alg variety.

An integral Langlands parameter for $GL_n(\mathbb{R})$ is

1. complex vector space V of dimension n;
2. involution $y \in \mathrm{Aut}(V)$ of order (one or) two;
3. bigrading $\{V_{p,q} \mid p, q \in \mathbb{Z}\}$, such that $y(V_{p,q}) = V_{q,p}$.

This is close to Hodge structure on cohom of smooth $X/\mathbb{R}$.

Langlands proved local Langlands conjecture:

THM : irr reps of $GL_n(\mathbb{R}) \longleftrightarrow$ equivalence classes of Langlands params .

Why wasn't that the last slide?

Recall that an integral Langlands parameter for $GL_n(\mathbb{R})$ is (Adams-Barbasch-V 1992)

1. complex vector space V of dimension n;
2. involution $y \in \mathrm{Aut}(V)$ of order (one or) two;
3. bigrading $\{V_{p,q} \mid p, q \in \mathbb{Z}\}$, such that $y(V_{p,q}) = V_{q,p}$.

and this is close to Hodge structure on cohom of smooth $X/\mathbb{R}$.

Two reasons to keep going:

aesthetic: non-smooth X lack such Hodge structure;

practical: (Langlands params$/\mathbb{R}$) lacks interesting geometry.

An integral geometric parameter for $GL_n(\mathbb{R})$ is

1. complex vector space V of dimension n;
2. involution $y \in \mathrm{Aut}(V)$ of order (one or) two;
3. filtration $\{\cdots F_{p-1}V \subset F_pV \subset F_{p+1}V \cdots\}$

and this is in the spirit of the Hodge filtration on cohom of any $X/\mathbb{R}$.

Linear algebra exercise:

$$(y, (V_{p.q})) \mapsto (y, \sum_{p'<p,q} V_{p',q})$$

is a bijection from equiv classes of integral Langlands params to equiv classes of geom params.

COROLLARY : irr reps of $GL_n(\mathbb{R}) \longleftrightarrow$ equivalence classes of geometric params .

What do you do with this?

$GL_n(\mathbb{R})$ reps correspond to integral geometric params:

1. complex vector space V of dimension n;
2. involution $y \in \mathrm{Aut}(V)$ of order (one or) two;
3. filtration $\{\cdots F_{p-1}V \subset F_pV \subset F_{p+1}V \cdots\}$

Equiv class of filtrations $\leftrightsquigarrow$ collection of nonnegative integers

$$m_p = \dim(F_pV/F_{p-1}V), \qquad \sum m_p = n.$$

Set of filtrations $\leftrightsquigarrow$ (complex projective) partial flag variety

$$GL_n(\mathbb{C})/P_{(m_p)} \qquad (P_{(m_p)} = \text{parabolic subgroup}).$$

Equiv class of involutions $\leftrightsquigarrow$ nonneg pairs $(a, b), a + b = n$.

Set of involutions $\leftrightsquigarrow$ $GL_n(\mathbb{C})/(GL_a(\mathbb{C}) \times GL_b(\mathbb{C}))$.

$$\begin{matrix}\text{equiv classes of}\\ \text{integral geom params}\end{matrix} \longleftrightarrow \begin{matrix}\text{orbits of } GL_a \times GL_b \text{ on}\\ \text{flag variety } GL_n/P_{(m_p)}\end{matrix}.$$

This is the beginning of detailed study of reps of $GL_n(\mathbb{R})$:

$$\begin{matrix}\text{intersection cohom}\\ \text{of orbit closures}\end{matrix} \longleftrightarrow \text{characters of irr reps.}$$

Left side was computed by George Lusztig.

Footnotes

Here are some details that didn't fit on earlier slides.

How to remove qualifier integral from def of Langlands params: Langlands, "On the classification of irreducible representations of real algebraic groups," 1970.

How to define Langlands params for any reductive G: same.

How to define geometric params for any reductive G: Adams-Barbasch-Vogan, *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, 1992.

Intersection cohomology of symmetric subgroup orbit closures: Lusztig-V, "Singularities of closures of K-orbits on flag manifolds," 1983.

Computing unitary representations using geometric params: Adams-van Leeuwen-Trapa-Vogan, "Unitary representations of real reductive groups," 2020.

Computer implementation: du Cloux-van Leeuwen, `atlas` software, `http://www.liegroups.org/software/`.