3. Hochschild-Serre spectral sequence for $SL(3,\mathbb{R})$

The first big chart here is the E_1 term of the Hochschild-Serre spectral sequence for computing the $\mathfrak{n}$-cohomology of a Harish-Chandra module X for $SL(3,\mathbb{R})$. Each column shows the contribution of one representation of $K = SO(3)$; the numbers are the weights of the representation of $T_1 = SO(2)$. For example, the column "5-diml" says that each copy of the 5-dimensional representation of $SO(3)$ occurring in X contributes a three-dimensional piece to the degree one part of E_1; the corresponding weights of T_1 are 1, 0, and -3.

degree	1-diml		3-diml		5-diml		7-diml		9-diml		11-diml	
0	0		1		2		3		4		5	
1	$-1,-2$	-1	$0,-1$	-2	$1,0$	-3	$2,1$	-4	$3,2$	-5	$4,3$	-6
2	-3	$-2,-3$	-2	$-3,-4$	-1	$-4,-5$	0	$-5,-6$	1	$-6,-7$	2	$-7,-8$
3		-4		-5		-6		-7		-8		-9

Suppose we concentrate for a moment on the -3 weight space (which has one dimension from each copy of the 5-dimensional representation of K). This weight space might fail to survive to E_∞ in two ways. First, it could be in the image of a differential from degree 0. But we see from the table (extrapolated infinitely to the right) that the weight -3 does not appear in degree 0; so this is impossible. Second, our weight space could have a non-zero image in the degree 2. By inspection of the degree 2 row of the table, we see that the weight -3 appears there only if X has a copy of the $SO(3)$ representation of dimension 1 or the $SO(3)$ representation of dimension 3.

So inspection of this spectral sequence leads to the following conclusion.

Proposition 1 *Suppose X is a Harish-Chandra module for $SL(3,\mathbb{R})$, and that the $SO(3)$ representations of dimensions 1 and 3 do not occur in X. Then the T_1 weight -3 appears in $H^\bullet(\mathfrak{n}, X)$ only in degree 1, where it has multiplicity equal to the multiplicity of the 5-dimensional representation of $SO(3)$ in X.*

An identical argument proves

Proposition 2 *Suppose that $2m+1 \geq 7$, and that the $SO(3)$ representations of dimensions $2m-1$, $2m-3$, and $2m-5$ do not occur in X. Then the T_1 weight $-m-1$ appears in $H^\bullet(\mathfrak{n}, X)$ only in degree 1, where it has multiplicity equal to the multiplicity of the $2m+1$-dimensional representation of $SO(3)$ in X.*

How was this spectral sequence constructed? Recall that the E_1 term in general is built from pieces $\bigwedge^p(\mathfrak{n}\cap\mathfrak{s})^* \otimes H^q(\mathfrak{n}_\mathfrak{k}, X)$, and that the degree is $p+q$. The top entries in the two columns for each representation of K are the weights in $H^0(\mathfrak{n}_\mathfrak{k}, \bullet)$ and $H^1(\mathfrak{n}_\mathfrak{k}, \bullet)$ for the indicated representation of K. (Those are provided by the Bott-Kostant theorem.) The rest of each column comes from adding the weights of $\bigwedge^\bullet(\mathfrak{n}\cap\mathfrak{s})^*$, which are tabulated below.

degree	$\bigwedge^\bullet(\mathfrak{n}\cap\mathfrak{s})^*$
0	0
1	$-1,-2$
2	-3