What's special about special?

David Vogan

University of Chicago Colloquium
April 23, 2025

Outline

What this talk is about

An infinite-diml irreducible representation π of a reductive group G is very complicated.

$\pi \rightsquigarrow$ simpler invt (Jantzen-Zuckerman):

1. rep σ of a Weyl group, vector $v(\pi)$ in space of σ.
2. vector $v(\pi) \rightsquigarrow$ subquotient rep $\sigma(\pi)$ of σ.
3. $\sigma(\pi)$ has natural irreducible quotient $\sigma^J(\pi)$ (Joseph).

Parts (1) and (2) are defined over $\mathbb{Z}$.

I learned at Chicago (from Jon Alperin and George Glauberman) that definition over $\mathbb{Z}$ is a good and interesting thing.

What this talk is about (continued)

Our story so far: infinite dimensional representation

$$\pi \text{ of reductive } G \rightsquigarrow \sigma(\pi) \twoheadrightarrow \sigma^J(\pi)$$

rep and irreducible quotient of finite Weyl group W.

$GL(n)$: $\sigma(\pi) = \sigma^J(\pi)$ is irreducible; any $\sigma \in \widehat{W}$ can occur.

Other G: only certain values of $\sigma^J(\pi)$ are possible.

FACT (Lusztig): $\sigma^J(\pi)$ must be a special rep of W

Lusztig (1979) defined special reps of W as part of his classification of irr reps of finite Chevalley groups.

PLAN(1): outline the definition of $\sigma(\pi)$
(coherent families: Schmid, Jantzen, Zuckerman).

PLAN(2): sketch some elementary integrality properties of the rep $\sigma(\pi)$ (tau invariants: Borho, Jantzen, Duflo).

PLAN(3): show how these integrality properties lead to Lusztig's special reps in rank two.

Character formulas: compact case

K compact conn Lie, max torus T, $\pi \in \widehat{K}$.

$\Theta_\pi(k) = \operatorname{tr} \pi(k)$ smooth on K; Θ_π determines π.

Weyl: there is a dominant regular character $\gamma \in \widehat{T}$ so that

$$\Theta_\pi(t) = \Big(\sum_{w \in W} \operatorname{sgn}(w)(w \cdot \gamma)(t) \Big) / D(t) \qquad (t \in T).$$

Summary: the character formula for an irreducible representation π of K is sum over a W-orbit of chars of T, coefficients given by representation $\operatorname{sgn} \in \widehat{W}$.

We'll see that $\sigma(\pi) = \sigma^J(\pi) = \operatorname{sgn}$.

Character formulas: noncompact case

Harish-Chandra showed how to extend Weyl's formula to real reductive $G \supset H$ maximal torus, $\pi \in \widehat{G}$.

$\Theta_\pi(g) = \mathrm{tr}\,\pi(g)$ generalized fn on G; Θ_π determines π.

HC: $\exists$ dominant weight $\gamma \in \mathfrak{h}^*_{\mathbb{C}}$ so that

$$\Theta_\pi(h) = \Big(\sum_{w\in W} \sum_{\substack{\phi\in\widehat{H} \\ d\phi = w\cdot\gamma}} a_{w,\phi}\cdot\phi(h)\Big)/D(h) \quad (h\in H;\, a_{w,\phi}\in\mathbb{Z}).$$

HC: differential eqns satisfied by $\Theta_\pi \rightsquigarrow \gamma$, exponentials.

Summary: character formula on H for irreducible representation π of G is sum over characters of H with differentials in a single W-orbit of characters of $\mathfrak{h}^*_{\mathbb{C}}$.

Roughly: $\sigma(\pi) \in \widehat{W}$ generated by function $w \mapsto a_w$ on W.

Difficulty: $a_{w,\phi}$ depends not just on w but also on $\phi \in \widehat{H}$.

(Not) making W act on characters of H.

Our story so far: $\pi \in \widehat{G} \rightsquigarrow \Theta_\pi$ distribution character of π:

$$\Theta_\pi(h) = \Big(\sum_{w \in W} \sum_{\substack{\phi \in \widehat{H} \\ d\phi = w\cdot\gamma}} a_\phi \cdot \phi(h)\Big)/D(h) \quad (h \in H; a_\phi \in \mathbb{Z}).$$

Want W to "act" on Θ_π: pretend Θ_π is "function" of γ. . . :

$$x \cdot \Theta_\pi(h) \stackrel{?}{=} \Big(\sum_{w \in W} \sum_{\substack{\phi \in \widehat{H} \\ d\phi' = wx^{-1}\cdot\gamma}} a_\phi \cdot \phi'(h)\Big)/D(h).$$

Dominant weight $\gamma \in \mathfrak{h}^*_{\mathbb{C}}$ called infinitesimal character of π

To make this precise, we need to pass from the character ϕ with $d\phi = w \cdot \gamma$ to a character ϕ' with $d\phi' = wx^{-1} \cdot \gamma$.

Should act on ϕ by $wx^{-1}w^{-1}$.

But W need not preserve the real Cartan subgroup H, so need not act on characters of H.

The integral Weyl group to the rescue.

To make Weyl group W act on a distribution character for G, needed W act on some characters of a real torus H.

Where W acts is on rational characters τ of H: if $\alpha \in \mathfrak{h}^*$ is a root, and $\alpha^\vee \in \mathfrak{h}$ the corresponding coroot, then

$$s_\alpha(\tau) = \tau - d\tau(\alpha^\vee)\alpha.$$

Rationality of τ means $d\tau(\alpha^\vee) \in \mathbb{Z}$; action of s_α just translates τ by a multiple of the root α.

This doesn't work in our setting: γ is not a rational character, so perhaps $\gamma(\alpha^\vee) \notin \mathbb{Z}$.

So we define the problem out of existence. . .

Definition. The integral Weyl group for $\gamma \in \mathfrak{h}^*_{\mathbb{C}}$ is

$$W(\gamma) = \{x \in W \mid x\gamma - \gamma = \text{integer combination of roots}\}$$

Jantzen-Zuckerman translation (A)

Recall that we had a formula for a distribution character

$$\Theta_\pi(h) = \Big(\sum_{w\in W}\sum_{d\phi = w\cdot\gamma} a_{w,\phi}\cdot\phi(h)\Big)/D(h)$$

and wished to define (for x in W)

$$x\cdot\Theta_\pi(h) \stackrel{?}{=} \Big(\sum_{w\in W}\sum_{d\phi' = wx^{-1}\cdot\gamma} a_\phi\cdot\phi'(h)\Big)/D(h).$$

For each character ϕ of differential $w\gamma$ in the character formula, and $x\in W(\gamma)$, we can define $\phi' = \phi + w(x\cdot\gamma - \gamma)$; here $x\cdot\gamma-\gamma$ is a sum of roots, so $w(x\cdot\gamma-\gamma)$ is as well, and therefore a well-defined (rational) character of H.

Theorem (Jantzen-Zuckerman): For $x\in W(\gamma)$, $x\cdot\Theta_\pi$ is an integer combination of chars of reps π' of infl char γ:

$$x\cdot\Theta_\pi = \sum_{\pi' \text{ infl char } \gamma} m_{\pi',\pi}(x)\Theta_{\pi'}.$$

Jantzen-Zuckerman translation (B)

$\gamma \in \mathfrak{h}^*$ dom regular, $W(\gamma) =$ integral Weyl group.

$\widehat{G}_\gamma =$ irr reps, infinitesimal char γ (finite set).

$\mathbb{Z}\widehat{G}_\gamma =$ virtual reps, infl char γ (finite rank lattice).

Recall Jantzen-Zuckerman action on virtual characters

$$x \cdot \Theta_\pi = \sum_{\pi' \text{ infl char } \gamma} m_{\pi',\pi}(x)\Theta_{\pi'}.$$

Theorem (Jantzen-Zuckerman): The integer matrices

$$M(x) = (m_{\pi',\pi}(x)) \qquad (x \in W(\gamma))$$

define a representation of $W(\gamma)$ over $\mathbb{Z}$, with basis $\widehat{G}_\gamma$.

Understanding this integer rep of $W(\gamma)$ is a large step toward character formulas for irreducible reps.

Cones and cells

$\pi, \pi' \in \widehat{G}_\gamma$; write $\pi' \le_L \pi$ if $\exists x \in W(\gamma)$ with Θ'_π in $x \cdot \Theta_\pi$.

Equivalent: $\exists F$ alg rep of G_{ad}, π' subquo of $\pi \otimes F$.

$\pi' \le_L \pi$ is directed graph structure on $\widehat{G}_\gamma$.

Left cone of π is $\overline{C}(\pi) = \{\pi' | \pi' \le_L \pi\}$; $W(\gamma) \circlearrowright \mathbb{Z}\overline{C}(\pi)$.

Left sub of π is $\overline{C}_0(\pi) = \{\pi' | \pi' \le_L \pi \not\le_L \pi'\}$; $W(\gamma) \circlearrowright \mathbb{Z}\overline{C}_0(\pi)$.

Get equiv relation $\pi \sim_L \pi'$ iff $\pi' \le_L \pi \le_L \pi'$. Equiv classes are called left cells: $C(\pi) = \overline{C}(\pi) - \overline{C}_0(\pi)$.

Left cell rep $\sigma(\pi)$ of $W(\gamma)$ is $\mathbb{Z}\overline{C}(\pi)/\mathbb{Z}\overline{C}_0(\pi)$.

Free $\mathbb{Z}$ module with basis $C(\pi)$.

Reason for the term left: Kazhdan and Lusztig introduced two relations $\le_L$, $\le_R$, on W, related to left and right multiplication in the Hecke algebra of W. This notion generalizes the KL definition of $\le_L$.

The Borho-Jantzen-Duflo τ invariant

$W(\gamma)$ = integral Weyl group $\supset S(\gamma)$ simple reflections.

Def: τ-invt of $\pi \in \widehat{G}_\gamma$ is $\tau(\pi) = \{s \in S(\gamma) \mid s \cdot \pi = -\pi\}$.

Write $\widehat{G}^s_\gamma = \{\pi \in \widehat{G}_\gamma \mid s \in \tau(\pi)\}$, $\quad \widehat{G}^{S_0}_\gamma = \cap_{s \in S_0} \widehat{G}^s_\gamma$.

We know everything about the action of simple reflections in the τ-invariant.

Next theorem tells something about the action of simple reflections not in the τ-invariant.

Theorem (BJDZ?) Suppose $\pi \in \widehat{G}_\gamma$, and $s \in S(\gamma)$. Then

$$s \cdot \pi = \begin{cases} -\pi & (s \in \tau(\pi)) \\ \pi + \sum_{\pi' \in \widehat{G}^s_\gamma} m_{\pi',\pi}(s)\pi' & (s \notin \tau(\pi)). \end{cases}$$

Order $\widehat{G}_\gamma$ by putting $\widehat{G}^s_\gamma$ *last*: $\sigma(s) = \begin{pmatrix} I & 0 \\ M & -I \end{pmatrix}$.

Where does that theorem come from?

At the center of geometric representation theory for $G(\mathbb{C})$ is the smooth projective algebraic variety

$$\mathcal{B} = \text{ Borel subgroups of } G(\mathbb{C}).$$

We have $\dim H^*(\mathcal{B}, \mathbb{C}) = \#W$: $\mathcal{B}$ is a geometrization of W.

W is generated by finite set S of simple reflections.

$s \in S \rightsquigarrow$ variety $\mathcal{P}_s$ of parabolic subgroups of type s.

Smooth fibration $p_s \colon \mathcal{B} \to \mathcal{P}_s$ makes $\mathcal{B}$ a $\mathbb{P}^1$ bundle.

These $\mathbb{P}^1$ bundles control the geometry of $\mathcal{B}$.

Geometric representation theory $\rightsquigarrow$ rep theory of G "fibers" over the rep theory of $SL(2)$ for each simple s.

Borho-Jantzen-Duflo-Zuckerman theorem does that.

τ invariants and cells

$$s \cdot \pi = \begin{cases} -\pi & (s \in \tau(\pi)) \\ \pi + \sum_{\pi' \in \widehat{G}^s_\gamma} m_{\pi',\pi}(s)\pi' & (s \notin \tau(\pi)) \end{cases}.$$

Corollary Suppose $\pi \in \widehat{G}_\gamma$.

1. If $\tau(\pi) = S_\gamma$ (all simple reflections) then
$$C(\pi) = \overline{C}(\pi) = \{\pi\},$$
and $\sigma(\pi) = \mathrm{sgn}_{W(\gamma)}$. (Say π is minimal.)
2. If $\tau(\pi) = \emptyset$, then
$$\overline{C}(\pi) = \{\pi\} \cup \{\pi' | \tau(\pi') \neq \emptyset\}, \qquad C(\pi) = \{\pi\}$$
and $\sigma(\pi) = \mathrm{trivial}_{W(\gamma)}$. (Say π is generic.)

Interesting/difficult case: $\emptyset \subsetneq \tau(\pi) \subsetneq S(\gamma)$.

Minimal includes the representations usually called minimal, like the Segal-Shale-Weil metaplectic representation.

Generic is (nearly) the automorphic form notion of generic.

Example: $U(2,1)$

Say $G = U(2,1)$, γ = half sum of positive roots.

$W(\gamma) = W = S_3, \quad S(\gamma) = \{s,t\} = \{(1,2),(2,3)\}.$

$\widehat{G}_\gamma$ consists of 6 representations:

1. A = generic disc ser, $\tau(A) = \emptyset$, F = triv, $\tau(F) = \{s,t\}$.
2. B = hol ds, $\tau(B) = \{s\}$, C = antihol ds, $\tau(C) = \{t\}$.
3. D = hol $A_{\mathfrak{q}}$, $\tau(D) = \{t\}$, E = antihol $A_{\mathfrak{q}}$, $\tau(E) = \{s\}$.

Action of W

π	$s \cdot \pi$	$t \cdot \pi$	left cell $C(\pi)$
A	$A + E$	$A + D$	$\{A\}$
B	$-B$	$B + D$	$\{B, D\}$
C	$C + E$	$-C$	$\{C, E\}$
D	$D + B + F$	$-D$	$\{B, D\}$
E	$-E$	$E + C + F$	$\{C, E\}$
F	$-F$	$-F$	$\{F\}$

Aside: the action of W on principal series reps is much simpler. This is how to prove that principal series reps are reducible.

Example: A_2

Suppose $s, t \in S(\gamma)$, $(st)^3 = 1$. Write

$$\widehat{G}_\gamma^{+-} = \{\pi \mid s \notin \tau(\pi), t \in \tau(\pi)\} = \{A_i\}$$
$$G_\gamma^{-+} = \{s \in \tau(\pi), t \notin \tau(\pi)\} = \{B_j\}$$

$$s \cdot A_i = A_i + \sum_j m_{ji} B_j + \sum_k y_{ki} Z_k, \qquad t \cdot B_j = B_j + \sum_j n_{ij} A_i + \sum_k x_{kj} Z_k.$$

Here the Zs have both s and t in τ. If we divide by the span of the Zs, we find matrix representations

$$\sigma(s) = \begin{pmatrix} I & 0 \\ M & -I \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} -I & N \\ 0 & I \end{pmatrix}$$

$$\sigma(sts) = \begin{pmatrix} -I + NM & -N \\ M(-2 + NM) & -MN + I \end{pmatrix},$$

$$\sigma(tst) = \begin{pmatrix} I - NM & N(-2 + MN) \\ -M & -I + MN \end{pmatrix}.$$

Integer linear algebra in type A_2

When $W(\gamma)$ is type A_2, we found formulas

$$\sigma(s) = \begin{pmatrix} I & 0 \\ M & -I \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} -I & N \\ 0 & I \end{pmatrix}$$

with M and N integer matrices with nonnegative entries.

Braid relation $sts = tst \leftrightsquigarrow MN = I$, $NM = I$; so $N = M^{-1}$.

Proposition. If M and N are nonnegative integer matrices with $NM = I$, $MN = I$, they are permutation matrices.

Corollary. Suppose s, t type A_2, $s \notin \tau(A)$, $t \in \tau(A)$. Then there is a unique B appearing in $s \cdot A$ with $t \notin \tau(B)$.

Cells in type A_2

Theorem. Suppose $W(\gamma)$ is of type A_2, with generators $\{s, t\}$. Cells in $\widehat{G}_\gamma$ are of three types:

1. Singletons $\{X\}$ with $\tau(X) = \emptyset$, $\sigma(X) = \text{trivial}$.
2. Singletons $\{Z\}$ with $\tau(Z) = \{s, t\}$, $\sigma(Z) = \text{sign rep}$.
3. Pairs $\{A, B\}$ with $\tau(A) = \{t\}$, $\tau(B) = \{s\}$, $\sigma(\text{cell}) = \text{reflection rep}$,

$$\sigma(s) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This description follows just from the integral structure given by the natural $\mathbb{Z}$-basis of irreducible G-representations; the Borho-Jantzen-Duflo dichotomy about the τ-invariant; and the braid relation $sts = tst$.

Exercise for the bored. This is a real representation of the finite group S_3. Why aren't $\sigma(s)$ and $\sigma(t)$ orthogonal?

Cells in type BC_2

Suppose $s, t \in S(\gamma)$, $(st)^4 = 1$. One can begin to analyze this case as in type A_2: one finds again

$$\sigma(s) = \begin{pmatrix} I & 0 \\ M & -I \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} -I & N \\ 0 & I \end{pmatrix}$$

but now with the braid relation $stst = tsts$.

Linear algebra over $/\mathbb{Z}$ as in A_2 suggests ten possible cells that are neither generic nor minimal. Two candidates are cells of three representations: for example $\{A, A', B\}$, with

$$\sigma(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad \sigma(t) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding W representation is the sum (over $\mathbb{Q}$) of the reflection representation (spanned by $A + A'$ and B), and a one-dimensional spanned by $A - A'$.

Another possibility is a cell $\{A, B\}$, with

$$\sigma(s) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

The point of this talk is to explain why such two-element cells cannot arise in representation theory.

Cells in type BC_2

Proposition. Suppose $\{A, B\} \subset \widehat{G}_\gamma$, $s \notin \tau(A), s \in \tau(B)$. Then the multiplicity of B in $s \cdot A$ is equal to

$$\dim \operatorname{Ext}^1(A, B) = \dim \operatorname{Ext}^1(B, A).$$

This ought to be elementary; but the only proof I know involves a complete reducibliity result coming from perverse sheaves (Beilinson/Bernstein/Deligne).

Identification $\operatorname{Ext}^1(A, B) \simeq \operatorname{Ext}^1(B, A)$ is elementary: existence of contravariant "duality" functor on G reps fixing irreducibles.

Corollary. The candidate cell $\{A, B\}$ in type BC_2 with

$$\sigma(s) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \sigma(t) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

cannot arise.

Proof. Applying the Proposition to (A, B, s) gives $\dim Ext^1(A, B) = 1$. Applying it to (B, A, t) gives $\dim Ext^1(A, B) = 2$.

Cells in type BC_2

Theorem. Suppose $W(\gamma)$ is of type BC_2, with generators $\{s, t\}$. Cells in $\widehat{G}_\gamma$ are of four types:

1. Singletons $\{X\}$ with $\tau(X) = \emptyset$, $\sigma(X) =$ trivial.
2. Singletons $\{Z\}$ with $\tau(Z) = \{s, t\}$, $\sigma(Z) =$ sign rep.
3. Triples $\{A, A'B\}$ with $\tau(A) = \tau(A') = \{t\}$, $\tau(B) = \{s\}$,
$$\sigma(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad \sigma(t) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$
4. Triples $\{A, B, B'\}$ with $\tau(A) = \{t\}$, $\tau(B) = \tau(B') = \{s\}$,
$$\sigma(s) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \sigma(t) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

I have explained why a cell containing the reflection rep of $W(\gamma)$ must also contain a one-dimensional rep

$$\mu(s) = 1,\ \mu(t) = -1 \quad \text{or} \quad \tau(s) = -1,\ \tau(t) = 1.$$

These arguments do not exclude (for example) candidate cells with a single representation M, $s \cdot M = M$, $t \cdot M = -M$.

But it's only a colloquium; I can omit something.

So what do cells look like?

Lusztig's description of representations of finite Chevalley groups used a partition of $\widehat{W}$ into families.

Each family $\mathcal{F}$ a has a unique special representation $\sigma_0(\mathcal{F})$, and some additional representations $\sigma_i'(\mathcal{F})$.

Lusztig proved: families $=$ the sets of W reps defined (with Kazhdan) by left-right cells in W.

Every cell rep of W is $\sigma_0(\mathcal{F}) + \sum_i m_i \sigma_i'(\mathcal{F})$.

Using deep results about Hecke algebras, Lusztig calculated his families completely in all cases.

Arguments above prove that the families for $W(BC_2)$ are

$$\{\text{trivial}\}, \quad \{\text{sgn}\}, \quad \{\text{reflection}, \mu, \tau\}.$$

Similar argument (using Ext^2 in addition to Ext^1) calculates families in $W(D_4)$.

Hope: characterize cell reps of W using integrality, positivity, symmetry properties like those above.