What's special about special?

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Linear Algebraic Groups: their Structure, Representations, and Geometry
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Outline

Introduction

Defining $AV(M)$

Structure of nilpotent orbits

Meaning of integral structure

Lusztig’s definition of special

Special nilpotents and integral representations

Section titles are just getting longer. Glad that was the last one
What this talk is about

\g complex reductive Lie algebra.

\( M \) irreducible (usually \( \infty \)-diml) \( U(\g) \)-module.

\( I(M) = \text{Ann}(M) \subset U(\g) \) two-sided (primitive) ideal.

Study \( I(M) \leadsto AV(I(M)) \), the simplest geom invt of \( I(M) \).

\( AV(I(M)) \subset \g^* \), \( G \)-invariant closed cone.

\( AV(I(M)) \) encodes interesting information about \( M \).

1950s algebra: \( G \) has finite \# nilp orbits on \( \g^* \).

1950s algebra: \( AV(I(M)) = \) finite union of nilp orbits.

FACT (Lusztig): \( M \) “integral” \( \implies \) \( AV(I(M)) \) special.

PLAN(1): sketch definitions, sketch Geck, Dong-Yang integral characterization of special.

PLAN(2): ask for proof of FACT using Geck, Dong-Yang characterization of special.
Associated varieties

$M U(g)$-module generated by fin. diml. $M_0 \subset M$.

$M_n =_{\text{def}} U_n(g) \cdot M_0, \quad M_0 \subset M_1 \subset M_2 \subset \cdots$

$\text{gr } M$ is a fin gen graded $S(g)$-module.

$AV(M) =_{\text{def}} \text{Supp } \text{gr}(M) \subset g^* = \text{Spec } S(g)$.

Big idea for controlling $AV(M)$:

$M$ irreducible $\leadsto \text{Ann}(M) \supset \text{max ideal } I_M \subset \text{Cent } U(g) \\
\leadsto AV(M)) \subset AV(\text{gr } I_M)$

$\text{gr } I_M =$ homogeneous polys of positive degree in $S(g)^G$.

Nilpotent cone (where $AV(M)$, $AV(\text{Ann}(M)$ must live!) is

$N^* = \{ \lambda \in g^* \mid p(\lambda) = 0 \ (p \in gS(g)^G \text{ homogeneous}) \}$.

$N^*/G \text{ finite } \implies AV(\text{Ann}(M)) = \text{finite union of } G \text{) orbits.}$
Structure of orbits: Jacobson-Morozov

\[ B = TN \] Borel subgroup. Rational coweights are

\[ X_\ast(T) = \text{Hom}_{\text{alg}}(\mathbb{C}^\times, T). \]

\( d \in X_\ast(T) \mapsto \) Lie algebra \( \mathbb{Z} \)-grading

\[ g = \sum_{n \in \mathbb{Z}} g_d(n), \quad t \subset g_d(0). \]

\( \mapsto \) parabolic \( P_d = L_d U_d, \quad L_d = G^d, \quad u = \sum_{n > 0} g_d(n). \)

Jacobson-Morozov: nilpotent orbits \( \leftrightarrow \) dominant cowts...

Nilpotent orbit \( \mathcal{O} \mapsto \) unique dominant \( d \in X_\ast^+(T) \) so

\( \mathcal{O} \) meets \( g_d^*(2) \) in open, \( d \in [g_d(2), g_d(-2)]. \)

Note: Levi \( L \) acts on \( g_d^*(2) \) with finitely many orbits.
Symplectic structure on orbits

Nilpotent $\mathcal{O} \rightsquigarrow$ dominant $d \in \text{Hom}_{\text{alg}}(\mathbb{C}^\times, T)$,

$\mathcal{O}$ meets $g_d^*(2)$ in open, $d \in [g_d(2), g_d(-2)]$.

$\lambda \in \mathcal{O} \cap g_d^*(2) \implies G^\lambda \subset P_d = L_d U_d$, and

$G^\lambda = L_d^\lambda \cdot U_d^\lambda$ \hspace{1cm} \text{(Levi decomposition)}

$T_{eG^\lambda}(G \cdot \lambda) = g_d(-1) + \sum_{m \geq 0} \left[ g_d(-m - 2) + g_d(m)/g_d(m)^\lambda \right]$.

$\mathcal{O}$ is a symplectic algebraic variety: nondegenerate form

$\omega_\lambda : g/g^\lambda \times g/g^\lambda \to \mathbb{C}$

$\omega_\lambda(X + g^\lambda, Y + g^\lambda) = \lambda([X, Y])$

$[g_d(-m - 2)]^* \cong_{\omega_\lambda} g_d(m)/g_d(m)^\lambda$ \hspace{1cm} $(m \geq 0)$

$\omega_\lambda$ nondegenerate on $g_d(-1)$.

Kirillov-Kostant: $\omega_\lambda$ relates $\mathcal{O} \rightsquigarrow$ representation theory.

Geck conjecture: $\mathcal{O}$ is special $\iff$ $\omega_\lambda$ integral.

Lusztig to be explained
Integral structures on $\mathfrak{g}$

Integral structure on $N$-diml Lie algebra $\mathfrak{g}$ over char 0 field $k$ is free rank $N$ lattice $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}$ subject to

$$\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_{\mathbb{Z}} k, \quad [\mathfrak{g}_\mathbb{Z}, \mathfrak{g}_\mathbb{Z}] \subset \mathfrak{g}_\mathbb{Z}.$$ 

Equivalent: basis $\{X_1, \ldots, X_N\}$ subject to

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \quad c_{ij}^k \in \mathbb{Z}.$$ 

Example: $\mathfrak{g} = sl(2)$, basis (this is the one we’ll generalize)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$ 

Example: $\mathfrak{g} = so(3)$, basis (but this is worth more study!)

$$U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

Chevalley integral structure

\[ g \supset b \supset t \text{ complex reductive Lie algebra} \]

roots \( \Delta(g, t) \subset t^* \), coroots \( \Delta^\vee(g, t) \subset t \).

Integral structure is called **split** if

1. Have integral basis \( = \text{basis } \{X_1, \ldots, X_\ell\} \) of \( t \), root vectors \( X_\alpha \) for each root; and
2. \( [X_\alpha, X_{-\alpha}] \) is equal to the coroot \( H_\alpha = \alpha^\vee \).

Chevalley: in a split integral structure, set of root vecs up to sign \( \{\pm X_\alpha\} \) is determined up to \( \text{Ad}(T) \), so should be thought of as unique.

Still in a split integral structure,

\[ \mathbb{Z} \Delta^\vee \subset t_\mathbb{Z} \subset \{t \in t \mid \alpha(t) \in \mathbb{Z} \quad (\alpha \in \Delta)\}; \]

and any such lattice \( t_\mathbb{Z} \) is allowed.

These \( t_\mathbb{Z} \) are the \( X_* (T) \leftrightarrow \) root data for alg \( G \), \( \text{Lie}(G) = g \).

If \( g \) semisimple, split integral structure (unique up to \( \text{Ad}(T) \)) with \( t_\mathbb{Z} = \mathbb{Z} \Delta^\vee \) is the **Chevalley integral structure**.
Integral linear functionals

\[ \text{split int str } g_Z \subset g \xrightarrow{\sim} g_Z^* =_{\text{def}} \text{Hom}_\mathbb{Z}(g_Z, \mathbb{Z}) \subset g^*. \]

\( \mathcal{O} \) is weakly integral if \( \mathcal{O} \cap g_Z^* \neq \emptyset \); includes all nilpotent \( \mathcal{O} \).

Fix \( \lambda \in \mathcal{O} \cap g_{d,Z}^*(2) \). Symplectic form \( \omega_\lambda \) defines

\[ \omega_{\lambda,Z} : g_Z/g_Z^\lambda \hookrightarrow [g_Z/g_Z^\lambda]^*. \]

Nondegen/\( \mathbb{C} \) \( \implies \) im(\( \omega_{\lambda,Z} \)) has finite index \( N_\lambda \).

Grading by \( d \) factors \( \omega_{\lambda,Z} \) as direct sum of maps

\[ \omega_{\lambda,Z}(m) : g_{d,Z}(m-1)/g_{d,Z}(m)^\lambda \hookrightarrow [g_{d,Z}(-m-1)]^* \quad (m \geq 1), \]

\[ \omega_{\lambda,Z}(0) : g_{d,Z}(-1) \hookrightarrow g_{d,Z}(-1)^*. \]

Each of these has finite index \( N_\lambda(m) \) in its image, and

\[ N_\lambda = N_\lambda(0) \cdot \prod_{m \geq 1} N_\lambda(m). \]

\( \lambda \) is strongly integral if \( N_\lambda = 1 \); i.e., \( \omega_{\lambda,Z} \) nondeg/\( \mathbb{Z} \).

\( \lambda \) is Geck integral if \( N_\lambda(0) = 1 \); i.e., \( \omega_{\lambda,Z}(0) \) nondeg/\( \mathbb{Z} \).
Springer (1978) defined inclusion $j$

$$j: \text{nilpotent orbits in } g^* \leftrightarrow \widehat{W}, \quad \mathcal{O} \leftrightarrow j(\mathcal{O}).$$

Springer (1978) also defined surjection $p$ ($p \circ j = id$)

$$p: \widehat{W} \rightarrow \text{nilpotent orbits in } g^*, \quad \sigma \mapsto p(\sigma).$$

KL theory partitions $\widehat{W}$ in families (two-sided cells).

**Theorem (Lusztig)**

1. Each family $\mathcal{F} \subset \widehat{W}$ has unique special rep $\sigma_s(\mathcal{F})$.
2. $\sigma_s(\mathcal{F})$ is $j(\mathcal{O}(\mathcal{F}))$, special nilpotent orbit.
Geck conjecture/Dong-Yang theorem

\[ G \supset B \supset T, \mathcal{O} \subset g^* \sim \text{dominant} \ d \in X_*(T): \]
\[ d \in [g_d(2), g_d(-2)], \quad \mathcal{O} \cap g_d^*(2) \text{ open in } g_d^*(2) \]
\[ \sim \omega_\lambda \text{ symplectic on } g/g^\lambda, \quad \omega_\lambda(0) \text{ on } g_d(-1) \subset g/g^\lambda. \]

Fix also split integral structure \( g_Z \subset g \sim g_Z^* \subset g^*. \)

May choose representative \( \lambda_Z \in \mathcal{O} \cap g^*_d(2). \)

Conj (Geck 2018) \( \mathcal{O} \text{ special iff } \exists \lambda_Z \text{ so } \omega_{\lambda_Z}(0) \text{ nondegenerate/}\mathbb{Z}. \)


Proof is case-by-case using enumeration of special nilps.

Recall that hypothesis Geck integral in Geck conjecture is weaker than natural hypothesis strongly integral.

Hope: Geck integral is equivalent to strongly integral.
Lusztig theorem on special nilpotent orbits

**Theorem (Lusztig)** If $L(\gamma) =$ simple highest weight module, highest weight $\gamma \in X^*(T)$, then

$$AV(\text{Ann}(L(\gamma))) = \text{closure of special nilpotent } \mathcal{O} \subset g^*.$$  

Proof is by KL theory, properties of families in $\hat{W}$.  

**Hope (point of talk):** there is a direct/conceptual path

If $\gamma \in X^*(T)$ then $AV(L(\gamma)) \supset$ dense set of strongly integral $\lambda$. 

Such a path would give a proof

$(\mathcal{O} \text{ special}) \implies (\mathcal{O} \text{ strongly integral}) \implies (\mathcal{O} \text{ Geck integral}).$

which is half of Geck's conjecture.
Thank you!