What's special about special?

David Vogan

Linear Algebraic Groups: their Structure,
Representations, and Geometry
March 19, 2022

Outline

What this talk is about

$\mathfrak{g}$ complex reductive Lie algebra.

M irreducible (usually ∞-diml) $U(\mathfrak{g})$-module.

$I(M) = \mathrm{Ann}(M) \subset U(g)$ two-sided (primitive) ideal.

Study $I(M)) \rightsquigarrow AV(I(M))$, the simplest geom invt of $I(M)$.

$AV(I(M)) \subset \mathfrak{g}^*$, G-invariant closed cone.

$AV(I(M))$ encodes interesting information about M.

1950s algebra: G has finite # nilp orbits on $\mathfrak{g}^*$.

1950s algebra: $AV(I(M)) =$ finite union of nilp orbits.

FACT (Lusztig): M "integral" $\implies AV(I(M))$ special.

PLAN(1): sketch definitions, sketch Geck, Dong-Yang integral characterization of special.

PLAN(2): ask for proof of FACT using Geck, Dong-Yang characterization of special.

Associated varieties

M $U(\mathfrak{g})$-module generated by fin. diml. $M_0 \subset M$.

$$M_n =_{def} U_n(\mathfrak{g}) \cdot M_0, \qquad M_0 \subset M_1 \subset M_2 \subset \cdots$$

$\mathrm{gr}\, M$ is a fin gen graded $S(\mathfrak{g})$-module.

$AV(M) =_{def} \mathrm{Supp}\, \mathrm{gr}(M) \subset \mathfrak{g}^* = \mathrm{Spec}\, S(\mathfrak{g})$.

Big idea for controlling $AV(M)$:

$$M \text{ irreducible} \rightsquigarrow \mathrm{Ann}(M) \supset \text{max ideal } I_M \subset \mathrm{Cent}\, U(\mathfrak{g})$$
$$\rightsquigarrow AV(M)) \subset AV(\mathrm{gr}\, I_M)$$

$\mathrm{gr}\, I_M =$ homogeneous polys of positive degree in $S(\mathfrak{g})^G$.

Nilpotent cone (where $AV(M)$, $AV(Ann(M)$ must live!) is

$$\mathcal{N}^* = \{\lambda \in \mathfrak{g}^* \mid p(\lambda) = 0 \ (p \in \mathfrak{g}S(\mathfrak{g})^G \text{ homogeneous})\}.$$

$\mathcal{N}^*/G$ finite $\Longrightarrow$ $AV(Ann(M)) =$ finite union of G) orbits.

Structure of orbits: Jacobson-Morozov

$B = TN$ Borel subgroup. Rational coweights are

$$X_*(T) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^\times, T).$$

$d \in X_*(T) \rightsquigarrow$ Lie algebra $\mathbb{Z}$-grading

$$\mathfrak{g} = \sum_{n \in \mathbb{Z}} \mathfrak{g}_d(n), \qquad \mathfrak{t} \subset \mathfrak{g}_d(0).$$

$$\rightsquigarrow \text{parabolic } P_d = L_d U_d, \quad L_d = G^d, \quad \mathfrak{u} = \sum_{n>0} \mathfrak{g}_d(n).$$

Jacobson-Morozov: nilpotent orbits $\hookrightarrow$ dominant cowts...

Nilpotent orbit $\mathcal{O} \rightsquigarrow$ unique dominant $d \in X_*^+(T)$ so

$$\mathcal{O} \text{ meets } \mathfrak{g}_d^*(2) \text{ in open}, \quad d \in [\mathfrak{g}_d(2), \mathfrak{g}_d(-2)].$$

Note: Levi L acts on $\mathfrak{g}_d^*(2)$ with finitely many orbits.

Symplectic structure on orbits

Nilpotent $\mathcal{O} \rightsquigarrow$ dominant $d \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^\times, T)$,

$\mathcal{O}$ meets $\mathfrak{g}_d^*(2)$ in open, $\quad d \in [\mathfrak{g}_d(2), \mathfrak{g}_d(-2)]$.

$\lambda \in \mathcal{O} \cap \mathfrak{g}_d^*(2) \implies G^\lambda \subset P_d = L_d U_d$, and

$$G^\lambda = L_d^\lambda \cdot U_d^\lambda \qquad \text{(Levi decomposition)}$$

$$T_{eG^\lambda}(G\cdot\lambda) = \mathfrak{g}_d(-1) + \textstyle\sum_{m\ge 0}\left[\mathfrak{g}_d(-m-2) + \mathfrak{g}_d(m)/\mathfrak{g}_d(m)^\lambda\right].$$

$\mathcal{O}$ is a symplectic algebraic variety: nondegenerate form

$$\omega_\lambda\colon \mathfrak{g}/\mathfrak{g}^\lambda \times \mathfrak{g}/\mathfrak{g}^\lambda \to \mathbb{C}$$

$$\omega_\lambda(X + \mathfrak{g}^\lambda, Y + \mathfrak{g}^\lambda) = \lambda([X, Y])$$

$$[\mathfrak{g}_d(-m-2)]^* \simeq_{\omega_\lambda} \mathfrak{g}_d(m)/\mathfrak{g}_d(m)^\lambda \quad (m \ge 0)$$

ω_λ nondegenerate on $\mathfrak{g}_d(-1)$.

Kirillov-Kostant: ω_λ relates $\mathcal{O} \rightsquigarrow$ representation theory.

Geck conjecture: $\mathcal{O}$ is $\underbrace{\text{special}}_{\text{Lusztig}} \iff \omega_\lambda \underbrace{\text{integral}}_{\text{to be explained}}$.

Integral structures on $\mathfrak{g}$

Integral structure on N-diml Lie algebra $\mathfrak{g}$ over char 0 field k is free rank N lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ subject to

$$\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k, \qquad [\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}] \subset \mathfrak{g}_{\mathbb{Z}}.$$

Equivalent: basis $\{X_1, \ldots, X_N\}$ subject to

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \qquad c_{ij}^k \in \mathbb{Z}.$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$, basis (this is the one we'll generalize)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Example: $\mathfrak{g} = \mathfrak{so}(3)$, basis (but this is worth more study!)

$$U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$[U, V] = W, \qquad [V, W] = U, \qquad [W, U] = V.$$

Chevalley integral structure

$\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ complex reductive Lie algebra

roots $\Delta(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^*$, coroots $\Delta^\vee(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}$.

Integral structure is called split if

1. Have integral basis = basis $\{X_1, \ldots, X_\ell\}$ of $\mathfrak{t}$, root vectors X_α for each root; and
2. $[X_\alpha, X_{-\alpha}]$ is equal to the coroot $H_\alpha = \alpha^\vee$.

Chevalley: in a split integral structure, set of root vecs up to sign $\{\pm X_\alpha\}$ is determined up to $\mathrm{Ad}(T)$, so should be thought of as unique.

Still in a split integral structure,

$$\mathbb{Z}\Delta^\vee \subset \mathfrak{t}_{\mathbb{Z}} \subset \{t \in \mathfrak{t} \mid \alpha(t) \in \mathbb{Z} \quad (\alpha \in \Delta)\};$$

and any such lattice $\mathfrak{t}_{\mathbb{Z}}$ is allowed.

These $\mathfrak{t}_{\mathbb{Z}}$ are the $X_*(T) \leftrightsquigarrow$ root data for alg G, $\mathrm{Lie}(G) = \mathfrak{g}$.

If $\mathfrak{g}$ semisimple, split integral structure (unique up to $\mathrm{Ad}(T)$) with $\mathfrak{t}_{\mathbb{Z}} = \mathbb{Z}\Delta^\vee$ is the Chevalley integral structure.

Integral linear functionals

split int str $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g} \rightsquigarrow \mathfrak{g}^*_{\mathbb{Z}} =_{\text{def}} \text{Hom}_{\mathbb{Z}}(\mathfrak{g}_{\mathbb{Z}}, \mathbb{Z}) \subset \mathfrak{g}^*$.

$\mathcal{O}$ is weakly integral if $\mathcal{O} \cap \mathfrak{g}^*_{\mathbb{Z}} \neq \emptyset$; includes all nilpotent $\mathcal{O}$.

Fix $\lambda \in \mathcal{O} \cap \mathfrak{g}^*_{d,\mathbb{Z}}(2)$. Symplectic form ω_λ defines

$$\omega_{\lambda,\mathbb{Z}} \colon \mathfrak{g}_{\mathbb{Z}}/\mathfrak{g}^\lambda_{\mathbb{Z}} \hookrightarrow \left[\mathfrak{g}_{\mathbb{Z}}/\mathfrak{g}^\lambda_{\mathbb{Z}}\right]^*.$$

Nondegen$/\mathbb{C} \implies$ $\text{im}(\omega_{\lambda,\mathbb{Z}})$ has finite index N_λ.

Grading by d factors $\omega_{\lambda,\mathbb{Z}}$ as direct sum of maps

$$\omega_{\lambda,\mathbb{Z}}(m) \colon \mathfrak{g}_{d,\mathbb{Z}}(m-1)/\mathfrak{g}_{d,\mathbb{Z}}(m)^\lambda \hookrightarrow [\mathfrak{g}_{d,\mathbb{Z}}(-m-1)]^* \qquad (m \geq 1),$$

$$\omega_{\lambda,\mathbb{Z}}(0) \colon \mathfrak{g}_{d,\mathbb{Z}}(-1) \hookrightarrow \mathfrak{g}_{d,\mathbb{Z}}(-1)^*.$$

Each of these has finite index $N_\lambda(m)$ in its image, and

$$N_\lambda = N_\lambda(0) \cdot \textstyle\prod_{m \geq 1} N_\lambda(m).$$

λ is strongly integral if $N_\lambda = 1$; *i.e.*, $\omega_{\lambda,\mathbb{Z}}$ nondeg$/\mathbb{Z}$.

λ is Geck integral if $N_\lambda(0) = 1$; *i.e.*, $\omega_{\lambda,\mathbb{Z}}(0)$ nondeg$/\mathbb{Z}$.

Lusztig's notion of special nilpotent orbits

Springer (1978) defined inclusion j

$$j\colon \text{nilpotent orbits in } \mathfrak{g}^* \hookrightarrow \widehat{W}, \qquad \mathcal{O} \mapsto j(\mathcal{O}).$$

Springer (1978) also defined surjection p ($p \circ j = id$)

$$p\colon \widehat{W} \twoheadrightarrow \text{nilpotent orbits in } \mathfrak{g}^*, \qquad \sigma \mapsto p(\sigma).$$

KL theory partitions $\widehat{W}$ in families (two-sided cells).

Theorem (Lusztig)

1. Each family $\mathcal{F} \subset \widehat{W}$ has unique special rep $\sigma_s(\mathcal{F})$.
2. $\sigma_s(\mathcal{F})$ is $j(\mathcal{O}(\mathcal{F}))$, special nilpotent orbit.

Geck conjecture/Dong-Yang theorem

$G \supset B \supset T$, $\mathcal{O} \subset \mathfrak{g}^* \rightsquigarrow$ dominant $d \in X_*(T)$:

$$d \in [\mathfrak{g}_d(2), \mathfrak{g}_d(-2)], \quad \mathcal{O} \cap \mathfrak{g}^*_d(2) \text{ open in } \mathfrak{g}^*_d(2)$$

$\rightsquigarrow \omega_\lambda$ symplectic on $\mathfrak{g}/\mathfrak{g}^\lambda$, $\omega_\lambda(0)$ on $\mathfrak{g}_d(-1) \subset \mathfrak{g}/\mathfrak{g}^\lambda$.

Fix also split integral structure $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g} \rightsquigarrow \mathfrak{g}^*_{\mathbb{Z}} \subset \mathfrak{g}^*$.

May choose representative $\lambda_{\mathbb{Z}} \in \mathcal{O} \cap \mathfrak{g}^*_{d,\mathbb{Z}}(2)$.

Conj (Geck 2018) $\mathcal{O}$ special iff $\exists \lambda_{\mathbb{Z}}$ so $\omega_{\lambda_{\mathbb{Z}}}(0)$ nondegenerate/$\mathbb{Z}$.
Proved by Geck (types *EFG*), Dong-Yang (2019) (types *ABCD*).
Proof is case-by-case using enumeration of special nilps.

Recall that hypothesis Geck integral in Geck conjecture is weaker than natural hypothesis strongly integral.

Hope: Geck integral is equivalent to strongly integral.

Lusztig theorem on special nilpotent orbits

Theorem (Lusztig) If $L(\gamma)$ = simple highest weight module, highest weight $\gamma \in X^*(T)$, then

$$AV(\mathrm{Ann}(L(\gamma))) = \text{closure of special nilpotent } \mathcal{O} \subset \mathfrak{g}^*.$$

Proof is by KL theory, properties of families in $\widehat{W}$.

Hope (point of talk): there is a direct/conceptual path

$$\text{If } \gamma \in X^*(T) \text{ then } AV(L(\gamma)) \supset \text{ dense set of strongly integral } \lambda.$$

Such a path would give a proof

$$(\mathcal{O} \text{ special}) \implies (\mathcal{O} \text{ strongly integral}) \implies (\mathcal{O} \text{ Geck integral}).$$

which is half of Geck's conjecture.

Thank you!