1. Representations of $SL(2,\mathbb{R})$

These notes describe the irreducible representations of the group $G = SL(2,\mathbb{R})$ of two by two real matrices of determinant one. The ideas go back mostly to Bargmann's paper [VB]. The definitions needed to discuss non-unitary representations are from Harish-Chandra [HC], and the details are copied from the account in [Green]. There is no homework in this class, but I've included some exercises that would be good for your soul (if you did them, or if you had one, whatever). The "1" in the title is wishful thinking, of course.

Thanks to Ben Harris (2007) for some corrections.

I began in class by considering a very reducible representation of G on the space $W = C^\infty(\mathbb{R}^2 - 0)$ of smooth functions on the punctured plane. Obviously G acts on the punctured plane by matrix multiplication, so we get a representation on functions by

$$\pi(g)f(x) = f(g^{-1}x) \qquad (g \in G, f \in W, x \in \mathbb{R}^2 - 0). \tag{1.1}$$

(The inverse is needed to make $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$; without it the factors would be reversed on the right side of this equality.) A long discussion in class led to a family of G-invariant closed subspaces of W, parametrized by a complex number ν and a parity $\epsilon \in \mathbb{Z}/2\mathbb{Z}$:

$$W_\nu^\epsilon = \{f \in W \mid f(tx) = t^\nu f(x), f(-x) = (-1)^\epsilon f(x)(t > 0, x \in \mathbb{R}^2 - 0)\}. \tag{1.2}$$

That is, W_ν^0 consists of the even functions homogeneous of degree ν, and W_ν^1 is the corresponding odd functions. The G-invariance of these subspaces is fairly clear. The representation π_ν^ϵ of G on W_ν^ϵ is called a *principal series representation* of G.

Exercise 1.3. Write S^1 for the unit circle in $\mathbb{R}^2 - 0$. The space $C^\infty(S^1)$ is the direct sum of the subspaces $C^\infty(S^1)^\epsilon$ of even and odd functions. Show that restriction to the circle defines a vector space isomorphism

$$W_\nu^\epsilon \simeq C^\infty(S^1)^\epsilon. \tag{1.4}$$

This identification is called the "compact picture" of the principal series. (The "noncompact picture" arises by restricting functions to something like the line $x_2 = 1$; it's more difficult to describe the image of the restriction map in that case.)

Since these notes are not constrained like fifty-minute classes, I can safely insert a couple of asides here. First, one can ask about parallel representations for $SL(2,F)$, where F is any topological field. The group $SL(2,F)$ acts on the left on the space F^2 of column vectors. The group therefore acts on any reasonable space W of complex-valued functions on $F^2 - 0$. (It makes sense to talk about representations over any base field k, in which case one should talk about k-valued functions on $F^2 - 0$; but I'll stick with complex representations.) The dilation action of $F^\times$ on functions commutes with the $SL(2,F)$ action. If ξ is any continuous homomorphism from $F^\times$ to $\mathbb{C}^\times$, then one can look at the space of homogeneous functions of degree ξ:

$$W_\xi = \{f \in W \mid f(tx) = \xi(t)f(x)(t \in F^\times, x \in F^2 - 0)\}. \tag{1.5}$$

The space W_ξ is a representation of $SL(2,F)$, called a *principal series representation.*

Exercise 1.6. Show that the parameters ν and ϵ above determine a continuous homomorphism of $\mathbb{R}^\times$ to $\mathbb{C}^\times$, and that all such homomorphisms arise in this way.

A second natural question is how this family of representations might be generalized to Lie groups other than $SL(2,\mathbb{R})$. Liberally interpreted, this question includes more or less all of the representation theory of reductive groups; so I'll try to narrow it a bit. Let's look first at the group $SL(n,\mathbb{R})$ of $n \times n$ real matrices of determinant 1. There is an obvious analogue of $\mathbb{R}^2 - 0$, namely $\mathbb{R}^n - 0$. One can construct a family of representations on homogeneous functions on this space, parametrized by a complex number and a sign. Unfortunately, these representations turn out to be rather special, and not of such general interest for $n \geq 3$. What's going on is that we ought to be looking at something closer to complete flags in our vector space. (Recall that a complete flag in an n-dimensional vector space is an increasing family of subspaces F_k, with

F_k of dimension k.) In a two-dimensional space, the only interesting element of a flag is the one-dimensional subspace. Here at any rate is a first approximation to the right n-dimensional version of $\mathbb{R}^2 - 0$:

$$X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \text{ is a non-zero vector in } \mathbb{R}^n/\langle x_1, x_2, \ldots, x_{i-1}\rangle\}. \tag{1.7}$$

That is, x_1 is a non-zero vector in $\mathbb{R}^n$; $\langle x_1\rangle$ is the line spanned by x_1, and x_2 is a non-zero vector in the $(n-1)$-dimensional space $\mathbb{R}^n/\langle x_1\rangle$; $\langle x_1, x_2\rangle$ is the two-dimensional preimage of the line through x_2 back in $\mathbb{R}^n$; and so on. I don't know a good name for the points of X_n, so I'll call them *fat flags*. Obviously every element of X_n defines a complete flag (with $F_k = \langle x_1, \ldots, x_k\rangle$); but a point of X_n has slightly more information. The first term x_1 is a column vector; x_2 is a column vector defined up to a multiple of x_1; x_3 is defined up to a linear combination of x_1 and x_2; and so on.

Now $SL(n,\mathbb{R})$ acts on X_n, by acting on representative column vectors for x_1, x_2, and so on. The difficulty is that this action is not quite transitive. Here is why. Given a point x of X_n, we can form an $n \times n$ matrix $A(x)$ whose columns are representatives of x_1, x_2, and so on. This matrix is not quite well-defined: the first column is defined, but the second is defined only up to adding a multiple of the first, and so on. Nevertheless the determinant $\det A(x)$ *is* well-defined; and the resulting function det on X_n is invariant under the action of $SL(n,\mathbb{R})$. We may therefore define

$$SX_n = \{x \in X_n \mid \det(x) = 1\}, \tag{1.8}$$

the space of *special fat flags*.

Exercise 1.9. Show that SX_n is a homogeneous space for $SL(n,\mathbb{R})$. Show that the isotropy group (the subgroup fixing a chosen base point) may be chosen to be the group of upper triangular matrices with 1s on the diagonal. Show that SX_2 may be identified with $\mathbb{R}^2 - 0$.

We can now define $W = C^\infty(SX_n)$, the space of smooth functions on special fat flags. Just as in the case of $SL(2)$, we'd like to identify G-invariant subspaces by homogeneity conditions. Consider the $(n-1)$-dimensional abelian Lie group

$$A = \{t = (t_1, \ldots, t_n) \mid t_i \in \mathbb{R},\ t_i > 0,\ t_1 \cdots t_n = 1\}. \tag{1.10}$$

Then A acts on SX_n on the right by scaling each vector x_i:

$$t \cdot x = (t_1 x_1, \ldots, t_n x_n) \qquad (t \in A, x \in SX_n).$$

This action commutes with the $SL(n,\mathbb{R})$ action. Write

$$\mathfrak{a}^* = \{\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n \mid \sum \nu_i = 0\}. \tag{1.11}$$

For $t \in A$ and $\nu \in \mathfrak{a}^*$, we can define

$$t^\nu = t_1^{\nu_1} \cdots t_n^{\nu_n} \in \mathbb{C}^\times.$$

For any $\nu \in \mathfrak{a}^*$, we define the space of homogeneous functions of degree ν,

$$W_\nu = \{f \in C^\infty(SX_n) \mid f(t \cdot x) = t^\nu f(x) (t \in A, x \in SX_n)\}. \tag{1.12}$$

Similarly, we define a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$:

$$M = \{m = (m_1, \ldots, m_n) \mid m_i \in \pm 1,\ m_1 \cdots m_n = 1\}. \tag{1.13}$$

Then M acts on SX_n on the right by scaling each vector x_i:

$$m \cdot x = (m_1 x_1, \ldots, m_n x_n) \qquad (m \in M, x \in SX_n).$$

This action commutes with the $SL(n,\mathbb{R})$ and A actions. Write

$$\widehat{M} = (\mathbb{Z}/2\mathbb{Z})^n/\{(0, \ldots, 0), (1, \ldots, 1)\} = \{\xi = (\xi_1, \ldots, \xi_n) \mid \xi_i \in \{0, 1\}\} / \sim;$$

here $\sim$ is the equivalence relation that identifies ξ and ξ' if their coordinates are either all equal or all different. For $m \in M$ and $\xi \in \widehat{M}$, we can define

$$m^\xi = m_1^{\xi_1} \cdots m_n^{\xi_n} \in \pm 1.$$

This depends only on the equivalence class of ξ. For any $\xi \in \widehat{M}$, we define the space of functions of parity ξ,

$$W^\xi = \{f \in C^\infty(SX_n) \mid f(m \cdot x) = m^\xi f(x)(m \in M, x \in SX_n)\}. \tag{1.14}$$

It is easy to check that the whole space W is a direct sum of these 2^{n-1} subspaces W^ξ, and that W_ν is the direct sum of the corresponding subspaces W_ν^ξ. The representation π_ν^ξ of $SL(n,\mathbb{R})$ on W_ν^ξ is called a *principal series representation* of $SL(n,\mathbb{R})$.

Exercise 1.15. Rewrite this discussion replacing the separate groups A and M by one group H isomorphic to a product of $n-1$ copies of $\mathbb{R}^\times$. In this form it makes sense with $\mathbb{R}$ replaced by any topological field. (The separate treatment of A and M has strong historical roots, which is why I've kept it.)

Exercise 1.16. Let $SO(n)$ be the compact group of $n \times n$ orthogonal matrices of determinant one. Show that $SO(n)$ embeds naturally in SX_n; and that restriction to $SO(n)$ defines a vector space isomorphism

$$W_\nu \simeq C^\infty(SO(n)).$$

What is the corresponding statement for the principal series representation W_ν^ξ? Can you find an analogous statement for principal series representations over a p-adic field F?

Exercise 1.17. (This one is easy.) Find a reasonable analogue of SX_n for the group $GL(n,\mathbb{R})$, and define principal series for this group.

Exercise 1.18. (This one is harder.) Let $Sp(2n,\mathbb{R})$ be the group of linear transformations of $\mathbb{R}^{2n}$ preserving your favorite standard symplectic form ω. Find a reasonable analogue of SX_n for this group, and define principal series representations. (Hint: the definition looks a lot like the one for $GL(n,\mathbb{R})$, but you should require that the subspace $\langle x_1, \ldots, x_n\rangle$ be isotropic for ω.)

Exercise 1.19. (This one is even harder.) Let $U(p,q)$ be the group of complex-linear transformations of $\mathbb{C}^{p+q}$ preserving the indefinite Hermitian form $|z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_{p+q}|^2$. Find a reasonable analogue of SX_n for this group, and define principal series. (On the basis of what I've said so far, it isn't clear what "reasonable" means for a compact group like $U(n)$. The answer I'm looking for in that case has the space equal to $U(n)$, with $U(n)$ acting on the left. The analogue of the group M is $U(n)$ acting on the right, and the analogue of A is trivial. A principal series representation is parametrized by an irreducible representation of $M = U(n)$. Now interpolate between this example and $SL(n,\mathbb{R})$.)

This is the end of the digression; we'll now return to talking about representations of $G = SL(2,\mathbb{R})$. Our goal is (more or less) to show that most representations of G look like principal series representations. In order to do that, we need to know what principal series representations look like in a convenient basis. I'll use the identification of the space W_ν^ϵ with (even or odd) functions on the circle given by Exercise 1.3. As a basis for functions on the circle I'll use the various

$$w_m(\cos\theta, \sin\theta) = \exp(im\theta); \tag{1.20}$$

here $m \equiv \epsilon \pmod 2$. These are obviously linearly independent functions on the circle; of course they are a basis only in a topological sense, which we will not yet make precise. The problem is to see how the group acts on this basis. Experience with finite-dimensional representation theory (see the description of representations of $\mathfrak{sl}(2)$ in [Hum], for example) suggests that it is much easier to describe explicitly the action of the Lie algebra than that of the group. The Lie algebra of $SL(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{R})$, the two by two real matrices of trace zero. This Lie algebra has dimension three; a standard basis is

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.21)(a}$$

These satisfy the commutation relations

$$[D,E]=2E, \qquad [D,F]=-2F, \qquad [E,F]=D. \tag{1.21(b)}$$

In class I showed how to compute the differential operators by which these basis elements act on $C^\infty(\mathbb{R}^2-0)$; they are

$$\pi(D)=-x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}, \qquad \pi(E)=-x_2\frac{\partial}{\partial x_1}, \qquad \pi(F)=-x_1\frac{\partial}{\partial x_2}. \tag{1.21(c)}$$

We want now to see how these Lie algebra elements act on smooth functions on the circle, and in particular on our basis elements $\exp(im\theta)$. The difficulty is that the vector fields in (1.21)(c) are not tangent to the circle. One way to compute is to change from the basis $\{\partial/\partial x_i\}$ to the basis $\{\partial/\partial\theta, \partial/\partial r\}$ for vector fields near the circle. On the space W_ν, $\partial/\partial r$ acts by multiplication by ν. The change of basis is easily computed to be

$$\frac{\partial}{\partial x_1}=-x_2\frac{\partial}{\partial\theta}+x_1\frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial x_2}=x_1\frac{\partial}{\partial\theta}+x_2\frac{\partial}{\partial r}. \tag{1.21(d)}$$

Putting these formulas and the identities

$$\frac{\partial}{\partial r}=\nu, \qquad x_1=\cos\theta, \qquad x_2=\sin\theta$$

into (1.21)(c) gives

$$\pi_\nu(D)=2\sin\theta\cos\theta\frac{\partial}{\partial\theta}+(-\cos^2\theta+\sin^2\theta)\nu, \tag{1.21(e)}$$

$$\pi_\nu(E)=\sin^2\theta\frac{\partial}{\partial\theta}+(-\cos\theta\sin\theta)\nu, \quad \pi_\nu(F)=-\cos^2\theta\frac{\partial}{\partial\theta}+(-\cos\theta\sin\theta)\nu. \tag{1.21(f)}$$

From the formulas in (1.21)(e–f) it is not too hard to see that each Lie algebra element carries a basis vector w_m to a linear combination of w_{m-2}, w_m, and w_{m+2}; but it is difficult to understand more. The problem is that this basis of the Lie algebra is adapted to weight vectors for the diagonal Cartan subalgebra, and the representation W_ν has no such weight vectors. What *does* act simply is the rotation matrix $E-F$, which acts by $\partial/\partial\theta$. Taking this as a starting point, we introduce a new basis of the complexified Lie algebra:

$$H=-i(E-F)=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \tag{1.22(a)}$$

$$X=\frac{1}{2}(D+iE+iF)=\begin{pmatrix}1/2 & i/2\\ i/2 & -1/2\end{pmatrix}, \qquad Y=\frac{1}{2}(D-iE-iF)=\begin{pmatrix}1/2 & -i/2\\ -i/2 & -1/2\end{pmatrix}. \tag{1.22(b)}$$

These elements satisfy the same commutation relations as D, E, and F:

$$[H,X]=2X, \qquad [H,Y]=-2Y, \qquad [X,Y]=H. \tag{1.22(c)}$$

Inserting the formulas (1.22)(a–b) in (1.21)(e–f) gives

$$\pi_\nu(H)=\frac{1}{i}\frac{\partial}{\partial\theta}, \quad \pi_\nu(X)=\frac{e^{2i\theta}}{2i}\frac{\partial}{\partial\theta}-\frac{e^{2i\theta}}{2}\nu, \quad \pi_\nu(Y)=\frac{-e^{-2i\theta}}{2i}\frac{\partial}{\partial\theta}-\frac{e^{-2i\theta}}{2}\nu. \tag{1.22(d)}$$

These operators act very simply on the functions $w_m=e^{im\theta}$:

$$\pi_\nu(H)w_m=mw_m, \qquad \pi_\nu(X)w_m=\frac{1}{2}(m-\nu)w_{m+2}, \qquad \pi_\nu(Y)w_m=\frac{1}{2}(-m-\nu)w_{m-2}. \tag{1.22(e)}$$

Exercise 1.23. Define V_ν^ϵ to be the algebraic span of the vectors w_m with $m\equiv\epsilon \pmod 2$ inside W_ν^ϵ, a vector space of countable dimension. Find all the subspaces of V_ν^ϵ that are preserved by the Lie algebra action (1.22)(e). (Here are some hints. First of all, show that unless ν is an integer with the property that $\nu \equiv \epsilon \pmod 2$, then V_ν^ϵ is irreducible. Next, suppose that ν is an integer congruent to ϵ. Show that the subspace spanned by $w_\nu, w_{\nu-2}, w_{\nu-4}, \ldots$ is invariant. In the same way, the subspace spanned by $w_{-\nu}, w_{-\nu+2}, w_{-\nu+4}, \ldots$ is invariant. Find all the possibilities for sums and intersections of these two subspaces, and show that in this way one gets all proper invariant subspaces of V_ν^ϵ.)

Having delayed for as long as possible, I must now turn to a discussion of general representations of $SL(2,\mathbb{R})$.

Definition 1.24. Suppose G is a topological group and V is a complex topological vector space. A *representation of G on V* is a homomorphism $\pi\colon G \to GL(V)$ with the property that the map

$$G \times V \to V, \qquad (g, v) \mapsto \pi(g)v$$

is continuous. The representation is said to be *irreducible* if V has exactly two closed G-invariant subspaces (which must then be 0 and V).

Suppose that G is a Lie group. A vector $v \in V$ is said to be *smooth* if the map $g \mapsto \pi(g)v$ from G to V is smooth. The collection of smooth vectors is a G-invariant subspace V^∞ of V (usually not closed in V). The action π^∞ of G on V^∞ differentiates to define a Lie algebra representation of $\mathfrak{g}_0 = \mathrm{Lie}(G)$ on V^∞, which we also denote π^∞.

In order to make much progress, we need to restrict attention to nice topological vector spaces. Harish-Chandra worked with Banach spaces. The disadvantage is that even if V is a Banach space, V^∞ is usually not. (This phenomenon appears with Sobolev spaces: one can make nice Hilbert spaces of functions having a finite number of L^2 derivatives, but defining smooth functions requires bounding an infinite collection of derivatives.) Recall that a *seminorm* on V is a function obeying all the axioms for a norm except that non-zero vectors may have norm zero. A topological vector space is *locally convex* if its topology is defined by a (possibly infinite) family of seminorms; that is, if every neighborhood of zero contains the intersection of a finite number of balls around zero defined by seminorms. In order to define vectors by nice limit processes, we also want to require that V be complete; roughly speaking, that sequences which are Cauchy in every seminorm must converge.

Proposition 1.25. *Suppose π is a representation of a Lie group G on a complete locally convex topological vector space V. Then V^∞ is a dense subspace of V, and carries a natural complete locally convex topology making π^∞ continuous.*

I'll omit the proof, but here are some hints. The main point is to introduce the "smooth group algebra" $M_c^\infty(G)$ of compactly supported smooth measures $d\mu$ on G. (A smooth measure on a manifold is one that looks like a smooth multiple of Lebesgue measure in local coordinates.) The algebra structure arises by convolution of measures. The first thing to prove is that π defines a representation of the algebra $M_c^\infty(G)$ by the formula

$$\pi(\mu)v = \int_G \pi(g) v \, d\mu(g).$$

The integral can be defined as a limit of Riemann sums taking values in V; the existence of the limit requires completeness of V. Using approximate identities in $M_c^\infty(G)$, it is easy to show that $\pi(M_c^\infty(G))V$ is dense in V. (This subspace is called the *Gårding subspace* of V.) It's also easy to check that

$$\pi(M_c^\infty(G))V \subset V^\infty,$$

because the differentiation can be moved from v to $d\mu$. This concludes the hints.

We have now constructed from our representation (π, V) of a Lie group G (with V complete locally convex) a smooth representation (π^∞, V^∞). Write

$$\mathfrak{g} = (\mathrm{Lie}(G)) \otimes_{\mathbb{R}} \mathbb{C}$$

for the complexified Lie algebra of G, and $U(\mathfrak{g})$ for its universal enveloping algebra. Smoothness of π^∞ means that there is an associative algebra homomorphism

$$\pi^\infty\colon U(\mathfrak{g}) \to \mathrm{End}(V^\infty),$$

so that V^∞ is a $U(\mathfrak{g})$-module. The adjoint action of G on its Lie algebra extends to an action on $U(\mathfrak{g})$ by complex algebra automorphisms. All of these things are related by a compatibility condition

$$\pi^\infty(\mathrm{Ad}(g)(u)) = \pi^\infty(g)\pi^\infty(u)\pi^\infty(g^{-1}). \tag{1.26)(a}$$

We are interested in this compatibility relation in a very special case. Define

$$\mathfrak{Z}(\mathfrak{g}) = \{z \in U(\mathfrak{g}) \mid \mathrm{Ad}(g)(z) = z, \text{all } g \in G\}. \tag{1.26)(b}$$

This is complex subalgebra of the enveloping algebra. If G is connected, it is precisely the center; in general it may be smaller. What we want from (1.26)(a) is

$$\text{for all } z \in \mathfrak{Z}(\mathfrak{g}),\ \pi^\infty(z) \text{ commutes with all operators } \pi^\infty(g) \text{ on } V^\infty. \tag{1.26)(c}$$

The idea is that (1.26)(c) forces the eigenspaces of $\pi^\infty(z)$ to be G-invariant closed subspaces of V^∞. If π^∞ is to be irreducible, this means that the only possible eigenspaces are 0 and V^∞. The latter possibility means that $\pi^\infty(z)$ is a scalar operator. We would like to use some kind of spectral theorem to rule out the former. For unitary representations, this is possible (see the theorem below). For general Banach space representations, it is not: Soergel has constructed in [WS] an example of an irreducible representation π of $SL(2,\mathbb{R})$ for which the Casimir operator does not act by scalars in π^∞. Here is the positive result.

Theorem 1.27 (Segal [IES]). *Suppose $(\pi, \mathcal{H})$ is an irreducible unitary representation of a Lie group G on a Hilbert space $\mathcal{H}$. Then for every $z \in \mathfrak{Z}(\mathfrak{g})$ (see (1.26)(b)) the operator $\pi^\infty(z)$ is a scalar.*

Motivated by this theorem, Harish-Chandra in [HC] made the following definition.

Definition 1.28. A representation (π, V) of a Lie group G on a complete locally convex vector space V is called *quasisimple* if $\pi^\infty(z)$ is a scalar for every $z \in \mathfrak{Z}(\mathfrak{g})$ (see (1.26)). The resulting algebra homomorphism

$$\chi_\pi \colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$$

is called the *infinitesimal character of* π.

Theorem 1.27 says that every irreducible unitary representation is quasisimple. We are going to study only quasisimple irreducible representations; an inspection of [WS] should convince you that the others might reasonably be regarded as pathological.

We return now to our assumption that $G = SL(2,\mathbb{R})$. As one might gather from our discussion of the principal series representations, we'll pay a lot of attention to the subgroup

$$K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}. \tag{1.29)(a}$$

The matrix appearing in the definition of K will be written $k(\theta)$. In terms of the Lie algebra element H of (1.22)(a), we have

$$k(\theta) = \exp(i\theta H). \tag{1.29)(b}$$

For the basis vectors w_m of W_ν, it follows from (1.29)(b) and (1.22)(e) that

$$\pi_\nu(k(\theta))w_m = e^{im\theta}w_m. \tag{1.29)(c}$$

Suppose now that (π, V) is a representation of G on a complete locally convex topological vector space. We know from Proposition 1.25 how to get a smooth representation π^∞ of G on the dense subspace V^∞. Our next goal is to find vectors in V^∞ like the basis vectors w_m of the principal series.

Definition 1.30. For any integer m, the *mth K-type of* V^∞ is the subspace

$$V_m^\infty = \{v \in V^\infty \mid \pi^\infty(k(\theta))v = e^{im\theta}v\ (\theta \in \mathbb{R})\}.$$

These subspaces are linearly independent. The *Harish-Chandra module of* V is their direct sum

$$V_K^\infty = \sum_{m \in \mathbb{Z}} V_m^\infty.$$

Proposition 1.31. *Suppose (π, V) is a representation of $SL(2,\mathbb{R})$ on a complete locally convex topological vector space; use notation as above.*

a) The Harish-Chandra module V_K^∞ is dense in V^∞. A vector $v \in V^\infty$ belongs to the Harish-Chandra module if and only if the subspace spanned by the vectors $\pi^\infty(k)v$ (as k varies over K) is finite-dimensional.

b) The subspace V_m^∞ may be characterized by

$$V_m^\infty = \{v \in V^\infty \mid \pi^\infty(H)v = mv\}.$$

c) The operator $\pi^\infty(X)$ (cf. (1.22)(b)) carries V_m^∞ into V_{m+2}^∞, and $\pi^\infty(Y)$ carries V_m^∞ into V_{m-2}^∞. In particular, V_K^∞ is an invariant subspace for the Lie algebra representation π^∞.

d) Suppose W is a closed K-invariant subspace of V. Then W is equal to the closure of

$$W_K^\infty = \sum_{m \in \mathbb{Z}} V_m^\infty \cap W.$$

As usual I will offer only some hints towards the proof. For every integer m there is a complex-valued smooth measure

$$\mu_m = \frac{1}{2\pi} e^{-im\theta} d\theta$$

on the circle. It's not difficult to check that $\pi^\infty(\mu_m)$ is a projection operator from V^∞ onto V_m^∞. (That is, its image is contained in V_m^∞, and it acts by the identity there.) On the other hand, the theory of Fourier series allows us to construct approximate identities on K (smooth measures of total mass 1 which are small away from the unit element) that are finite linear combinations of the μ_m. It follows easily that V_K^∞ is dense in V^∞. The rest of the proposition is fairly easy.

This proposition is a step in the direction of an algebraic description of the representations of G. The candidate for the algebraic model of a representation V is the Harish-Chandra module V_K^∞. Part (d) shows that a closed invariant subspace W of V gives rise to a "sub-Harish-Chandra module" W_K^∞, from which W can be recovered. What is missing is a converse: a statement that the closure of a sub-Harish-Chandra module is a G-invariant subspace of V. This statement is not true in general, but it is for quasisimple representations. In order to formulate the result cleanly, it is helpful to have an abstract definition of Harish-Chandra module.

Definition 1.32. A *Harish-Chandra module for $SL(2,\mathbb{R})$* is a complex vector space W endowed with two additional structures: a representation of the complexified Lie algebra $\mathfrak{g}$, and a representation of the compact group K. It is traditional to use module notation for both of these structures, and so to write $k \cdot w$ or $Z \cdot w$ for $w \in W$, $k \in K$, and $Z \in \mathfrak{g}$. Sometimes to maintain compatibility with representation theory roots, we will write something like π for the representations, and so $\pi(k)w$ or $\pi(Z)w$ to mean exactly the same thing.

We impose three conditions on these representations. First, *the action of K should be locally finite.* This means that every vector $w \in W$ belongs to a finite-dimensional subspace $F(w)$ that is preserved by K, and on which K acts continuously. This first condition guarantees that the action of K is smooth (and even analytic), since continuous homomorphisms of Lie groups (in this case K and $GL(F(w))$) are analytic. The second requirement is *the differential of the K action is equal to the action of $\mathfrak{k}_0 \subset \mathfrak{g}$.* The third requirement relates the actions of K and $\mathfrak{g}$ and the adjoint action of K on $\mathfrak{g}$:

$$k \cdot (Z \cdot (k^{-1} \cdot w)) = [\mathrm{Ad}(k)(Z)] \cdot w.$$

As explained in the following exercise, this third condition is actually a consequence of the second, because K is connected. We include it to emphasize the analogy with a more general definition of Harish-Chandra module that will appear later.

Exercise 1.33. Show that the third condition in the definition of Harish-Chandra module is automatically satisfied. Show also that a Harish-Chandra module is the same thing as a complex vector space W endowed with two things: a $\mathbb{Z}$ grading

$$W = \sum_{m\in\mathbb{Z}} W_m,$$

and linear transformations X and Y satisfying

$$X(W_m) \subset W_{m+2}, \qquad Y(W_m) \subset W_{m-2}, \qquad (XY - YX)|_{W_m} = \text{ multiplication by } m.$$

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