The size of infinite-dimensional representations I

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Outline

Something to do during the talk

k_v local field, $G_v = G(k_v)$ reductive, $\mathfrak{g}_v = \mathrm{Lie}(G_v)$.

$\mathfrak{g}_v^* =$ lin fnls on $\mathfrak{g}_v$, $\mathcal{O}_v = G_v \cdot x_v$ coadjt orbit.

$N(\mathcal{O}_v) =_{\mathrm{def}} \overline{k_v \cdot \mathcal{O}_v} \cap \mathcal{N}_v^*$ asymp nilp cone of $\mathcal{O}_v$.

k global, $\pi = \otimes_v \pi_v$ automorphic rep of G reductive.

Conjecture (global coherence of WF sets)

1. $\exists$ coadjt orbit $G(k) \cdot x \subset \mathfrak{g}(k)^*$, $N(G_v \cdot x) = \mathrm{WF}(\pi_v)$.
2. $\exists$ global version of local char expansions for π_v.

Says $G(k) \cdot x \rightsquigarrow$ asymp of K-types at each place.

$\mathcal{O}_{\overline{k}} =_{\mathrm{def}} G(\overline{k}) \cdot x \rightsquigarrow N(\mathcal{O}_{\overline{k}}) = \overline{\overline{k} \cdot \mathcal{O}_{\overline{k}}} \cap \mathcal{N}_{\overline{k}}^*$

$N(\mathcal{O}_{\overline{k}}) =$ closure of one nilp orbit $\mathcal{M}$.

$N(G_v \cdot x) \subset N(\mathcal{O}_{\overline{k}})$, but possibly $N(G_v \cdot x) \cap \mathcal{M} = \emptyset$.

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic of lectures: what's an irreducible unitary representation look like?

Gelfand-Kirillov dimension

G topological group, $\pi\colon G \to \mathcal{H}_\pi$ irreducible unitary.

Study $\underbrace{\text{what's } \pi \text{ look like?}}_{\text{interesting, hard}}$ via $\underbrace{\text{how large is } \pi\text{?}}_{\text{duller, possible}}$

Goal: $\pi \rightsquigarrow \mathrm{Dim}(\pi) =$ Gelfand-Kirillov dimension.

Desiderata:

1. $\mathrm{Dim}(\pi)$ integer, $0 \le \mathrm{Dim}(\pi) \le (\dim G)/2$;
2. π finite-diml $\iff \mathrm{Dim}(\pi) = 0$;
3. $\pi \simeq$ secs of bundle on $X \implies \mathrm{Dim}(\pi) = \dim(X)$.

So far vague about what G is. But (1) makes sense only if G is Lie group, or algebraic over local field k.

Good news: (3) makes sense if $X =$ mfld, alg var$/k$.

Bad news: (3) is not possible.

Why Dim π can't be dim X.

Most important representation in the world is oscillator representation ω.

Defined on Hilbert space $\mathcal{H}^1 = L^2(\mathbb{R})$.

Three groups of unitary operators on $L^2(\mathbb{R})$:

translation	$(T^1_t f)(x) = f(x - t)$	$X^1 = d/dx$
multiplication	$(M^1_\xi f)(x) = e^{-2\pi i \xi x} f(x)$	$Y^1 = 2\pi i x$
phase shift	$(P^1_\theta f)(x) = e^{-2\pi i \theta} f(x)$	$Z^1 = 2\pi i$

These generate three-dimensional Heisenberg group H; elements X, Y, Z span Lie algebra.

ω lives on secs of bdle on homog space $X = \mathbb{R}$.

Desideratum says $\mathrm{Dim}\,\omega \stackrel{?}{=} \dim \mathbb{R} = 1$.

Notice $\mathcal{H}^{1,\infty} = \mathcal{S}(\mathbb{R})$, Schwartz space of $\mathbb{R}$.

So far so good. But $\exists$ other realizations of ω...

Another realization of ω

On two-diml torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ define line bundle $\mathcal{L}$:

Sections of $\mathcal{L}$ defined to be

$$\{F\colon \mathbb{R}^2 \to \mathbb{C} \mid F(x+p, y+q) = e^{-2\pi iqx}F(x,y)\}.$$

Three gps of unitary ops on $\mathcal{H}^2 = L^2(\mathbb{T}^2, \mathcal{L})$:

x trans	$(T^2_t F)(x,y) = e^{-2\pi ity}F(x-t,y)$	$X^2 = \partial/\partial x + 2\pi iy$
y trans	$(M^2_\xi F)(x,y) = F(x, y+\xi)$	$Y^2 = \partial/\partial y$
phase	$(P^2_\theta F)(x,y) = e^{-2\pi i\theta}F(x,y)$	$Z^2 = 2\pi i$

There's isomorphism $\mathcal{S}(\mathbb{R}) \to C^\infty(\mathbb{T}^2, \mathcal{L})$,

$$f \mapsto F, \qquad F(x,y) = \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi i(x+n)y}.$$

Extends to Hilb space isom $L^2(\mathbb{R}) \to L^2(\mathbb{T}^2, \mathcal{L})$

Second realization suggests $\mathrm{Dim}\,\omega \overset{?}{=} \dim \mathbb{T}^2 = 2$.

One representation of $GL(V)$

Dim(secs over X) $\stackrel{?}{=}$ dim X OK for G reductive...

Begin with $G = GL(V(k))$ invertible linear transformations of n-diml vector space $V(k)$.

Stay vague about (locally compact) ground field k.

Ex. G acts on $(n-1)$-diml (over k) proj variety

$$X_{1,n-1}(k) = \{\text{1-diml subspaces of } V(k)\}$$

$\rightsquigarrow$ G acts by irr rep $\rho_{1,n-1}$ on Hilbert space

$$\mathcal{H}_{1,n-1}(k) = \{L^2 \text{ half-densities on } X_{1,n-1}(k)\}$$

$$\text{Dim}\, \rho_{1,n-1} = \dim X_{1,n-1} = n-1.$$

More general reps of $GL(V)$

Continue with n-diml $V/$ loc cpt k, $G = GL(V)$
$p = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; G acts on

$$X_p = \{0 = S_0 \subset S_1 \subset \cdots S_m = V(k), \\ \text{subspace chains}, \ \dim(S_j/S_{j-1}) = p_j\}$$

G acts on proj variety X_p/k,

$$\dim X_p = (n^2 - \sum p_i^2)/2.$$

$\rightsquigarrow$ rep $\rho_p(\mathcal{E})$ on secs of bdle $\mathcal{E} \to X_p$ has

$$\mathrm{Dim}(\rho_p(\mathcal{E})) = (n^2 - \sum p_i^2)/2.$$

So big repns $\leftrightsquigarrow$ partitions p with small parts.
To define $\mathrm{Dim}\,\pi$ for general π, need repn-theoretic

$$\rho_p(\mathcal{E}) \overset{?}{\rightsquigarrow} (n^2 - \sum p_i^2)/2.$$

Lessons from real analysis

X compact d-diml Riemannian, Δ_X Laplacian

$$\mathcal{H}^X = L^2(X), \qquad \mathcal{H}^X_\lambda = \lambda\text{-eigenspace of } \Delta_X.$$

Theorem (Weyl)

If $\mathcal{H}^X(N) = \sum_{\lambda \le N^2} \mathcal{H}_\lambda$, *then* $\dim \mathcal{H}^X(N) \sim c_X N^d$.

Same conclusion for secs of vector bdle $\mathcal{E} \to X$.

Conclude: $\dim X \leftrightsquigarrow$ asymp distn of Δ_X eigenvalues

Example: $X = \mathbb{RP}^{n-1}$, $C^\infty(X)$ = homog even fns on $\mathbb{R}^n$.

$$\dim \mathcal{H}^X_{2k(2k+(n-1))} = \frac{[(2k+1)(2k+2)\cdots(2k+n-3)][4k+n-2]}{(n-2)!},$$

polynomial in k of degree $n-2$.

$$\mathcal{H}^X\left(2k\sqrt{1+\frac{n-1}{2k}}\right) \simeq S^{2k}(\mathbb{R}^n)$$

$$\dim \mathcal{H}^X\left(2k\sqrt{1+\frac{n-1}{2k}}\right) = \binom{n+2k-1}{n-1},$$

polynomial in k of degree $n-1$.

Rep-theoretic desc of eigenvalue asymptotics $\rightsquigarrow$ general def of $\mathrm{Dim}(\pi)$.

Eigenvalue asymptotics in representations

G Lie group, $\langle \cdot, \cdot \rangle$ neg def on $\mathfrak{g}$, A_i onb of $\mathfrak{g}$,

$$\Delta_G = \sum A_i^2 \in U(\mathfrak{g}).$$

G acts on $\mathcal{E} \to X$ bdle on cpt homog $X \rightsquigarrow$ action of Δ_G on $C^\infty(X, \mathcal{E})$ satisfies Weyl asymptotics.

Conclusion: can hope to define $\operatorname{Dim} \pi$ using eigval asymptotics of Δ_G on $\mathcal{H}_\pi^\infty$.

Ex. If G is the Heisenberg group, can choose

$$\begin{aligned} \Delta_G &= -X^2 - Y^2 - Z^2 \quad \text{in } U(\mathfrak{g}) \\ &\rightsquigarrow -d^2/dx^2 + 4\pi^2 x^2 + 4\pi^2 \quad \text{in } L^2(\mathbb{R}) \\ &\rightsquigarrow -\partial^2/\partial x^2 - \partial^2/\partial y^2 - 4\pi i y \partial/\partial x + 4\pi^2 \quad \text{in } L^2(\mathbb{T}^2, \mathcal{L}). \end{aligned}$$

Eigvals in ω are $4\pi(k + 1 + \pi)$ for nonneg int k. Number to N^2 is $(N^2/4\pi) - \pi$.

Eigvals suggest (true) that ω lives on two-diml compact X, and (false) that $\operatorname{Dim}(\omega) = 2$.

Asymptotics to infinity and beyond

Lie gp G, neg def inner prod on $\mathfrak{g} \rightsquigarrow \Delta_G = \sum A_i^2 \in U(\mathfrak{g})$.

Problem: Δ_G has disc spec in $\mathcal{H}_\pi^\infty$, any irr unitary π?

If true, eigval asymptotics $\overset{?}{\rightsquigarrow}$ Dim π.

Ex: $G = GL(V)$, V n-diml real vector space.

$\mathfrak{g}$ has G-invt symm bilinear

$$B(X,Y) =_{\text{def}} \text{tr}(XY):$$

pos def on $\mathfrak{s} =_{\text{def}}$ symm matrices,

neg def on $\mathfrak{k} =_{\text{def}}$ skew symm matrices.

Define $\theta(g) = {}^t g^{-1} \quad (g \in G)$, $\theta X = -{}^t X \quad (X \in \mathfrak{g})$.

$$\langle X, Y\rangle =_{\text{def}} \text{tr}(X\theta(Y)) \qquad \text{negative definite on } \mathfrak{g}.$$

Thm. $\Delta_{GL(V)}$ has discrete spectrum on any $\mathcal{H}_\pi^\infty$;
eigvals $\leq N^2 \sim N^d$, nonneg int $d =_{\text{def}} \text{Dim}(\pi)$.

General representations over $\mathbb{R}$

$(\pi, \mathcal{H}_\pi)$ arbitrary irr rep of $G(\mathbb{R}) \simeq GL(n, \mathbb{R})$.
Restriction to cpt subgp $O(n)$ decomposes

$$\mathcal{H}_\pi \simeq \sum_{\mu \in \widehat{O(n)}} m_\pi(\mu)\mu \qquad (m_\pi(\mu) \text{ non-neg integer}).$$

$\mathcal{H}_\pi = L^2(X_p)$ ($p = (p_1, \ldots, p_r)$, $\sum p_i = n$) suggests defining

$$\mathcal{H}_\pi(N) =_{\text{def}} \sum_{\mu(\Omega) \leq N^2} m_\rho(\mu)\mu.$$

Theorem

There is partition $p(\pi)$ of n, pos const $a(\pi)$ so that

$$\dim \mathcal{H}(N) \sim a(\pi) N^{p(\pi}.$$

Recall that $\dim \mathcal{H}_\pi(N) \sim a(\pi) N^{d(p)}$.

Definition

For π irr rep of $G(\mathbb{R})$, the Gelfand-Kirillov dimension of π is the non-neg integer $\mathrm{Dim}(\pi) = d(p(\pi))$; measures asymp distn of eigenvalues of Casimir $\Omega_{O(n)}$ in π.

(First) moral of the real story

$G(\mathbb{R}) = GL(V(\mathbb{R}))$ has compact subgroup $O(n)$.

irr rep of $G(\mathbb{R}) \rightsquigarrow$ partition $\pi(\rho)$ of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep on $\mathcal{H} \approx$ functions on $X_\pi(\mathbb{R})$, cpt homog space for $G(\mathbb{R})$ and for $O(n)$. Precisely:

asymp distn of eigenvalues of Casimir $\Omega_{O(n)}$ in $\rho \rightsquigarrow$ eigenvals of Laplacian on $X_\pi(\mathbb{R})$.

Problems: what partition is attached to each irr rep?

what else does partition tell you about irr rep?

Other real reductive groups

$G(\mathbb{R})$ real reductive group, $K(\mathbb{R})$ maximal compact subgroup, $\Omega_{K(\mathbb{R})}$ Casimir operator for $K(\mathbb{R})$.

Example: $Sp(2n, \mathbb{R})$, $\mathbb{R}$-linear transf of $\mathbb{C}^n$ preserving symplectic form

$$\omega(v, w) = \mathrm{Im}\langle v, w\rangle$$

(imag part of std Herm form); $K(\mathbb{R}) = U(n)$.

Example: $O(p, q)$ linear transf of $\mathbb{R} \times \mathbb{R}^q$ preserving symmetric form

$$\langle (v_1, v_2), (w_1, w_2)\rangle_{p,q} = \langle v_1, w_1\rangle - \langle v_2, w_2\rangle;$$

$K(\mathbb{R}) = O(p) \times O(q)$.

(Al)most general example: $G(\mathbb{R}) \subset GL(N, \mathbb{R})$ closed subgp preserved by transpose, $K(\mathbb{R}) = G(\mathbb{R}) \cap O(N)$.

Big idea:

$G(\mathbb{R})$ rep "size" $\leftrightsquigarrow$ restriction to $K(\mathbb{R})$ asymptotics

GK dimension for other real reductive

$G(\mathbb{R})$ real reductive group, $K(\mathbb{R})$ maximal compact subgroup, $\Omega_{K(\mathbb{R})}$ Casimir operator for $K(\mathbb{R})$.

$(\rho, \mathcal{H})$ irr rep of $G(\mathbb{R})$; then (Harish-Chandra)

$$\mathcal{H} \simeq \sum_{\mu \in \widehat{K(\mathbb{R})}} m_\rho(\mu)\mu, \qquad (m_\rho(\mu) \text{ non-neg integer}).$$

As for $GL(n)$, can define

$$\mathcal{H}(N) =_{\text{def}} \sum_{\mu(\Omega_{K(\mathbb{R})}) \leq N^2} m_\rho(\mu)\mu.$$

Theorem

There is a non-negative integer $d(\rho)$ and a positive constant $b(\rho)$ so that

$$\dim \mathcal{H}(N) \sim b(\rho) N^{d(\rho)}.$$

Call $d(\rho)$ the Gelfand-Kirillov dimension of ρ.