Unitary representations of (real) reductive groups

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The unitary dual of local reductive groups

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Outline

Introduction and outline

Langlands classification

Real infinitesimal character

Unitarity at one piece of Langlands

The mysterious FPP

Slides at `http://www-math.mit.edu/~dav/paper.html`

Most interesting problem in mathematics…

…is understanding the unitary dual of G.

Meaning: all ways G can be auts of a Hilbert space.

Interesting since math is full of interesting Hilbert spaces.

Gelfand (1930s): should solve this for G loc cpt.

Mackey (1950s): enough to consider G simple.

(many parents) (1890–1970): best is G reductive.

My work is about unitary dual of real reductive G.

Still no satisfactory complete description of this set.

What we understand is **the shape of the answer**.

That's the subject for today.

Technicalities about the setting

Always work with connected reductive complex algebraic $G(\mathbb{C})$, defined over $\mathbb{R}$, and $G = G(\mathbb{R})$.

Not difficult to get parallel results for finite covers of $G(\mathbb{R})$...

...but KL theory, algorithms for unitarity not available.

p-adic groups have some analogous behavior, but the differences and difficulties are more serious.

$G(\mathbb{C})$ has algebraic involution θ: Cartan involution

$G(\mathbb{R})^\theta = K(\mathbb{R})$ is a max cpt subgroup of $G(\mathbb{R})$, having complexification the complex reductive alg group

$$K(\mathbb{C}) = G(\mathbb{C})^\theta.$$

Analytic rep theory of $K(\mathbb{R})$ = alg rep theory of $K(\mathbb{C})$.

This is key to relating analytic rep theory of $G(\mathbb{R})$ $\leftrightsquigarrow$ alg geom of $K(\mathbb{C})$ action on $G(\mathbb{C})$.

I'll flash through this slide really fast

Definition. A Harish-Chandra module for $G(\mathbb{R})$ is a complex vector space V carrying

1. a finite length rep of $\mathrm{Lie}(G(\mathbb{R}))$, and
2. a locally finite representation of $K(\mathbb{R})$,

subject to some easy compatibility. EQUIV:

1. a finite length rep of $\mathrm{Lie}(G(\mathbb{C}))$, and
2. an algebraic representation of $K(\mathbb{C})$.

An invariant Hermitian form on V is a hermitian pairing $\langle , \rangle_V$ on V such that

$$\langle X \cdot w_1, w_2 \rangle_V + \langle w_1, X \cdot w_2 \rangle_V = 0 \qquad (X \in \mathrm{Lie}(G(\mathbb{R})),$$
$$\langle k \cdot w_1, k \cdot w_2 \rangle_V = \langle w_1, w_2 \rangle_V \qquad (k \in K(\mathbb{R}))$$

Theorem (HC). There is a natural bijection

irr HC modules with pos def invt Herm form $\longleftrightarrow$ irr unitary reps of $G(\mathbb{R})$

How to find all irr unitary reps of $G(\mathbb{R})$

Step one: find all irr HC modules.

($\approx$ answer): ctble union of cplx vec spaces $\mathfrak{a}^*_d$
Each def/$\mathbb{Q}$, so $\mathfrak{a}^*_d = \mathfrak{a}^*_d(\mathbb{R}) + i\mathfrak{a}^*_d(\mathbb{R})$.
Langlands (1973), using HC's deepest work.

Step two: find which of these are Hermitian.

($\approx$ answer) finite number of real forms

$$\mathfrak{a}^*_{d,e}(\mathbb{R}) = \mathfrak{a}^*_{d,e,-}(\mathbb{R}) + i\mathfrak{a}^*_{d,e,+}(\mathbb{R}),$$

including $i\mathfrak{a}^*_d(\mathbb{R})$.
Knapp and Zuckerman (1977): easy if you're smart enough to *see* that it's easy.

Step three: find which Herm ones are unitary.

($\approx$ answer) let $M_{d,e}(\mathbb{R})A_{d,e,+}(\mathbb{R}) = \mathrm{cent}_{G(\mathbb{R})}(\mathfrak{a}_{d,e,+})$,
Levi factor of a real parabolic subgroup of $G(\mathbb{R})$.

1. Then Hermitian rep of $G(\mathbb{R})$

$$V^G_{d,e}(\nu_-, i\nu_+) = \mathrm{Ind}^{G(\mathbb{R})}_{P_{d,e}}(V^{M_{d,e}(\mathbb{R})}(\nu_-) \otimes e^{i\nu_+})$$

is unitarily ind; unitary iff $V^{M_{d,e}(\mathbb{R})}(\nu_-)$ unitary.

2. In Knapp's *Overview* text (1986), Thm 16.10.

Could you repeat that?

Step one: find all irr HC modules.

for a θ-stable representative $H(\mathbb{R}) = T(\mathbb{R})A$ of each conjugacy class of Cartan subgroups of $G(R)$, form real Levi subgroup $L(\mathbb{R}) = MA = \text{cent}_{G(\mathbb{R})}(A)$. $T(\mathbb{C}) = H(\mathbb{C})^\theta$, $A(\mathbb{C}) = H(\mathbb{C})^{-\theta}$

Ctble index set is pairs $(A, \text{disc ser rep } \delta \in \widehat{M})$; $\approx$ (HC) by $\widehat{T(\mathbb{R})}$. Cplx vec space is $\mathfrak{a}(\mathbb{C})^*$.

For $\nu \in \mathfrak{a}(\mathbb{C})^*$, HC module(s) are Langlands subquots of $I(P, \delta, \nu) = \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\delta \otimes e^\nu)$.

Langlands subquots characterized by **EITHER**

having largest possible growth of matrix coeffs (char by $\nu_{\text{Re}} \in \mathfrak{a}(\mathbb{R})^*$), **OR EQUIVALENTLY**
containing a smallest possible rep of $K(\mathbb{R})$ (char by HC param $\lambda \in i\mathfrak{t}(\mathbb{R})^*$).

Langlands: each irr HC module is Langlands subquot of some $I(P, \delta, \nu)$, unique mod “obvious” equiv.

And what came next?

Step two/three: find all irr Herm/unitary HC mods.

So we have cuspidal Levi MA, $\delta \in \widehat{M}$ discrete series,

$$\nu = \nu_{\mathrm{Re}} + i\nu_{\mathrm{Im}} \in \mathfrak{a}(\mathbb{R})^* + i\mathfrak{a}(\mathbb{R})^* = \mathfrak{a}^*$$

Set $L_{\mathrm{Im}} = M_{\mathrm{Im}} A_{\mathrm{Im}} = \mathrm{cent}_{G(\mathbb{R})}(\nu_{\mathrm{Im}})$, Levi of real parabolic containing MA.

Unitary induction from L_{Im} to $G(\mathbb{R})$ is bijection from irr consts of $I^{L_{\mathrm{Im}}(\mathbb{R})}(\delta, \nu)$ to irr consts of $I^{G(\mathbb{R})}(\delta, \nu)$, taking Hermitian onto Hermitian and unitary onto unitary.

In particular, unitary Langlands subquots of $I^{G(\mathbb{R})}(\delta, \nu)$ $\leftrightarrow$ unitary Langlands subquots of $I^{L_{\mathrm{Im}}(\mathbb{R})}(\delta, \nu_{\mathrm{Re}})$

Reduces construction/classification of unitary reps to case of real infinitesimal character: $\nu = \nu_{\mathrm{Re}} \in \mathfrak{a}(\mathbb{R})^*$

What's "infinitesimal character"?

Write $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{g}(\mathbb{C}))$ = **center** of univ env alg $U(\mathfrak{g}(\mathbb{C}))$.

HC Theorem: If $\mathfrak{h}(\mathbb{C})$ any Cartan subalg of $\mathfrak{g}(\mathbb{C})$, then

$$\mathfrak{Z}(\mathfrak{g}(\mathbb{C})) \simeq S(\mathfrak{h}(\mathbb{C}))^W,$$

with W the Weyl group of $\mathfrak{h}(\mathbb{C})$ in $\mathfrak{g}(\mathbb{C})$. Any alg hom from $\mathfrak{Z}$ to $\mathbb{C}$ is given by evaluation at some $\gamma \in \mathfrak{h}(\mathbb{C})^*/W$.

$\mathfrak{Z}$ acts on any irr HC module V by such an alg hom $\chi_\gamma \colon \mathfrak{Z} \to \mathbb{C}$, called **infl char of V**.

$H(\mathbb{R}) = T(\mathbb{R})A$ is a real CSG, and

1. V = **Langlands subquot of** $I(\delta, \nu)$, with
2. δ disc series of HC param $\lambda \in i\mathfrak{t}^*$ and $\nu \in \mathfrak{a}(\mathbb{C})^*$.

Then V has infl char $\lambda + \nu \in i\mathfrak{t}^* + \mathfrak{a}(\mathbb{C})^*$.

Langlands param can be **constructed** from disc series rep δ and infl char γ, using Cartan inv θ on H.

Construction is $\nu = (1 - \theta) * \gamma/2$; succeeds exactly when $\gamma \in$ affine space $\lambda = (1 + \theta) * \gamma/2$.

What's "real infl character"?

Alg group $G(\mathbb{C}) \rightsquigarrow$ character lattice $X^* \subset \mathfrak{h}(\mathbb{C})^*$.

Lattice makes $\mathfrak{h}^*$ defined over $\mathbb{Z}$, and in fact over any any subfield (like $\mathbb{R}$) of $\mathbb{C}$. Define canonical real form

$$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} X^* \subset \mathbb{C} \otimes_{\mathbb{Z}} X^* \simeq \mathfrak{h}^*(\mathbb{C}).$$

If $H(\mathbb{R}) = T(\mathbb{R})A$ is def over $\mathbb{R}$, then this is **not** $\mathfrak{h}(\mathbb{R})^*$:

$$\mathfrak{h}(\mathbb{R})^* = \mathfrak{t}(\mathbb{R})^* + \mathfrak{a}(\mathbb{R})^*,$$
$$\mathfrak{h}^*_{\mathbb{R}} = i\mathfrak{t}(\mathbb{R})^* + \mathfrak{a}(\mathbb{R})^*.$$

$V =$ **subquot of** $I(\delta, \nu)$ **has real infl char iff** $\nu \in \mathfrak{a}(\mathbb{R})^*$;

that is, iff inducing discrete series δ is twisted by **real-valued** character ν of A.

These reps are as **far from unit induced as possible**.

Knapp: these are the reps **whose unitarity we need**.

All the infinitesimal characters: W

Start with $G(\mathbb{C})$ complex semisimple $\supset H(\mathbb{C})$ Cartan, $\mathfrak{h}^*_{\mathbb{R}}$ canonical real form.

$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}$-span of roots $R(G, H)$, $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}$-span of coroots $R^\vee(G, H)$, $W = W(G, H)$ Weyl group acts on $\mathfrak{h}_{\mathbb{R}}$.

Fix also pos roots and coroots $R^+(G, H)$, etc.

What everybody knows about W:

1. coroot hyperplanes $E_{\alpha^\vee} = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) = 0\}$ partition their complement into Weyl chambers.
2. pos Weyl cham $C = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) \geq 0 \ (\alpha^\vee \in R^{\vee,+})\}$ is a fund domain for W.
3. W acts simply transitively on the chambers in $\mathfrak{h}^*_{\mathbb{R}}$.
4. C = disjt union of 2^{rank} fundamental faces indexed by subsets of simple coroots $= 0$.
5. $\mathfrak{h}^*_{\mathbb{R}}$ = disjoint union of faces = W-orbits of fund faces.

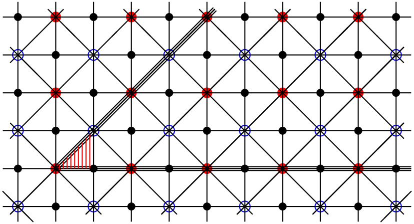
All the infinitesimal characters: W_{aff}

Induced reps can be reducible only if infl char take int value on some coroot. Hence interested in the affine coroot hyperplanes

$$E_{\alpha^\vee, m} = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) = m\}$$

Here is a picture for $SO(5)$, coroots $\{(\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1)\}$.

Pos Weyl cham in **triple lines**, fund alcove with red lines.



Each open triangle is a face, called an alcove. An alcove has three kinds of 1-diml faces as edges, and three kinds of 0-diml faces as vertices.

Relevant group is $W_{\mathrm{aff}} = W \rtimes ((\text{root lattice}))$, generated by reflections in affine coroot hyperplanes.

Fundamental alcove $A = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \le \gamma(\alpha^\vee) \le 1 \ (\alpha^\vee \in R^{\vee,+}\}$, product of one r-diml simplex for each rank r simple factor of G.

$\mathfrak{h}^*_{\mathbb{R}} = \coprod(\text{aff faces}) = W_{\mathrm{aff}}$-orbits of fund alcove faces.

Back to local Langlands...

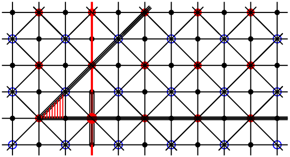
θ-stable CSG $H(\mathbb{R}) = T(\mathbb{R})A$, real Levi subgroup $MA = \text{cent}_{G(\mathbb{R})}(A)$, disc ser $\delta \in \widehat{M}$, HC param $\lambda \in i\mathfrak{t}(\mathbb{R})^*$.

Langlands params of real infl char are (δ, ν), with $\nu \in \mathfrak{a}(\mathbb{R})^*$.

infl char $= \gamma = (\lambda, \nu) \in i\mathfrak{t}(\mathbb{R})^* + \mathfrak{a}(\mathbb{R})^*$.
$\nu = (1-\theta)\cdot\gamma/2$; allowed γ are those satisfying $(1+\theta)\cdot\gamma/2$.
Set $\tau(\gamma) = -\theta\cdot\gamma + 2\cdot\lambda$, inv aut of $\mathfrak{h}^*_{\mathbb{R}}$ perm aff hyps $E_{\alpha^\vee,m}$.

Allowed γ are fixed points of $\tau = \lambda + (-1 \text{ eig of } \theta)$.
Get decomp of allowed γ from W_{aff} decomp of all γ: face is intersection with fixed pts of τ on a τ-fixed face of $\mathfrak{h}^*_{\mathbb{R}}$.

Example: $G(\mathbb{R}) = SO(3,2)$, $MA = GL(2,R)$,
$\theta(a,b) = (-b,-a)$, $\lambda = (1/2,-1/2)$,
$\tau(a,b) = (1+b,-1+a)$, allowed $\gamma = \{(a+1/2, a-1/2)\}$.



Red line = allowed γ, red circ $= \lambda$. 0-diml faces are the alt blue and red vertices on the line, and 1-diml faces are open intervals.

Unitary points are the vertical closed interval in black.

What shape is the (real) unitary dual?

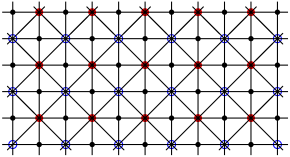
Start with $H(\mathbb{R} = T(\mathbb{R})A$ and $\delta \in \widehat{M}$ (discrete series).

$(H(\mathbb{R}), \delta) \approx$ rep $\mu \in \widehat{K}(\mathbb{R})$; reps indexed by $(\delta, \nu) \approx$ reps of lowest $K(\mathbb{R})$-type μ.

Reps also indexed by $\gamma = (\lambda, \nu) \in \mathfrak{h}(\mathbb{C})^*$, and γ also indexes infl char.

Set of real γ is an affine subspace of $\mathfrak{h}^*_{\mathbb{R}}$; inherits from $\mathfrak{h}^*_{\mathbb{R}}$ a decomp into (products of) simplices.

So **Langlands params for reps of real infl char, fixed LKT** is affine with ratl structure, tiling similar to this:



Ancient results of Birgit Speh and friends: **on each face, every rep is unitary or no rep is unitary.**

Real unitary = **cpt polyhedron** made from these faces.

Houston, we have a problem.

Gave finite description of unitary reps of LKT $\mu \in \widehat{K(\mathbb{R})}$.

How to **compute** it: left for Jeff.

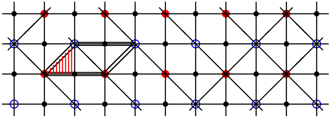
But there are **infinitely many μ**.

Solution analogous to Knapp solution for non-real infl char: **non-real is parab induced from real**.

Work of **Barbasch** identified key compact set of real infl chars: **FPP = fundamental parallelpiped =**

$$\{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) \in [0,1] \quad (\alpha^\vee \text{ simple})\}$$

Here's the $SO(5)$ picture, with FPP in triple lines:



FPP $= \#W/\#Z^\sim$ alcoves ($Z^\sim = \text{cent}(G^{\text{der,sc}})$).

Theorem(Davis, Mason-Brown) If real infl chars at LKT μ do not meet *FPP*, then all unitary reps of LKT μ are good range coh ind from unitary on proper θ-stable Levi.

What's the happy ending?

Given semisimple $G(\mathbb{R})$, list all irr reps $\mu \in \widehat{K(\mathbb{R})}$ so that there exist irr reps with real infl char in the FPP and LKT μ.

For each such μ, FPP_μ = real infl chars in FPP for LKT μ = cpt cvx polytope, (fixed points of affine involution) $\cap$ FPP.

FPP_μ inherits simplicial decomp from $\mathfrak{h}^*_{\mathbb{R}}$.

Unitary set is compact poly in FPP_μ, built from simplices.

Here are some $G(\mathbb{R})$, corr number of FPP-K-types.

Group	# LKTs for FPP	Group	# LKTs for FPP
$Sp(2,\mathbb{R})$	5	$F4_s$	497
$Sp(4,\mathbb{R})$	25	$E6_s$	466
$Sp(6,\mathbb{R})$	128	$E6_q$	7832
$Sp(8,\mathbb{R})$	697	$E7_s$	68466
$Sp(10,\mathbb{R})$	3825	$E7_q$	34690
$E8_q$	189606		

Reasonable step $\rightsquigarrow$ unitary dual: characterize these μ.