Unitary representations of reductive Lie groups

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Outline

Gelfand's abstract harmonic analysis

Quantum mechanics and Hilbert spaces

Classical mechanics and symplectic geometry

Problem: Unitary repns of simple Lie groups

Classical repn theory: Coadjoint orbits

Problem: Quantization for coadjoint orbits

Three kinds of coadjoint orbit and two kinds of quantization

Towards a third kind of quantization

Abstract Harmonic Analysis

Setting: G (topological) group acting on X (topological) space; class of "interesting questions" about X

Step 1. Attach to X a (topological) vector space V, so questions about X become questions about V.

Step 2. Find (finest possible) G-equivariant decomposition $V = \sum_i V_i$. Hope each V_i is an *irreducible* representation of G.

Step 3. Understand each irreducible representation of G. (Topic for today!)

Step 4. Assemble information to answer questions about X.

First example

$G = S_n$, symmetric group on n letters;

$X =$ p-element subsets of $\{1, \ldots, n\}$.

Question: what is the cardinality of X?

Step 1. $V =$ functions on X; card $X = \dim V$.

Step 2. First decompose

$$V = \sum_{\tau \text{ partition of } n} m_\tau V_\tau.$$

Say $p \le n - p$. Then $m_\tau = 0$ unless $\tau = (r, s)$ with $s \le p$ (when $m_\tau = 1$). Have

$$\dim V = \sum_\tau m_\tau \dim V_\tau$$

Step 3. Calculate

$$\dim V_\tau = \text{ \# std Young tableaux of shape } \tau$$

$$\dim V_{(r,s)} = \binom{n}{s} \left(\frac{r - s + 1}{r + 1} \right)$$

Step 4. Conclude (for $p \le n - p$)

$$\text{card}(X) = \sum_{s=0}^{p} \binom{n}{s} \left(\frac{n - 2s + 1}{n - s + 1} \right) = \binom{n}{p}$$

Second example

$$G = \text{ real reductive Lie} \supset \Gamma \text{ discrete cocompact}$$

$$X = G/\Gamma, \quad Z = \text{ loc. symm. space } K\backslash G/\Gamma$$

Problem: understand de Rham cohomology $H^p(Z)$.

Step 1. $V = C^\infty(X)$; $H^p(Z) \simeq H^p(\mathfrak{g}, K; V)$ (relative Lie algebra cohomology).

Step 2. First decompose

$$V = \sum_{\pi \in \widehat{G}} m_\pi(\Gamma) V_\pi.$$

Sum is over irr. unitary reps. π of G. Have

$$H^p(Z) = \sum_\pi m_\pi(\Gamma) \dim H^p(\mathfrak{g}, K; V_\pi)$$

Step 3. Classify irr. unitary reps. π with $H^p(\mathfrak{g}, K; V_\pi)$ not zero. Main ex. is $V_\pi = \mathbb{C}$,

$$H^p(\mathfrak{g}, K; \mathbb{C}) \simeq H^p(U), \qquad U = \text{ cpt. dual of } G/K.$$

Other $\pi \longleftrightarrow$ smaller cpt. symm. space U_π,

$$H^p(\mathfrak{g}, K; V_\pi) \simeq H^{p+d_\pi}(U_\pi), \quad d_\pi = \frac{1}{2}(\dim U - \dim U_\pi).$$

Step 4. Conclude $H^p(K\backslash G/\Gamma) \simeq H^p(U)$, $\quad (p \le \min_{\pi \ne 1} d_\pi)$.

Quantum mechanics

Physical system $\longleftrightarrow$ complex Hilbert space $\mathcal{H}$

States $\longleftrightarrow$ lines in $\mathcal{H}$

Observables $\longleftrightarrow$ operators $\{A_j\}$ on $\mathcal{H}$

Expectation of observable on state $v \longleftrightarrow \langle Av, v\rangle$

Energy $\longleftrightarrow$ skew-adjoint operator A_0

Time evolution $\longleftrightarrow$ unitary gp $\exp(tA_0)$

Evolution of $A \longleftrightarrow \exp(-tA_0)A\exp(tA_0)$

Observable A conserved $\longleftrightarrow [A_0, A] = 0$

Classical mechanics

Symplectic manifold is manifold M with Lie algebra structure $\{,\}$ on $C^\infty(M)$ satisfying

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

and a nondegeneracy condition. Function f defines Hamiltonian vector field $\xi_f = \{f, \cdot\}$.

Physical system $\longleftrightarrow$ symplectic manifold M

States (pos + velocities) $\longleftrightarrow$ points in M

Observables $\longleftrightarrow$ smooth fns $\{a_j\} \subset C^\infty(M)$

Value of observable a on state m $\longleftrightarrow$ $a(m)$

Energy $\longleftrightarrow$ real-valued function a_0

Time evolution $\longleftrightarrow$ flow of vector field ξ_{a_0}

Evolution of a $\longleftrightarrow$ $da/dt = \{a_0, a\}$

Observable a conserved $\longleftrightarrow$ $\{a_0, a\} = 0$

Unitary reps of Lie group G

A *unitary representation* of G is a Hilbert space $\mathcal{H}_\pi$ endowed with a continuous action

$$G \times \mathcal{H}_\pi \to \mathcal{H}_\pi, \qquad (g,v) \mapsto \pi(g)v$$

resp. inner product: $\langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle$.

An *invariant subspace* is closed subspace $\mathcal{V}$ preserved by action; then $\mathcal{V}^\perp$ is invariant, and $\mathcal{H}_\pi = \mathcal{V} \oplus \mathcal{V}^\perp$. Say π is *irreducible* if it has exactly two invt subspaces.

Unitary dual problem: Determine the set $\widehat{G}_u$ of irr unitary reps of G.

Want to emphasize the analogy

unitary reps $\longleftrightarrow$ quantum mech systems

Put $\mathfrak{g}_0 = \mathrm{Lie}(G)$, Lie alg of G. From $X \in \mathfrak{g}_0$ get skew-adjt op $d\pi(X)$: $\pi(tX) = \exp(td\pi(X))$.

skew-adjoint ops $\{d\pi(X)\} \longleftrightarrow$ observables

“Classical” representations

after Kirillov and Kostant

A *Hamiltonian G-space* is a symplectic manifold M with smooth action

$$G \times M \to M, \qquad (g, m) \mapsto g \cdot m$$

resp. symplectic structure. Each $X \in \mathfrak{g}_0$ defines vector fld $\xi(X)$ on M. Require also G-eqvt Lie algebra homomorphism

$$\mathfrak{g}_0 \to C^\infty(M), \qquad X \mapsto f(X)$$

satisfying $\xi(X) = \xi_{f(X)} = \{f(X), \cdot\}$.

Classical analogue of “irreducible” is that M is homogeneous. Analogy here is

Ham. G-spaces $\longleftrightarrow$ classical mech systems

functions $f(X)$ $\longleftrightarrow$ classical observables

Classification of "classical" reps

Define $\mathfrak{g}_0^* =$ dual space of Lie algebra. Elements of $\mathfrak{g}_0$ are (linear) functions on $\mathfrak{g}_0^*$; Lie bracket on these elements extends to Poisson bracket $\{,\}$ on $C^\infty(\mathfrak{g}_0^*)$.

Fix $M \subset \mathfrak{g}_0^*$ orbit of G (a *coadjoint orbit*). Poisson bracket on $\mathfrak{g}_0^*$ restricts to (G-invt) symplectic structure on M. Elements $X \in \mathfrak{g}_0$ restrict to functions $f(X)$ on M; makes $M = G/H$ a homogeneous Hamiltonian G-space. Structure is inherited by any equivariant covering space $\widetilde{M} = G/H_1$, with $H \supset H_1 \supset H_0$.

Theorem (Kirillov-Kostant) *Each homogeneous Hamiltonian G-space is a G-equivariant cover of a coadjoint orbit.*

Philosophy of coadjoint orbits

Diagram of analogies and wishful thinking:

$$\begin{array}{ccc} \text{irr unitary reps} & \longleftrightarrow & \text{quant mech} \\ \uparrow & & \uparrow \\ \text{coadjt orbit covers} & \longleftrightarrow & \text{classical mech} \end{array}$$

Right vert arrow is “quantization;” should exist under some integrality constraint on classical system (since the world exists, and is quantum-mechanical).

Left vert arrow is the wishful thinking part: should exist by analogy with right.

Quantization problem: construct unitary reps from (suitably integral) coadjt orbits.

This won’t produce all irreducible unitary reps, but it does better than any other known approach. (Second prize to Arthur’s conjectures, based on Langlands’ philosophy and automorphic repn theory.)

Three kinds of coadjoint orbit...

Nice reductive Lie group G has nice map

$$G \to GL(n, \mathbb{R})$$

with discrete kernel. Leads to nice inclusion

$$\mathfrak{g}_0^* \subset n \times n \text{ real matrices.}$$

Element $\lambda \in \mathfrak{g}_0^*$ is called...

- **hyperbolic** if corr matrix is diagonalizable
- **elliptic** if diag/$\mathbb{C}$, purely imag eigenvalues
- **nilpotent** if corr matrix is nilpotent

General coadjt orbits built in a simple way from these three types.

...and two kinds of quantization

Write $M = G \cdot \lambda \simeq G/G^\lambda$, coadjt orbit.

Theorem *If λ hyperbolic, there is a G-eqvt fibration $M \to Z$ with Lagrangian fibers and Z compact.*

Can attach unitary rep to M, on space of L^2 sections of a line bundle on Z.

Theorem *If λ elliptic, there is G-eqvt cplx structure on M making M (indefinite) Kähler.*

Under integrality constraint, can attach unitary rep to M, on Dolbeault cohomology of M with coeffs in holomorphic line bundle.

Theorem *If λ nilpotent, then M is a cone (closed under positive dilations in $\mathfrak{g}_0^*$).*

Don't know how to attach unitary rep to a nilpotent coadjoint orbit.

Quantizing nilpotent orbits

Define $\mathcal{N}^*_{\mathbb{R}}$ = cone of nilpotent elements in $\mathfrak{g}_0^*$. Finitely union of orbits of G (still a reductive group). Want to attach unitary reps to these orbits. Two guiding principles. . .

First, G has maximal compact subgroup K; understand unitary representations of K very well. Arise from "classical" picture using invt complex structure. (Example: Borel-Weil theorem.) Such reps extend naturally to complexification $K_{\mathbb{C}}$ of K.

Idea: replace $\mathcal{N}^*_{\mathbb{R}}$ with a complex algebraic variety $\mathcal{N}^*_\theta$ carrying algebraic $K_{\mathbb{C}}$ action, so that actions of K on $\mathcal{N}^*_{\mathbb{R}}$ and $\mathcal{N}^*_\theta$ are (more or less) equiv.

Construction: $\mathfrak{g}^*$ = cplx dual of $\mathfrak{g}_0$,

$$\mathcal{N}^*_\theta = \{\lambda \in \mathfrak{g}^* \mid \lambda\,|_{\mathfrak{k}} = 0 \text{ and } \lambda \text{ nilpotent}\}.$$

Kostant-Rallis: this is a complex algebraic variety on which $K_{\mathbb{C}}$ acts with finitely many orbits.

Sekiguchi: orbits in 1–1 corr with G orbits on $\mathcal{N}^*_{\mathbb{R}}$

Vergne: corr orbits are K-eqvtly diffeomorphic

Conclusion: reps attached to nilpotent orbits should be realized (as reps of K) as (global sections of) $K_{\mathbb{C}}$-eqvt sheaves of modules on $\mathcal{N}^*_\theta$.

The classical limit

Second guiding principle: so far considered only constructing quantum system from classical one. Physics also considers "classical limit:" let Planck's constant go to zero. Amounts to making operators more commutative. Example is symbol calculus for diff ops on mfld X, replacing $\mathcal{D}(X)$ by $C^\infty(T^*(X))$.

Idea: use Poincaré-Birkhoff-Witt isom $\operatorname{gr} U(\mathfrak{g}) \simeq S(\mathfrak{g})$. Try to replace reps (modules for $U(\mathfrak{g})$) by modules for poly ring $S(\mathfrak{g})$.

Construction: V irr HC module for reductive G. Choose K-invt good filtration

$$V_0 \subset V_1 \subset \cdots \quad \bigcup_n V_n = V \quad U_p(\mathfrak{g}) \cdot V_q \subset V_{p+q}.$$

Then $\operatorname{gr} V$ is fin gen $K_{\mathbb{C}}$-eqvt $S(\mathfrak{g}/\mathfrak{k})$-module supported on $\mathcal{N}_\theta^*$.

Conclusion: reps V attached to nilpotent orbits should have $\operatorname{gr} V$ an uncomplicated module with specified supp in $\mathcal{N}_\theta^*$.

Unitarity

Given nilpotent orbit, seek unitary rep of G so corr HC module V makes gr V an uncomplicated module with specified support. Kazhdan-Lusztig conjectures show how to construct many HC modules V' with specified support. Two (related) problems: gr V' may be complicated, and V' may not be unitary.

Manifestation of first problem: multiplicity of gr V' (along generic part of support) may be bigger than 1.

Manifestation of second problem: V' may carry Hermitian form with indefinite signature.

Idea: show that second problem forces first to occur. Rule out first problem by algebra (for special V), and conclude that such V come from unitary representations.

Signatures

Theorem *Suppose V irr HC module for reductive G admitting invt Hermitian form. Then there are $K_{\mathbb{C}}$-eqvt fg graded $S(\mathfrak{g}/\mathfrak{k})$-modules $M^{\pm}$ supp on $\mathcal{N}^*_\theta$, with following properties:*

1. *gr$V = M^+ + M^-$ in Grothendieck group.*
2. *generic mult(gr$V$) = gen mult(M^+)+gen mult(M^-).*
3. *$M^- = 0$ iff form on V is positive definite.*
4. *There are virtual HC modules $V^{\pm}$ satisfying*
   - *(a) gr$V^{\pm} = M^{\pm}$ in Grothendieck group.*
   - *(b) $|$infl char($V^{\pm}$)$| \leq |$infl char(V)$|$.*

Conclusion: if generic multiplicity of gr V is 1, then support of gr V^- must be smaller than support of gr V.

Theorem *Suppose V virtual HC module, and supp gr$V$ is small. Then infl char(V) is large.*

Problem: can't yet make "large" and "small" precise.

Conclusion: if generic multiplicity of gr V is 1, and infl char(V) is small, then form on V is positive definite; so V comes from unitary rep of G.