Regular polyhedra and Coxeter groups

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Outline

Introduction

Ideas from linear algebra

Flags in polyhedra

Reflections and relations

Relations satisfied by reflection symmetries

Presentation and classification

Counting faces of regular polyhedra

What's the plan?

Goal: understand classification of regular polyhedra.
Path to goal:

1. Regular polyhedra $\leftrightsquigarrow$ big symmetry groups.
2. Big symmetry groups $\overset{\text{Coxeter}}{\leftrightsquigarrow}$ generators and relations.

   Analogy: matrix groups $\overset{\text{Serre}}{\leftrightsquigarrow}$ generators and relations.
   This is what you teach as Gaussian elimination.

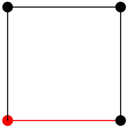
3. So far: regular polyhedra $\longleftrightarrow$ finite Coxeter groups.
4. Finish: classify finite Coxeter groups.

Matrix group building block: 2×2 matrices.

Coxeter group building block: $\mathbb{Z}/2\mathbb{Z}$.

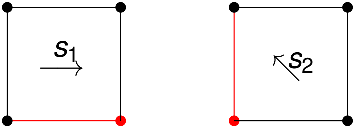
What's a regular polyhedron?

Something really symmetrical... like a square



FIX one vertex inside one edge inside square.

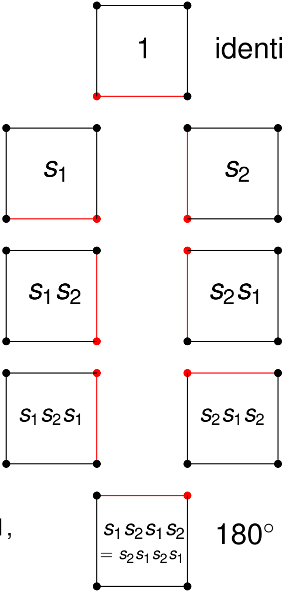
Two building block symmetries.



s_1 takes red vertex to adj vertex along red edge;

s_2 takes red edge to adj edge at red vertex.

More symmetries from building blocks



- 1: identity
- s_1: reflect in $x = 0$
- s_2: reflect in $y = x$
- $s_1 s_2$: 90° rotation
- $s_2 s_1$: 270° rotation
- $s_1 s_2 s_1$: reflect in $y = -x$
- $s_2 s_1 s_2$: reflect in $y = 0$
- $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$: 180° rotation

Presentation:
generators s_1, s_2;
relations $s_1^2 = s_2^2 = 1$,
$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

Understanding all regular polyhedra

Introduce a flag as a chain of faces like vertex $\subset$ edge in a square.

Introduce basic symmetries like s_1, s_2 which change a flag as little as possible.

Find a presentation of the symmetry group.

See how to recover polyhedron from presentation of symmetry group.

Decide which presentations are possible.

Most of linear algebra

V n-diml vec space $\rightsquigarrow$ $GL(V)$ invertible linear maps.

complete flag in V is chain of subspaces $\mathcal{F}$

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V, \quad \dim V_i = i.$$

Stabilizer $B(\mathcal{F})$ called Borel subgroup of $GL(V)$.

Example

$V = k^n$, $V_i = \{(x_1, \ldots, x_i, 0, \ldots, 0) \mid x_j \in k\} \simeq k^i$.
Stabilizer of this flag is upper triangular matrices.

Theorem

1. *$GL(V)$ acts transitively on flags.*
2. *Stabilizer of one flag is isomorphic to group of invertible upper triangular matrices.*

Rest of linear algebra

Fix integers $\mathbf{d} = (0 = d_0 < d_1 < \cdots < d_r = n)$

partial flag of type $\mathbf{d}$ is chain of subspaces $\mathcal{G}$

$$W_0 \subset W_1 \subset \cdots \subset V_{r-1} \subset W_r, \qquad \dim W_j = d_j.$$

Stabilizer $P(\mathcal{G})$ is a parabolic subgroup of $GL(V)$.

Example

$V = k^n$, $W_j = \{(x_1, \ldots, x_{d_j}, 0, \ldots, 0) \mid x_i \in k\} \simeq k^{d_j}$.

Stabilizer is block upper triangular matrices.

Theorem

1. *$GL(V)$ acts transitively on partial flags of type* $\mathbf{d}$.
2. *Stabilizer of one flag is isomorphic to group of invertible block upper triangular matrices.*
3. *Consider the $n-1$ partial flags obtained by omitting one proper subspace from a fixed complete flag:*

   $$\mathcal{G}_p = (V_0 \subset \cdots \subset \widehat{V_p} \subset \cdots \subset V_n) \quad 1 \le p \le n-1.$$

   Then $GL(V)$ is generated by the $n-1$ parabolic subgroups $P(\mathcal{G}_p)$, corresponding to block upper triangular matrices with a single 2×2 block.

Notation for polyhedra

Set C in $\mathbb{R}^N$ is convex if

$$c_i \in C, t_i \in [0,1], \sum t_i = 1 \Rightarrow \sum t_i c_i \in C.$$

Convex polyhedron P is intersection of half spaces

$$P = \{v \in \mathbb{R}^N \mid \lambda_i(v) \leq a_i, 1 \leq i \leq M\}.$$

Here $\lambda_i \in (\mathbb{R}^N)^*$ (dual space), $a_i \in \mathbb{R}$.

If P is nonempty, it generates an affine subspace

$$S(P) = \{t_1 q_1 + \cdots + t_r q_r \mid q_i \in P, t_i \in \mathbb{R}, \sum t_i = 1\};$$

say P is n-dimensional if $S(P)$ is n-diml.

Interior P^0 of P is topological interior of $P \cap S(P)$.

Boundary ∂P of P is $P - P^0$.

Theorem

Boundary of n-diml convex polyhedron P is finite union of $(n-1)$-diml convex polyhedra, the faces of P.

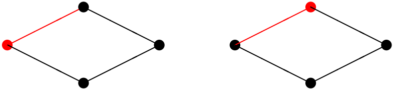
Flags

P_n compact n-dimensional convex polyhedron

A (complete) flag $\mathcal{F}$ in P is a chain

$$P_0 \subset P_1 \subset \cdots \subset P_n, \qquad \dim P_i = i$$

with P_{i-1} a face of P_i.



Two flags in two-diml P. Symmetry group (generated by reflections in x and y axes) is transitive on edges, not transitive on flags.

Definition

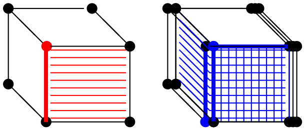
P regular if symmetry group acts transitively on flags.

Adjacent flags

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \qquad \dim P_i = i$$

complete flag in n-diml compact convex polyhedron.

A flag $\mathcal{F}' = (P_0' \subset P_1' \subset \cdots \subset P_n')$ is i-adjacent to $\mathcal{F}$ if $P_j = P_j'$ for all $j \neq i$, and $P_i \neq P_i'$.



Three flags adjacent to $\mathcal{F}$, $i = 0, 1, 2$.
$\mathcal{F}_0'$: move vertex P_0 only. $\mathcal{F}_1'$: move edge P_1 only.
$\mathcal{F}_2'$: move face P_2 only.

There is exactly one $\mathcal{F}'$ i-adjacent to $\mathcal{F}$ (each $i = 0, 1, \ldots, n-1$).

Stabilizing a flag

Lemma

Suppose $\mathcal{F} = (P_0 \subset P_1 \subset \cdots)$ complete flag in n-dimensional compact convex polyhedron P_n. Any affine map T preserving $\mathcal{F}$ acts trivially on P_n.

Proof. Induction on n. If $n = -1$, $P_n = \emptyset$ and result is true.

Suppose $n \geq 0$ and the the result is known for $n - 1$.

Write $p_n =$ center of mass of P_n. Since center of mass is preserved by affine transformations, $Tp_n = p_n$.

By inductive hypothesis, T acts trivially on $(n-1)$-diml affine $S(P_{n-1})$ spanned by P_{n-1}.

Easy to see that $p_n \notin S(P_{n-1})$, so p_n and $(n-1)$-diml $S(P_{n-1})$ must generate n-diml $S(P_n)$.

Since T trivial on gens, trivial on $S(P_n)$. Q.E.D.

Compactness matters; result fails for $P_1 = [0, \infty)$.

Symmetries and flags

Henceforth P_n is cpt cvx reg polyhedron with fixed flag

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \qquad \dim P_i = i$$

Write p_i = center of mass of P_i

Theorem

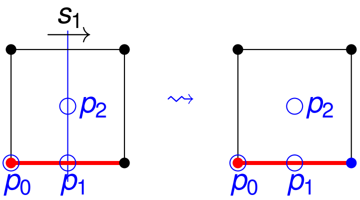
There is exactly one symmetry w of P_n for each complete flag $\mathcal{G}$, characterized by $w\mathcal{F} = \mathcal{G}$.

Corollary

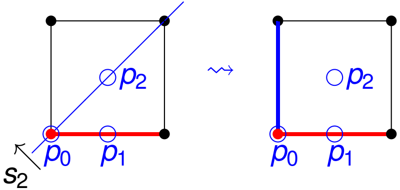
Define $\mathcal{F}'_{i-1}$ = unique flag $(i-1)$-adj to $\mathcal{F}$ $(1 \le i \le n)$. There is a unique symmetry s_i of P_n char by $s_i(\mathcal{F}) = \mathcal{F}'_{i-1}$. It satisfies

1. $s_i(\mathcal{F}'_{i-1}) = \mathcal{F}$, $s_i^2 = 1$.
2. s_i *fixes the* $(n-1)$*-diml hyperplane through the n points* $\{p_0, \ldots, p_{i-2}, \widehat{p_{i-1}}, p_i, \ldots, p_n\}$.

Examples of basic symmetries s_i



This is s_1, which changes $\mathcal{F}$ only in P_0, so acts trivially on the line through p_1 and p_2.



This is s_2, which changes $\mathcal{F}$ only in P_1, so acts trivially on the line through p_0 and p_2.

What's a reflection?

On vector space V (characteristic not 2), a linear map s with $s^2 = 1$, $\dim(-1 \text{ eigspace}) = 1$.

-1 eigenspace is line L_s; fix basis vector $\alpha^\vee \in V$

$$L_s = \{v \in V \mid sv = -v\} = \text{span}(\alpha^\vee).$$

$+1$ eigspace $=$ hyperplane $H_s =$ kernel of nonzero $\alpha \in V^*$

$$H_s = \{v \in V \mid sv = v\} = \ker(\alpha).$$

$$sv = s_{(\alpha,\alpha^\vee)}(v) = v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha, \alpha^\vee\rangle}\alpha^\vee.$$

Extend $\{\alpha^\vee\}$ to basis of V with basis of H_s:

$$\text{matrix of } s = \begin{pmatrix} -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ & & \ddots \end{pmatrix}$$

Orth reflections: quadratic form $\langle , \rangle$ identifies $V \simeq V^*$;

$$\alpha = \alpha^\vee \Rightarrow s \text{ orthogonal.}$$

Two reflections

$$sv = v - 2\frac{\langle \alpha_s, v\rangle}{\langle \alpha_s, \alpha_s^\vee\rangle}\alpha_s^\vee, \qquad tv = v - 2\frac{\langle \alpha_t, v\rangle}{\langle \alpha_t, \alpha_t^\vee\rangle}\alpha_t^\vee.$$

Assume $V = L_s \oplus L_t \oplus (H_s \cap H_t)$.

On subspace $L_s \oplus L_t$, basis $\{\alpha_s^\vee, \alpha_t^\vee\}$, $c_{st} = 2\langle \alpha_s, \alpha_t^\vee\rangle / \langle \alpha_s, \alpha_s^\vee\rangle$,

$$s = \begin{pmatrix} -1 & -c_{st} \\ 0 & 1 \end{pmatrix}, \ t = \begin{pmatrix} 1 & 0 \\ -c_{ts} & -1 \end{pmatrix}, \quad st = \begin{pmatrix} -1 + c_{st}c_{ts} & c_{st} \\ -c_{ts} & -1 \end{pmatrix}.$$

$\det(st) = 1$, $\mathrm{tr}(st) = -2 + c_{st}c_{ts}$,

$$\text{eigenvalues } \exp(\pm i \cos^{-1}(-1 + c_{st}c_{ts}/2)).$$

Proposition

Suppose $-1 + c_{st}c_{ts}/2 =$ real part of a prim mth root of 1, $m \geq 3$; or that $m = 2$, and $c_{st} = c_{ts} = 0$. Then st has order exactly m. Otherwise st has infinite order. In particular

1. *$m = 2$ if and only if $c_{st} = c_{st} = 0$;*
2. *$m = 3$ if and only if $c_{st}c_{ts} = 1$;*
3. *$m = 4$ if and only if $c_{st}c_{ts} = 2$;*
4. *$m = 6$ if and only if $c_{st}c_{ts} = 3$;*

Reflection symmetries

P_n compact convex regular polyhedron in $\mathbb{R}^n$, flag

$$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_k = k, \quad p_k = \text{ctr of mass}(P_k).$$

$s_k =$ nontriv symmetry preserving all P_j except P_{k-1}.

s_k must be orthogonal reflection in hyperplane

$$H_k = S(p_0, p_1, \ldots, \widehat{p_{k-1}}, p_k, \ldots, p_n)$$

(unique aff hyperplane containing these n points).

Write eqn of H_k

$$H_k = \{v \in \mathbb{R}^n \mid \langle \alpha_k, v \rangle = c_k\}.$$

α_k characterized up to positive scalar multiple by

$$\langle \alpha_k, p_j - p_n \rangle = 0 \quad (j \neq k-1), \qquad \langle \alpha_k, p_{k-1} - p_n \rangle > 0.$$

$$s_k(v) = v - \frac{2\langle \alpha_k, v - p_n \rangle}{\langle \alpha_k, \alpha_k \rangle} \alpha_k.$$

Good coordinates

P_n compact convex regular polyhedron in $\mathbb{R}^n$, flag
$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_i = i, \quad p_i = \text{ctr of mass}(P_i)$.

Translate so center of mass is at the origin: $p_n = 0$.

Rotate p_{n-1} to $\mathbb{R}^1 \subset \mathbb{R}^n$: $p_{n-1} = (a_n, 0, \ldots), a_n > 0$.

Now hyperplane $S(P_{n-1})$ is $\{x_1 = a_n\}$.

Rotate p_{n-2} (fixing p_{n-1}) to $\mathbb{R}^2 \subset \mathbb{R}^n$:
$p_{n-2} = (a_n, a_{n-1}, 0 \ldots), a_{n-1} > 0$.

$(n-2)$-plane $S(P_{n-2})$ is $\{x_1 = a_n, x_2 = a_{n-1}\}$.

$\vdots$

$p_{n-k} = (a_n, \ldots, a_{n-k+1}, 0 \ldots), a_{n-k+1} > 0$.

$(n-k)$-plane $S(P_{n-k}) = \{x_1 = a_n,\ x_2 = a_{n-1} \ldots x_k = a_{n-k+1}\}$.

Reflections in good coordinates

P_n cpt cvx reg polyhedron in $\mathbb{R}^n$, flag
$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n)$, $\dim P_i = i$, $p_i =$ ctr of mass(P_i).

$$p_k = (a_n, \ldots, a_{k+1}, 0 \ldots), a_{k+1} > 0.$$

k-plane $S(P_k)$ is $\{x_1 = a_n,\ x_2 = a_{n-1} \ldots x_{n-k} = a_{k+1}\}$.

Reflection symmetry s_k preserves all P_j *except* $P_{k-1}(1 \le k \le n)$, so fixes all p_j except p_{k-1}.

Fixes $p_n = 0$, so a reflection through the origin: $s_k = s_{\alpha_k}$, α_k orthogonal to all p_j except p_{k-1}.

Solve equations: $\alpha_k = (0, \ldots, a_k^{-1}, -a_{k-1}^{-1}, 0, \ldots, 0)$ (entries in coordinates $n-k+1$ and $n-k+2$).

To relate two reflections s_{k_1} and s_{k_2}, needed

$$c_{k_1,k_2} = 2\langle \alpha_{k_1}, \alpha_{k_2} \rangle / \langle \alpha_{k_1}, \alpha_{k_1} \rangle = 0 \qquad (|k_1 - k_2| > 1),$$

$$c_{k,k+1} = 2\langle \alpha_k, \alpha_{k+1} \rangle / \langle \alpha_k, \alpha_k \rangle = -2a_{k-1}^2/(a_k^2 + a_{k-1}^2),$$

$$c_{k+1,k} = 2\langle \alpha_{k+1}, \alpha_k \rangle / \langle \alpha_{k+1}, \alpha_{k+1} \rangle = -2a_{k+1}^2/(a_k^2 + a_{k+1}^2).$$

$$s_k s_{k+1} = \text{rot by } \cos^{-1}\left(\frac{a_{k-1}^2 a_{k+1}^2 - a_{k-1}^2 a_k^2 - a_k^2 a_{k+1}^2 - a_k^4}{a_{k-1}^2 a_{k+1}^2 + a_{k-1}^2 a_k^2 + a_k^2 a_{k+1}^2 + a_k^4}\right).$$

Example: n-cube

$$P_n = \{x \in \mathbb{R}^n \mid -1 \le x_i \le 1 \quad (1 \le i \le n)\}.$$

Choose flag $P_k = \{x \in P_n \mid x_1 = \cdots = x_{n-k} = 1\}$, ctr of mass $p_k = (1, \ldots, 1, 0 \ldots, 0) \quad (n-k \text{ 1s})$.

$$s_k = \text{refl in } \alpha_k = (0, \ldots, 1, -1, \ldots, 0) = e_{n-k+1} - e_{n-k+2}$$

$$= \text{exchange coords } n-k+1, n-k+2 \quad (k \ge 2).$$

$$s_1 = \text{refl in } \alpha_1 = (0, \ldots, 0, 1) = e_n$$

$$= \text{sign change of coord } n.$$

$$s_k s_{k+1} = \text{rot by } \cos^{-1}\left(\frac{1^4 - 1^4 - 1^4 - 1^4}{1^4 + 1^4 + 1^4 + 1^4}\right) = 2\pi/3 \quad (k \ge 2)$$

$$s_1 s_2 = \text{rot by } \cos^{-1}\left(\frac{1^4 - 1^4}{1^4 + 1^4}\right) = 2\pi/4$$

Symm grp = permutations, sign changes of coords

$$= \langle s_1, \ldots s_n \rangle / \langle s_k^2 = 1, (s_k s_{k+1})^3 = 1, (s_1 s_2)^4 = 1 \rangle$$

Angles and coordinates

$\mathcal{F} = (P_0 \subset P_1 \subset \cdots \subset P_n), \quad \dim P_i = i, \quad p_i = \text{ctr of mass}(P_i).$

$$p_k = (a_n, \ldots, a_{k+1}, 0 \ldots), a_{k+1} > 0.$$

Geom given by $n-1$ (strictly) positive reals $r_k = (a_{k+1}/a_k)^2$.

$s_k s_{k+1} =$ rotation by $\theta_k \in (0, \pi)$,

$$\cos(\theta_k) = \left(\frac{r_k - r_k r_{k-1} - r_{k-1} - 1}{r_k r_{k-1} + r_k + r_{k-1} + 1}\right).$$

When $k = 1$, some terms disappear:

$$\cos(\theta_1) = \frac{r_1 - 1}{r_1 + 1}, \qquad r_1 = \frac{1 + \cos(\theta_1)}{1 - \cos(\theta_1)}.$$

These recursion formulas give all r_k in terms of all θ_k.

Next formula is

$$r_2 = -\frac{\cos(\theta_2) + \cos(\theta_1)}{1 + \cos(\theta_2)}.$$

Formula makes sense (defines strictly positive r_2) iff $\cos(\theta_2) + \cos(\theta_1) < 0$.

Coxeter graphs

Regular polyhedron given by $n-1$ pos ratios $r_k = (a_{k+1}/a_k)^2$.

Symmetry group has n generators $s_1, \ldots, s_n$,

$$s_k^2 = 1, \quad s_k s_{k'} = s_{k'} s_k \ (|k-k'| > 1), \quad (s_k s_{k+1})^{m_k} = 1.$$

Here $m_k \geq 3$. Rotation angle for $s_k s_{k+1}$ must be

$$\theta_k = 2\pi/m_k \in \{120^\circ, 90^\circ, 72^\circ, 60^\circ \ldots\},$$

$$\cos(\theta_k) \in \left\{-\frac{1}{2},\ 0,\ \frac{\sqrt{5}-1}{4},\ \frac{1}{2}, \ldots\right\},$$

Group-theoretic information recorded in Coxeter graph

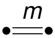
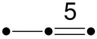
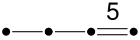
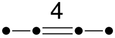
Recursion formulas give r_k from $\cos(\theta_k) = \cos(2\pi/m_k)$.

Condition $\cos(\theta_2) + \cos(\theta_1) < 0$ says

one of m_{k+1}, m_k must be 3; other at most 5.

Finite Coxeter groups with one line

Same ideas lead (Coxeter) to classification of all graphs for which recursion gives positive r_k.

type	diagram	G	$\|G\|$	regular polyhedron
A_n		symmetric group S_{n+1}	$n!$	n-simplex
BC_n		cube group	$2^n \cdot n!$	hyperoctahedron, hypercube
$I_2(m)$		dihedral group D_m	$2m$	m-gon
H_3		H_3	120	icosahedron, dodecahedron
H_4		H_4	14400	600-cell, 120-cell
F_4		F_4	1152	24-cell

For much more, see Bill Casselman's amazing website

`http://www.math.ubc.ca/~cass/coxeter/crm.html`

Reading geometry from the Coxeter diagram

H_4	•—•—•$\overset{5}{=}$•	H_4	14400	600-cell, 120-cell

Read either left to right (600-cell) or right to left (120 cell).

First k vertices $\longleftrightarrow$ (symmetry group of) k-diml face.

k-diml face also preserved by reflections for last $(n-k-1)$ vertices, which act trivially.

$$\#(k\text{-faces}) = \frac{\#(n\text{-vertex group})}{\#(\text{first } k\text{-vrtx grp}) \cdot \#(\text{last } (n-k-1)\text{-vrtx grp})}.$$

Here's the 600-cell:

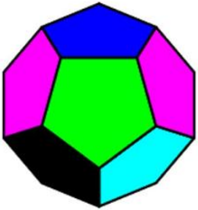
0. 0-face $=$ point $=$ 0-simplex (trivial symmetry)
   number of vertices $= 14400/(1 \cdot 120) = 120$.
1. 1-face $=$ interval $=$ 1-simplex (symmetry • $\longleftrightarrow S_2$)
   number of edges $= 14400/(2 \cdot 10) = 720$.
2. 2-face $=$ triangle $=$ 2-simplex (symmetry •—• $\longleftrightarrow S_3$)
   number of 2-faces $= 14400/(6 \cdot 2) = 1200$.
3. 3-face $=$ tetrahedron $=$ 3-simplex (symmetry •—•—• $\longleftrightarrow S_4$)
   number of 3-faces $= 14400/(24 \cdot 1) = 600$.

Once more for the 120 cell

H_4	$\bullet \overset{5}{=\!=} \bullet — \bullet — \bullet$	H_4	14400	120-cell, 600-cell

Read this reversed diagram left to right for the 120 cell):

0. 0-face = point = 0-simplex (trivial symmetry)
   number of vertices $= 14400/(1 \cdot 24) = 600$.
1. 1-face = interval = 1-simplex (symmetry $\bullet \longleftrightarrow S_2$)
   number of edges $= 14400/(2 \cdot 6) = 1200$.
2. 2-face = pentagon (symmetry $\bullet \overset{5}{=\!=} \bullet \longleftrightarrow$ dihedral D_5)
   number of 2-faces $= 14400/(10 \cdot 2) = 720$.
3. 3-face = dodecahedron (symmetry $\bullet \overset{5}{=\!=} \bullet — \bullet$) $\longleftrightarrow H_3$)
   number of 3-faces $= 14400/(120 \cdot 1) = 120$.



Glue 120 of these together along pentagons; the four dodecahedra meeting at each vertex need to be bent together a bit in four dimensions to close up.