Quaternionic groups

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The point of these notes is to relate some of the quaternionic groups discussed in class to complex matrix groups. In particular, the notation $Sp(n)$ for the quaternionic unitary group suggests that it has something to do with symplectic groups. This is true, and will be explained in (0.4h). There is some disagreement about whether symplectic groups should be labelled with $2n$ (the dimension of the vector space) or n (the "rank," which we'll talk about eventually). I think that for the noncompact groups $2n$ wins, but for the compact groups, there is more support for n. Anyway the result is that (0.4h) looks a bit funny, but that's life.

Recall the division ring of quaternions

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}. \tag{0.1a}$$

Multiplication is defined by

$$i^2 = j^2 = k^2 = -1, \qquad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik. \tag{0.1b}$$

A more succinct way to write this is

$$\mathbb{H} = \mathbb{C}[j], \qquad j^2 = -1, \quad jzj^{-1} = \overline{z}. \tag{0.1c}$$

That is, any quaternion h may be written uniquely as

$$h = z + jw \qquad (z, w \in \mathbb{C}); \tag{0.1d}$$

the multiplication rules are determined by (0.1c). There is an algebra anti-automorphism of the quaternions given by

$$\overline{a + bi + cj + dk} = a - bi - cj - dk, \qquad \overline{h_1 h_2} = \overline{h_2 h_1}. \tag{0.1e}$$

A right quaternionic vector space V is automatically a complex vector space, just by restricting scalar multiplication to the subring $\mathbb{C} \subset \mathbb{H}$. An

$\mathbb{H}$-linear map from V to V is automatically $\mathbb{C}$-linear, so we get an inclusion of $\mathbb{R}$-algebras

$$\mathrm{Hom}_{\mathbb{H}}(V,V) \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V), \tag{0.2a}$$

and an inclusion of Lie groups

$$GL_{\mathbb{H}}(V) \hookrightarrow GL_{\mathbb{C}}(V). \tag{0.2b}$$

$GL_{\mathbb{H}}(V)$ is *not* a complex Lie group, even though it has real dimension divisible by 2 (even 4!).

If

$$(e_1, \ldots, e_n) = \mathbb{H}\text{-basis of } V, \tag{0.2c}$$

then

$$(e_1, \ldots, e_n, e_1 j, \ldots, e_n j) = \mathbb{C}\text{-basis of } V. \tag{0.2d}$$

As explained in class, the choice of basis of V identifies

$$\mathrm{Hom}_{\mathbb{H}}(V,V) \simeq n \times n \text{ matrices over } \mathbb{H} = M_n(\mathbb{H}). \tag{0.2e}$$

The (p,q) entry H_{pq} of the matrix H is equal to the coefficient of e_p in the expression of He_q:

$$He_q = \sum_{p=1}^{n} e_p H_{pq}. \tag{0.2f}$$

Any quaternionic matrix H can be written (using (0.1d)

$$H = Z + jW \qquad (Z, W \in M_n(\mathbb{C}). \tag{0.2g}$$

The choices of basis above translate (0.2a) into

$$M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C}), \qquad Z + jW \mapsto \begin{pmatrix} Z & -\overline{W} \\ W & \overline{Z} \end{pmatrix} \tag{0.2h}$$

Another way to say this is that there is a $\mathbb{C}$-conjugate linear algebra automorphism of $M_{2n}(\mathbb{C})$

$$\sigma_{\mathbb{H}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \overline{D} & -\overline{C} \\ -\overline{B} & \overline{A} \end{pmatrix}. \tag{0.2i}$$

The subalgebra fixed by $\sigma_{\mathbb{H}}$ is $M_n(\mathbb{H})$:

$$M_n(\mathbb{H}) = M_{2n}(\mathbb{C})^{\sigma_{\mathbb{H}}}. \tag{0.2j}$$

The identification from left to right is the one written in (0.2h). Looking at the groups of invertible elements in these algebras gives the coordinate version of (0.2b):

$$GL(n,\mathbb{H}) \simeq GL(2n,\mathbb{C})^{\sigma_{\mathbb{H}}}. \tag{0.2k}$$

Now let's see how all this looks on the level of the compact subgroups. We start therefore with a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ on V. Recall that this means (in addition to bi-additivity) that

$$\langle v \cdot h, w \rangle = \overline{h} \langle v, w \rangle, \qquad \langle v, w \cdot h' \rangle = \langle v, w \rangle h' \tag{0.3a}$$

and that

$$\langle v, w \rangle = \overline{\langle w, v \rangle}. \tag{0.3b}$$

(This is just like the definition of a Hermitian form on a complex vector space, except that I've put the conjugate-linearity in the first variable instead of the second. The reason is to simplify a lot of formulas (in which the noncommutativity of $\mathbb{H}$ matters). Some mathematicians think that one should do the same thing with complex Hermitian forms, but I think that goes against too much tradition.) As long as V is finite-dimensional, any linear transformation H of V has an *adjoint* H^* defined by the requirement

$$\langle v, Hw \rangle = \langle H^* v, w \rangle. \tag{0.3c}$$

The conjugate-linear algebra antiautomorphism $*$ of $\mathrm{Hom}(V,V)$ gives rise to a group automorphism of order two

$$\theta_{\mathbb{H}}(g) = (g^*)^{-1} \qquad (g \in GL(V)). \tag{0.3d}$$

The group of fixed points of this involution is (very easily seen to be) the quaternionic unitary group (of $\mathbb{H}$-linear transformations preserving the Hermitian form):

$$U(V) = \{ g \in GL(V) \mid \theta_{\mathbb{H}}(g) = g \}. \tag{0.3e}$$

This kind of formula works for $\mathbb{R}$ and $\mathbb{C}$ as well: for example, if W is a finite-dimensional real vector space with a positive-definite quadratic form, then the adjoint operation for $\mathbb{R}$ (which is transpose on the level of matrices in an orthonormal basis) satisfies

$$\theta_{\mathbb{R}}(g) = (g^*)^{-1}, \qquad O(W) = \{ g \in GL(W) \mid \theta_{\mathbb{R}}(g) = g \}.$$

The positive definite Hermitian form on the quaternionic vector space V defines a positive definite Hermitian form on the underlying complex vector space V, by the formula

$$\langle v, w \rangle_{\mathbb{C}} = a + bi \quad \text{whenever} \quad \langle v, w \rangle_{\mathbb{H}} = a + bi + cj + dk. \tag{0.3f}$$

If $H \in \mathrm{Hom}_{\mathbb{H}}(V,V)$, then it's clear from this definition that the quaternionic adjoint of H is precisely equal to the complex adjoint H regarded as an element of $\mathrm{Hom}_{\mathbb{C}}(V,V)$ by (0.2a)

Assume now that the basis chosen in (0.2c) is *orthonormal*:

$$\langle e_p, e_q \rangle = \delta_{pq} \tag{0.3g}$$

The basis identifies V with column vectors $v \in \mathbb{H}^n$. In this identification, the Hermitian form becomes

$$\langle v, w \rangle = {}^t\overline{v} \cdot w = \sum_{p=1}^{n} \overline{v_p} w_p. \tag{0.3h}$$

From this equation and the easy fact about quaternionic matrices

$${}^t\overline{ST} = {}^t\overline{T}\,{}^t\overline{S},$$

we deduce that the matrix of H^* is the conjugate transpose of the matrix of H:

$$H^* = {}^t\overline{H}, \quad (Z + jW)^* = {}^t\overline{Z} + {}^t\overline{W}(-j) = {}^t\overline{Z} - j({}^tW). \tag{0.3i}$$

Recalling that adjoint for complex matrices is conjugate transpose (again assuming that the basis is orthonormal for the Hermitian form involved), we can write this as

$$(Z + jW)^* = Z^* - j({}^tW). \tag{0.3j}$$

If we plug this formula into (0.2h), we get

$$(Z + jW)^* = Z^* - j({}^tW) \mapsto \begin{pmatrix} Z^* & W^* \\ -\overline{W}^* & \overline{Z^*} \end{pmatrix} = \begin{pmatrix} Z & -\overline{W} \\ W & \overline{Z} \end{pmatrix}^*. \tag{0.3k}$$

That is, *the inclusion of $M_n(\mathbb{H})$ in $M_{2n}(\mathbb{C})$ respects the adjoint operation* $*$ (as long as the chosen basis of V is orthonormal).

So here is what we know.

1. $GL(n,\mathbb{H})$ is the group of fixed points of the involution $\sigma_{\mathbb{H}}$ acting on $GL(2n,\mathbb{C})$ (see (0.2k)).

2. $U(2n)$ is the group of fixed points of the involution $\theta_{\mathbb{C}}$ acting on $GL(2n,\mathbb{C})$ (complex version of (0.3e)).

3. $U(n,\mathbb{H}) = Sp(n)$ is the group of fixed points of the involution $\theta_{\mathbb{H}}$ acting on $GL(n,\mathbb{H})$ (0.3e), which is the same as the group of fixed points of $\theta_{\mathbb{C}}$ acting on $GL(n,\mathbb{H}) \subset GL(2n,\mathbb{C})$ (because of (0.3k)).

4. The involutions $\theta_{\mathbb{C}}$ and $\sigma_{\mathbb{H}}$ of $GL(2n,\mathbb{C})$ commute with each other.

The last fact is just a computation; Putting these facts together, we get

$$\begin{aligned} Sp(n) = U(n,\mathbb{H}) &= \{g \in GL(2n,\mathbb{C}) \mid \theta_{\mathbb{C}}(g) = g \text{ and } \sigma_{\mathbb{H}}(g) = g\} \\ &= GL(n,\mathbb{H}) \cap U(2n). \end{aligned} \tag{0.4a}$$

Because of the commutativity in (4) above, we can define a new involution

$$\tau_{Sp}\colon GL(2n,\mathbb{C}) \to GL(2n,\mathbb{C}, \qquad \tau_{Sp}(g) = \theta_{\mathbb{C}} \circ \sigma_{\mathbb{H}}(g). \tag{0.4b}$$

Then we can rewrite (0.4a) as

$$Sp(n) = \{g \in GL(2n,\mathbb{C}) \mid \theta_{\mathbb{C}}(g) = g \text{ and } \tau_{Sp}(g) = g\}. \tag{0.4c}$$

The involution τ_{Sp} can be computed using the formula in (0.2i) and (0.3d) (or rather the complex analogue):

$$\begin{aligned} \tau_{Sp}\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \theta_{\mathbb{C}}\begin{pmatrix} \overline{D} & -\overline{C} \\ -\overline{B} & \overline{A} \end{pmatrix} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}^{-1} \\ &= [J^{-1}\,{}^t\!\begin{pmatrix} A & B \\ C & D \end{pmatrix} J]^{-1}. \end{aligned} \tag{0.4d}$$

Here

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{0.4e}$$

The standard symplectic form on $\mathbb{C}^{2n}$ is

$$\omega(v,w) = {}^twJv; \tag{0.4f}$$

so it's easy to check that the fixed points of τ_{Sp} is precisely the symplectic group:

$$Sp(2n,\mathbb{C}) = Sp(\mathbb{C}^{2n},\omega) = \{g \in GL(2n,\mathbb{C}) \mid \tau(g) = g\}. \tag{0.4g}$$

Plugging this fact into (0.4c) gives

$$U(n,\mathbb{H}) = Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n): \tag{0.4h}$$

the quaternionic unitary group may be identified with the complex-linear transformations of $\mathbb{C}^{2n}$ preserving both the standard symplectic form and the standard (positive) Hermitian form.