18.100B Problem Set 8

Due in class Monday, April 27. You may discuss the problems with other students, but you should write solutions entirely on your own.

1. This is a version of problems 8 and 10 in the text (pages 166-167). Let $S = \{x_1, x_2, \dots\}$ be a countable subset of $(0, 1)$. (For example, S might consist of all the rational numbers between 0 and 1.) Define a real-valued function f on $[0, 1]$ by

$$f(x) = \sum_{\{n | x_n \leq x\}} 2^{-n}.$$

The notation means that the sum extends exactly over those positive integers n for which $x_n \leq x$.

a) Show that f is a well-defined increasing function on $[0, 1]$, that $f(0) = 0$, and $f(1) = 1$. (By Theorem 6.9, it follows that f is Riemann-integrable.)

b) Show that f is continuous at x if and only if $x \notin S$.

c) Suppose that $x \notin \overline{S}$. Prove that f is differentiable at x.

d) Suppose S is dense in $[0, 1]$. Does the derivative $f'(x)$ exist for any value of x? (In this case part (c) doesn't provide any places where the derivative exists. This question is quite a bit harder than any of the others; don't worry if you can't make any progress on it.)

2. Text, page 165, number 3.

3. Text, page 165, number 4.

4. Text, page 168, number 14. (If you're artistically inclined, you might try defining $x_N(t)$ and $y_N(t)$ to be the Nth partial sums of the series defining x and y, and then try to sketch the parametric curves $\Phi_N(t) = (x_N(t), y_N(t))$ for $N = 1$ and 2.)