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FUNCTIONS ON THE MODEL ORBIT IN E_8

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1. Introduction

Perhaps the most intriguing and difficult problem in the representation theory of reductive Lie groups is this: how can one attach unitary representations to nilpotent coadjoint orbits? (More extensive discussions of this problem may be found in [Vunip] and [Vorange].) One of the difficulties about the problem is knowing when you have solved it; that is, knowing when a particular unitary representation may reasonably be regarded as "attached" to a particular orbit. In this direction some help is available from the general philosophy of the orbit method. We will not try to describe that philosophy in any generality here, concentrating instead on a very special part of it. What we will see is that the orbit method can sometimes predict how the representation "attached" to an orbit should restrict to a maximal compact subgroup. This prediction can be used in connection with the difficulty above: given an orbit and a unitary representation, one can try to compare the restriction of the representation to a maximal compact subgroup with the prediction of the orbit method. Agreement is evidence that the representation might be attached to the orbit.

If we try to put this idea into practice, there are two computational difficulties. First, there is the problem of computing the restriction to a maximal compact subgroup of a representation. This problem is solved in principle by the Blattner Conjecture and the Kazhdan-Lusztig Conjecture (both proved); the calculation may be unpleasant but any particular case can be done. Second, we need to compute explicitly the prediction provided by the orbit method. This (as we shall explain) is a problem in the representation theory of algebraic groups. It has not been solved in general. Even in the cases when it has been solved, the solution is difficult to compute with.

This brings us at last to the subject of the paper. We look first at a complex algebraic group G of type E_8. There is a unique nilpotent coadjoint orbit $X_{\mathbb{R}}$ for G of (real) dimension 256. (Because G is complex, every coadjoint orbit is complex; so $X_{\mathbb{R}}$ is actually a complex manifold of dimension 128. Some of what we say applies equally well to real reductive groups, and we are overlooking the complex structure here in order to maintain that compatibility.) We are concerned with the problem of attaching a unitary representation $\pi(X_{\mathbb{R}})$ to this orbit. A natural candidate for $\pi(X_{\mathbb{R}})$ has been proposed (implicitly) by McGovern in [McGmodel], Theorem 2.1: it is the irreducible spherical representation of G with parameter one fourth the

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sum of the positive roots. (We do not know whether $\pi(X_{\mathbb{R}})$ is unitary.) McGovern calculated the restriction of $\pi(X_{\mathbb{R}})$ to a maximal compact subgroup K of G. (The group K is the compact form of E_8.) It is the sum of all irreducible representations of K, each occurring with multiplicity one.

In order to test whether $\pi(X_{\mathbb{R}})$ is really a good candidate for attachment to $X_{\mathbb{R}}$, one must also compute the orbit method's predicted K-multiplicities. This McGovern was unable to do. (He carried out analogous calculations for almost all other complex simple groups.) To explain the problem, let us write $K_{\mathbb{C}}$ for the complexification of the compact group K; this is an algebraic group. (Of course in our case it is isomorphic to G, but we maintain the notational distinction to allow us to discuss a more general situation.) Recall that algebraic representations of $K_{\mathbb{C}}$ may be identified with locally finite representations of K. Attached to $X_{\mathbb{R}}$ there is an algebraic homogeneous space X_θ for $K_{\mathbb{C}}$, whose complex dimension is half the real dimension of $X_{\mathbb{R}}$. (In our case X_θ may be identified with $X_{\mathbb{R}}$, regarded now as an algebraic homogeneous space for G.) As an algebraic homogeneous space, X_θ has an algebra of regular functions $R(X_\theta)$, which carries an algebraic representation of $K_{\mathbb{C}}$. The prediction of the orbit method is that this algebraic representation of $K_{\mathbb{C}}$, when regarded as a locally finite representation of K, is equivalent to the K-finite part of $\pi(X_{\mathbb{R}})$. That is, an irreducible representation τ of K should have the same multiplicity in $\pi(X_{\mathbb{R}})$ as in the ring of regular functions on X_θ. (We will formulate a more general version of this conjecture in section 2.) Our main result is that this prediction agrees with McGovern's calculation of the K-multiplicities in $\pi(X_{\mathbb{R}})$. Here is a statement.

Theorem 1.1. *Suppose G is a complex algebraic group of type E_8 and X is the nilpotent orbit of complex dimension 128. Then the algebra $R(X)$ of regular functions on X contains every irreducible (algebraic) representation of G with multiplicity one.*

We will complete the proof in section 6.

The statement of the theorem explains McGovern's terminology "model orbit" for X. According to the terminology introduced in [BGG], a model representation of G is one containing every irreducible representation of G exactly once. (The term is deliberately vague about the category of representations in question.) Theorem 1.1 says that $R(X)$ is a model (algebraic) representation of G.

The problem of calculating multiplicities in algebras of regular functions on nilpotent orbits has a long history. Perhaps the best results now available are those of McGovern in [McGmult]; here one can find an explicit description of the multiplicities as an alternating sum of certain partition functions. To pass from such a formula to Theorem 1.1 is a matter of finitely many calculations: too many to do by hand (which is why McGovern did not prove Theorem 1.1 in [McGmodel]) but probably not too many for a computer. Our contribution is a series of artful dodges making possible a proof of Theorem 1.1 by hand.

Suppose next that $G_{\mathbb{R}}$ is a connected and simply connected split real form of G. It turns out that there is a unique real form $Y_{\mathbb{R}}$ of the orbit X; this is a coadjoint orbit for $G_{\mathbb{R}}$ of real dimension 128. As we will explain in section 8 (Conjecture 8.14), there is a natural candidate for a unitary representation $\pi(Y_{\mathbb{R}})$ attached to this orbit. This representation is not nearly so well understood as McGovern's $\pi(X_{\mathbb{R}})$, however. It is not known to be unitary, and we do not know its restriction to a maximal compact subgroup $K_{\mathbb{R}} = Spin(16)$ of $G_{\mathbb{R}}$. Nevertheless, it makes sense to try to compute the prediction of the orbit method for this restriction, and this we do in section 8 (Theorem 8.10). Here we will describe the result only qualitatively. An obvious guess is that $\pi(Y_{\mathbb{R}})$ should be a model representation for $K_{\mathbb{R}}$: the sum of all the irreducible representations, each with multiplicity one. In fact it is much smaller than this. The irreducible representations of $Spin(16)$ are parametrized by the cone of D_8-dominant weights inside an eight-dimensional lattice. The orbit method predicts roughly that only those representations whose highest weights are dominant for E_8 should actually appear. (The predicted multiplicity is one.)

2. The multiplicity conjecture

In this section we will recall from [Vunip] some conjectures and theorems about K-multiplicities in representations attached to nilpotent orbits. In order to understand what we have proved for complex groups, the discussion before Theorem 1.1 is sufficient; but for the real E_8, and in order to understand what generalizations should be pursued, a larger context is useful. We begin with a real reductive Lie group G in Harish-Chandra's class ([HC], section 3). We write $\mathfrak{g}_0$ for the real Lie algebra of G and $\mathfrak{g}$ for its complexification; analogous notation will be used for other groups. Choose a Cartan involution θ of G, so that the group K of fixed points is a maximal compact subgroup of G. Define

$$\mathfrak{p}_0 = -1 \text{ eigenspace of } \theta \text{ on } \mathfrak{g}_0. \tag{2.1)(a}$$

We will make use of

$$K_{\mathbb{C}} = \text{complexification of } K; \tag{2.1)(b}$$

this is a complex algebraic group.

The "coadjoint orbits" of the orbit method consist of linear functionals on Lie algebras. We are interested first of all in

$$i\mathfrak{g}_0^* = \text{imaginary-valued } \mathbb{R}\text{-linear functionals on } \mathfrak{g}_0. \tag{2.1)(c}$$

A *coadjoint orbit* is by definition an orbit of G on $i\mathfrak{g}_0^*$. An orbit is called *nilpotent* (or $\mathbb{R}$-*nilpotent*) if its closure is a cone. Formally,

$$\mathcal{N}_{\mathbb{R}}^* = \{\lambda \in i\mathfrak{g}_0^* \mid \text{for all } t > 0,\, t\lambda \in G \cdot \lambda\}. \tag{2.1)(d}$$

We will also be concerned with the action of $K_{\mathbb{C}}$ on $(\mathfrak{g}/\mathfrak{k})^*$. The corresponding nilpotent cone is

$$\mathcal{N}_{\theta}^* = \{\lambda \in (\mathfrak{g}/\mathfrak{k})^* \mid \text{for all } t \in \mathbb{C}^\times,\, t\lambda \in K_{\mathbb{C}} \cdot \lambda\}. \tag{2.1)(e}$$

Elements of $\mathcal{N}_{\theta}^*$ may be called θ-*nilpotent* to distinguish them from elements of $\mathcal{N}_{\mathbb{R}}^*$.

We will be concerned first of all with the orbits of G on $\mathcal{N}_{\mathbb{R}}^*$. Now the orbit method seeks to attach unitary representations not to arbitrary coadjoint orbits, but only to those satisfying a certain additional condition. In the original work of Kirillov and Kostant, this additional condition, called *integrality*, was formulated in a straightforward way. All nilpotent coadjoint orbits are integral. But Duflo (see [Duflo]) later found that a treatment of more general groups was possible only if integrality was replaced by the more subtle requirement of *admissibility*. Not every nilpotent coadjoint orbit is admissible, so we need to understand this condition.

Definition 2.2 (see [Duflo] or [Vunip], Definition 7.2). Suppose G is a real Lie group, and $\lambda \in i\mathfrak{g}_0^*$. Write $G(\lambda)$ for the isotropy group of the coadjoint action at λ, so that the coadjoint orbit $X = G\cdot\lambda$ may be identified with $G/G(\lambda)$. Recall that the tangent space $\mathfrak{g}_0/\mathfrak{g}(\lambda)_0$ to X at λ carries a natural $G(\lambda)$-invariant imaginary-valued non-degenerate symplectic form ω_λ, defined by

$$\omega_\lambda(A + \mathfrak{g}(\lambda)_0, B + \mathfrak{g}(\lambda)_0) = \lambda([A,B]) \qquad (A, B \in \mathfrak{g}_0).$$

The isotropy action therefore gives a natural group homomorphism

$$j(\lambda)\colon G(\lambda) \to Sp(\omega_\lambda).$$

The symplectic group has a natural two-fold covering, the *metaplectic group*:

$$1 \to \{1, \epsilon\} \to Mp(\omega_\lambda) \xrightarrow{p} Sp(\omega_\lambda) \to 1.$$

This covering may be pulled back via the homomorphism $j(\lambda)$ to define the *metaplectic double cover*:

$$1 \to \{1, \epsilon\} \to \widetilde{G}(\lambda) \xrightarrow{p(\lambda)} G(\lambda) \to 1.$$

Explicitly,

$$\widetilde{G}(\lambda) = \{(g, m) \in G(\lambda) \times Mp(\omega_\lambda) \mid j(\lambda)(g) = p(m)\}.$$

A representation χ of $\widetilde{G}(\lambda)$ is called *genuine* if $\chi(\epsilon) = -I$. It is called *admissible* if it is genuine, and in addition the differential of χ is a multiple of λ. Explicitly,

$$\chi(\exp A) = \exp(\lambda(A)) \cdot I \qquad (A \in \mathfrak{g}(\lambda)_0).$$

An *admissible orbit datum* is a pair (λ, χ) with $\lambda \in i\mathfrak{g}_0^*$ and χ an irreducible unitary admissible representation of $\widetilde{G}(\lambda)$. The element λ (or the coadjoint orbit $G \cdot \lambda$) is called *admissible* if admissible orbit data (λ, χ) exist. When we need to distinguish these definitions from parallel ones involving $\mathcal{N}_\theta^*$, we will say $\mathbb{R}$-*admissible.*

The notion of admissibility is a bit involved, but fortunately it simplifies for nilpotent orbits.

Lemma 2.3 ([Vunip], Theorem 5.7 and Observation 7.4). *Suppose G is a real reductive Lie group. Then an element $\lambda \in i\mathfrak{g}_0^*$ is nilpotent (cf. (2.1)(d)) if and only if the restriction of λ to $\mathfrak{g}(\lambda)$ is zero.*

Suppose henceforth that λ is nilpotent. A representation χ of $\widetilde{G}(\lambda)$ is admissible if and only if $\chi(\epsilon) = -I$, and χ is trivial on the identity component of $\widetilde{G}(\lambda)$. Consequently λ is admissible if and only if the preimage under the metaplectic covering map $p(\lambda)$ of the identity component $G(\lambda)_0$ is disconnected.

In the case of complex groups, matters are even simpler.

Lemma 2.4. *Suppose G is a complex Lie group and $\lambda \in i\mathfrak{g}_0^*$. Then the metaplectic double cover of $G(\lambda)$ is trivial:*

$$\widetilde{G}(\lambda) \simeq \{1, \epsilon\} \times G(\lambda).$$

Suppose in addition that G is reductive and λ is nilpotent. Then λ is admissible, and the admissible representations of $\widetilde{G}(\lambda)$ are in one-to-one correspondence with irreducible representations of the group $G(\lambda)/G(\lambda)_0$ of connected components of $G(\lambda)$.

Here is a weak formulation of the orbit philosophy for nilpotent orbits.

Conjecture 2.5. Suppose G is a real reductive Lie group, and (λ, χ) is a nilpotent admissible orbit datum (cf. (2.1), Definition 2.2, and Lemma 2.3). Assume in addition that the boundary of the orbit closure $\overline{G \cdot \lambda}$ has codimension at least four. (This means that if $G \cdot \lambda'$ is contained in the closure of $G \cdot \lambda$, then $\dim G \cdot \lambda' \leq \dim G \cdot \lambda - 4$.) Then there is attached to λ and χ an irreducible unitary representation $\pi(\lambda, \chi)$ of G.

Some motivation for the codimension condition may be found in [Vunip], Theorem 4.6. When the condition fails, there may still be unitary representations attached to the orbit, but it is not so clear how to parametrize them.

As mentioned in the introduction, one of the difficulties with Conjecture 2.5 is the interpretation of the word "attached." Our main goal in this section is to formulate the multiplicity conjecture describing $\pi(\lambda, \chi)$ restricted to a maximal compact subgroup (in terms of λ and χ).

A second difficulty with Conjecture 2.5 is that it is false. There is a counterexample for the double cover of $SL(3, \mathbb{R})$, found by Torasso in [Tor], and considered further in [Vunip], Example 12.4. Because our concern here is essentially with evaluating candidates for $\pi(\lambda, \chi)$, we will not worry about the possibility that no candidates exist.

In order to formulate the multiplicity conjecture, we need first to recall the Sekiguchi correspondence. Here is some representation-theoretic motivation for it. We want to give a conjecture for the restriction to K of $\pi(\lambda, \chi)$. According to the general philosophy of the orbit method, this restriction should be obtained by applying geometric quantization to the action of K on the orbit $X_{\mathbb{R}} = G \cdot \lambda$ (with χ defining something like a bundle on $X_{\mathbb{R}}$). Geometric quantization requires a K-invariant polarization of the symplectic structure. In the present case this polarization will be a K-invariant complex structure on $X_{\mathbb{R}}$. The representation of K will then be on a space of holomorphic sections of the bundle related to χ.

It is convenient to begin with an analogue of Definition 2.2.

Definition 2.6 (see [Vunip], Definition 7.13). In the setting of (2.1), suppose $\lambda \in \mathcal{N}_\theta^*$. Write $K_{\mathbb{C}}(\lambda)$ for the isotropy group of the coadjoint action at λ, so that the orbit $X = K_{\mathbb{C}} \cdot \lambda$ may be identified with $K_{\mathbb{C}}/K_{\mathbb{C}}(\lambda)$. Define $\gamma(\lambda)$ to be the character by which $K_{\mathbb{C}}(\lambda)$ acts on top degree differential forms at λ:

$$\gamma(\lambda) \colon K_{\mathbb{C}}(\lambda) \to \mathbb{C}^\times, \qquad \gamma(\lambda)(k) = \det(\mathrm{Ad}(k)|_{(\mathfrak{k}/\mathfrak{k}(\lambda))^*}.$$

A representation χ of $K_{\mathbb{C}}(\lambda)$ is called *admissible* if the differential of χ is a multiple of $\frac{1}{2} d\gamma(\lambda)$. Explicitly,

$$\chi(\exp A) = \gamma(\lambda)(\exp(A/2)) \cdot I \qquad (A \in \mathfrak{k}(\lambda)_0).$$

An *admissible orbit datum* is a pair (λ, χ) with $\lambda \in \mathcal{N}_\theta^*$ and χ an irreducible admissible representation of $K_{\mathbb{C}}(\lambda)$. If (χ, V_χ) is an irreducible admissible representation, then we can define a $K_{\mathbb{C}}$-equivariant algebraic vector bundle

$$\mathcal{V}_\chi = K_{\mathbb{C}} \times_{K_{\mathbb{C}}(\lambda)} V_\chi$$

over $X \simeq K_{\mathbb{C}}/K_{\mathbb{C}}(\lambda)$. The element λ (or the orbit $K_{\mathbb{C}} \cdot \lambda$) is called *admissible* if admissible orbit data (λ, χ) exist. When we need to distinguish these notions from those of Definition 2.2, we will say θ-*admissible*.

Theorem 2.7 ([Seki], [Vergne]). *Suppose G is a real reductive Lie group in Harish-Chandra's class; use the notation of (2.1). Then there is a natural bijection from G orbits on $\mathcal{N}_{\mathbb{R}}^*$ to $K_{\mathbb{C}}$ orbits on $\mathcal{N}_\theta^*$. Suppose this bijection sends the G orbit $X_{\mathbb{R}} = G \cdot \lambda_{\mathbb{R}}$ to the $K_{\mathbb{C}}$ orbit $X_\theta = K_{\mathbb{C}} \cdot \lambda_\theta$. Then*

1. *There is a K-equivariant diffeomorphism $X_{\mathbb{R}} \simeq X_\theta$.*
2. *For some $\lambda_{\mathbb{R}} \in X_{\mathbb{R}}$, the subgroup $K(\lambda_{\mathbb{R}}) = G(\lambda_{\mathbb{R}}) \cap K$ is a maximal compact subgroup of $G(\lambda_{\mathbb{R}})$.*
3. *For some $\lambda_\theta \in X_\theta$, the subgroup $K(\lambda_\theta) = K_{\mathbb{C}}(\lambda_\theta) \cap G$ is a maximal compact subgroup of $K_{\mathbb{C}}(\lambda_\theta)$.*
4. *For $\lambda_{\mathbb{R}}$ and λ_θ as in 2 and 3 above, there is an isomorphism $K(\lambda_{\mathbb{R}}) \simeq K(\lambda_\theta)$, uniquely defined up to inner automorphisms.*

Notice that Theorem 2.7.1, which is due to Vergne, allows us to endow $X_{\mathbb{R}}$ with a K-invariant complex structure. According to the discussion after Conjecture 2.5, the restriction to K of $\pi(\lambda_{\mathbb{R}}, \chi_{\mathbb{R}})$ ought to be something like a space of holomorphic functions on X_θ. More precisely, it should be a space of sections of a bundle related to $\chi_{\mathbb{R}}$. The following result of James Schwartz provides a construction of the bundle. An equivariant bundle on a homogeneous space is the same thing as a representation of the isotropy group, and that is what the theorem provides.

Theorem 2.8 ([Schwartz], [Vunip], Theorem 7.14). *In the setting of Theorem 2.7, choose $\lambda_{\mathbb{R}}$ and λ_θ as in parts 2 and 3. Then there is a natural bijection from admissible representations $\chi_{\mathbb{R}}$ of $\widetilde{G}(\lambda_{\mathbb{R}})$ (Definition 2.2) to admissible representations χ_θ of $K_{\mathbb{C}}(\lambda_\theta)$ (Definition 2.6).*

We can now formulate the multiplicity conjecture.

Conjecture 2.9. Suppose G is a real reductive Lie group, and $(\lambda_{\mathbb{R}}, \chi_{\mathbb{R}})$ is a nilpotent $\mathbb{R}$-admissible orbit datum (cf. (2.1), Definition 2.2, and Lemma 2.3). Assume in addition that the boundary of the orbit closure $\overline{G \cdot \lambda_{\mathbb{R}}}$ has codimension at least four. Let $\pi(\lambda_{\mathbb{R}}, \chi_{\mathbb{R}})$ be the unitary representation of G conjecturally attached to $\lambda_{\mathbb{R}}$ and $\chi_{\mathbb{R}}$. Now let $(\lambda_\theta, \chi_\theta)$ be the corresponding θ-admissible data (Theorems 2.7 and 2.8), and $\mathcal{V}_{\chi_\theta}$ the corresponding vector bundle over the $K_{\mathbb{C}}$ orbit X_θ (Definition 2.6). Then the space of K-finite vectors of $\pi(\lambda_{\mathbb{R}}, \chi_{\mathbb{R}})$ to K is isomorphic to the space of algebraic sections of $\mathcal{V}_{\chi_\theta}$.

This conjecture may also be regarded as part of Conjecture 2.5, saying something about what is required of the conjectural unitary representations $\pi(\lambda_{\mathbb{R}}, \chi_{\mathbb{R}})$. In fact it is precisely this requirement that is used in [Vunip] as evidence that Conjecture 2.5 is false: for a certain admissible orbit datum (λ, χ) for the double cover of $SL(3, \mathbb{R})$, no unitary representation satisfies the requirement of Conjecture 2.9.

We conclude this section by outlining the simplifications in Conjecture 2.9 when G is a complex reductive group. We use the notation of (2.1). The assumption that G is complex means that there is an $\mathrm{Ad}(G)$-equivariant map (multiplication by i)

$$J\colon \mathfrak{g}_0 \to \mathfrak{g}_0, \qquad J^2 = -I, \qquad J\theta = -\theta J. \tag{2.10(a)}$$

From these facts it follows that the action of J defines an $\mathrm{Ad}(K)$-equivariant isomorphism

$$J\colon \mathfrak{k}_0 \to \mathfrak{p}_0. \tag{2.10(b)}$$

Now $\mathfrak{g}_0$ is the sum of $\mathfrak{k}_0$ and $\mathfrak{p}_0$, and $\mathfrak{p}$ (the complexification of $\mathfrak{p}_0$ is the sum of $\mathfrak{p}_0$ and $i\mathfrak{p}_0$. We can therefore construct an $\mathrm{Ad}(K)$-equivariant isomorphism

$$\psi\colon \mathfrak{g}_0 \to \mathfrak{p}, \qquad \psi(Z) = \frac{1}{2}(Z - \theta Z) - \frac{i}{2}(JZ - \theta JZ). \tag{2.10)(c}$$

The map ψ identifies $\mathfrak{p}_0 \subset \mathfrak{g}_0$ with the real part of $\mathfrak{p}$, and sends $\mathfrak{k}_0 \subset \mathfrak{g}_0$ isomorphically onto the imaginary part of $\mathfrak{p}$. One calculates immediately that

$$\psi(JZ) = i\psi(Z); \tag{2.10)(d}$$

that is, that ψ is complex-linear. By dualizing and so on one can easily construct from ψ another $\mathrm{Ad}(K)$-equivariant isomorphism

$$\psi\colon i\mathfrak{g}_0^* \to (\mathfrak{g}/\mathfrak{k})^*, \qquad \psi(JZ) = i\psi(Z). \tag{2.10)(e}$$

Now the group G is naturally isomorphic to the complexification $K_{\mathbb{C}}$ of K. Because the action of K on $(\mathfrak{g}/\mathfrak{k})^*$ respects the complex structure, it extends naturally and uniquely to a holomorphic action of $K_{\mathbb{C}}$; we already used this in (2.1). The action of G on $i\mathfrak{g}_0^*$ is holomorphic for the complex structure defined by J. It therefore follows from (2.1)(e) that the map ψ carries the action of G to the action of $K_{\mathbb{C}}$, using the isomorphism of G with $K_{\mathbb{C}}$. In particular,

$$\psi\colon \mathcal{N}_{\mathbb{R}}^* \xrightarrow{\simeq} \mathcal{N}_{\theta}^*, \tag{2.10)(f}$$

a K-equivariant diffeomorphism carrying G orbits to $K_{\mathbb{C}}$ orbits. By calculation in $SL(2,\mathbb{C})$, one sees that this diffeomorphism implements the Sekiguchi bijection of Theorem 2.7. We saw in Lemma 2.4 that the metaplectic covers were all trivial in this case, so that $\mathbb{R}$-admissible orbit data were identified with G-equivariant local systems on orbits. Similarly, the isomorphism (2.10)(f) guarantees that the $K_{\mathbb{C}}$ orbits are all symplectic, so the characters $\gamma(\lambda)$ of Definition 2.6 are all trivial, and θ-admissible orbit data are identified with $K_{\mathbb{C}}$-equivariant local systems on orbits. The identification of Theorem 2.8 is the obvious one given by (2.10)(f). Finally, since all the orbits have even complex dimension, the codimension conditions in Conjectures 2.5 and 2.9 are automatic. Here is how the conjectures finally look for complex groups.

Conjecture 2.11. Suppose G is a complex reductive Lie group, and $X = G\cdot\lambda \simeq G/G(\lambda)$ is a nilpotent coadjoint orbit. Suppose we are given an irreducible G-equivariant local system on X; equivalently, an irreducible representation χ, V_χ of the finite group $G(\lambda)/G(\lambda)_0$, or an indecomposable G-equivariant holomorphic vector $\mathcal{V}_\chi$ on X with a flat connection. Then there is attached to χ an irreducible unitary representation $\pi(\lambda,\chi)$ of G. The space of K-finite vectors of $\pi(\lambda,\chi)$ is isomorphic to the space of algebraic sections of the bundle $\mathcal{V}_\chi$.

We are not aware of counterexamples to Conjecture 2.11.

3. McGovern's method for computing $R(X)$.

In this section we will recall from [McGmult] McGovern's method for calculating rings of functions on nilpotent orbits. As an excuse for repeating these known results, we will formulate the ideas in the more general context of Conjecture 2.9, even though the critical vanishing theorems of McGovern have not been proven there. McGovern's method appears to depend on working not with nilpotent elements of $\mathfrak{g}^*$, but rather with nilpotent elements of $\mathfrak{g}$ itself. An invariant bilinear form on $\mathfrak{g}$ identifies these two pictures, so for calculations the distinction is not significant; but philosophically it is a serious weakness.

At any rate, we begin with the nilpotent cone

$$\mathcal{N}_\theta = \{e \in \mathfrak{p} \mid \ \mathrm{ad}(e) \text{ is nilpotent }\}. \tag{3.1)(a}$$

(Recall from (2.1) that $\mathfrak{p}_0$ is the -1 eigenspace of θ on $\mathfrak{g}_0$.) Elements of $\mathcal{N}_\theta$ may be called *θ-nilpotent.* We fix an $\mathrm{Ad}(G)$-invariant, θ-invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}_0$, positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$. This form provides an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ that carries $\mathcal{N}_\theta$ $K_{\mathbb{C}}$-equivariantly onto $\mathcal{N}_\theta^*$ (see for example [Vunip], Corollary 5.11). We could of course define $\mathcal{N}_{\mathbb{R}}$ in a parallel way, and find a G-equivariant isomorphism $\mathcal{N}_{\mathbb{R}} \simeq \mathcal{N}_{\mathbb{R}}^*$; but we will have no particular need for this.

We fix now a θ-nilpotent element

$$e \in \mathcal{N}_\theta \subset \mathfrak{p}. \tag{3.1)(b}$$

According to a version of the Jacobson-Morozov theorem due to Kostant and Rallis, we can find another θ-nilpotent element $f \in \mathfrak{p}$ and a semisimple element $h \in \mathfrak{k}$ so that

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h \tag{3.1)(c}$$

([KR], page 767). This means that there is a Lie algebra homomorphism

$$\phi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}, \qquad \phi\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = ah + be + cf. \tag{3.1)(d}$$

(After replacing e, h, and f by conjugates under $\mathrm{Ad}(K_{\mathbb{C}})$, it is possible to arrange

$$\overline{h} = -h, \qquad \overline{e} = f, \qquad \overline{f} = e.$$

Here bar denotes the complex conjugation for the real form $\mathfrak{g}_0$. In this case the map ϕ carries the real form $\mathfrak{su}(1,1)$ into $\mathfrak{g}_0$. This idea is at the heart of the proof of Theorem 2.7, but we will have no explicit need of it.) The operator $\mathrm{ad}(h)$ has integer eigenvalues (by the representation theory of $\mathfrak{sl}(2)$). Accordingly we may define

$$\mathfrak{g}(m) = \{z \in \mathfrak{g} \mid [h,z] = mz\}, \quad \mathfrak{k}(m) = \mathfrak{g}(m) \cap \mathfrak{k}, \quad \mathfrak{p}(m) = \mathfrak{g}(m) \cap \mathfrak{p}. \tag{3.1)(e}$$

These spaces define gradings of $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{p}$. We are particularly interested in the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{k}$ defined by

$$\mathfrak{q} = \sum_{m \geq 0} \mathfrak{k}(m), \qquad \mathfrak{u} = \sum_{m > 0} \mathfrak{k}(m), \qquad \mathfrak{l} = \mathfrak{k}(0).$$

Write $Q = L_{\mathbb{C}} U$ for the corresponding parabolic subgroup of $K_{\mathbb{C}}$. Here we define

$$L_{\mathbb{C}} = \{k \in K_{\mathbb{C}} \mid \mathrm{Ad}(k)h = h\}, \tag{3.1)(f}$$

and let U be the connected subgroup with Lie algebra $\mathfrak{u}$. (When h satisfies the condition $\overline{h} = -h$, then $L_{\mathbb{C}}$ is actually the complexification of the compact group

$$L = \{k \in K \mid \mathrm{Ad}(k)h = h\},$$

but in general the subscript $\mathbb{C}$ has no particular meaning.) Finally, we define

$$\mathfrak{v} = \sum_{m \geq 2} \mathfrak{p}(m). \tag{3.1)(g}$$

McGovern's idea is to study the action of $K_{\mathbb{C}}$ on $K_{\mathbb{C}} \cdot e$ by first studying $Q \cdot e$, then "inducing" from Q to $K_{\mathbb{C}}$. Here are the structural results necessary for this program.

Proposition 3.2. *Suppose we are in the setting (3.1); write $K_{\mathbb{C}}(e)$ for the stabilizer of e in the adjoint action.*

1. *The subgroup $L_{\mathbb{C}}(e)$ is equal to the stabilizer in $K_{\mathbb{C}}$ of the image of ϕ. It is therefore a reductive subgroup of $K_{\mathbb{C}}$.*
2. *The subgroup $U(e)$ is a connected unipotent algebraic group.*
3. *There is a Levi decomposition $K_{\mathbb{C}}(e) = L_{\mathbb{C}}(e)U(e)$. In particular, this isotropy group is contained in Q, and*

$$K_{\mathbb{C}}(e)/K_{\mathbb{C}}(e)_0 \simeq L_{\mathbb{C}}(e)/L_{\mathbb{C}}(e)_0.$$

4. *The orbit $Q \cdot e$ is a dense open subvariety of $\mathfrak{v}$.*

Proof. Parts 1–3 are essentially proved in [BV], Proposition 2.4. (What appears there is the complex analogue, but the present result follows easily.) Part 4 is at least implicit in [KR]; here is the argument. Clearly $\mathfrak{v}$ is preserved by Q, and $e \in \mathfrak{v}$ by (3.1)(e) and (3.1)(g). The claim of part 4 is therefore equivalent to $\mathrm{ad}(\mathfrak{q})(e) = \mathfrak{v}$. This in turn may be formulated as

$$\mathrm{ad}(e)\colon \mathfrak{k}(m) \to \mathfrak{p}(m+2) \text{ is surjective for } m \geq 0.$$

This last statement follows from the representation theory of $\mathfrak{sl}(2)$. □

Here is McGovern's construction. Because Q acts algebraically on the smooth variety $\mathfrak{v}$, we can form the fiber product

$$Z = K_{\mathbb{C}} \times_Q \mathfrak{v} = (K_{\mathbb{C}} \times \mathfrak{v})/\sim \tag{3.3)(a}$$

Here $\sim$ is the equivalence relation

$$(x, z) \sim (x', z') \Leftrightarrow x = x'q \text{ and } z' = \mathrm{Ad}(q)z \qquad (\text{some } q \in Q). \tag{3.3)(b}$$

The space Z is a smooth variety with an action of $K_{\mathbb{C}}$; it is the total space of a homogeneous vector bundle on the flag variety $K_{\mathbb{C}}/Q$. (A brief general discussion of this construction may be found in an appendix to [VDix].) It follows from (3.3)(b) that

$$(x, z) \sim (x', z') \Rightarrow \mathrm{Ad}(x)z = \mathrm{Ad}(x')z'. \tag{3.3)(c}$$

We therefore have an algebraic map

$$\pi\colon Z \to \mathcal{N}_\theta, \qquad (x, z) \mapsto \mathrm{Ad}(x)z. \tag{3.3)(d}$$

(Notice that $\mathfrak{v}$ obviously consists of nilpotent elements in $\mathfrak{p}$, and therefore the entire image of π does as well.)

Theorem 3.4. *Suppose we are in the setting of* (3.1) *and* (3.3). *Then the map* π *is proper and birational, with image equal to the closure of* $K_{\mathbb{C}} \cdot e$. *Consequently the algebra* $R(Z)$ *of regular functions on* Z *is naturally isomorphic to the normalization of* $R(\overline{K_{\mathbb{C}} \cdot e})$.

Assume in addition that the boundary of the orbit closure $\overline{K_{\mathbb{C}} \cdot e}$ *has complex codimension at least two. Then* $R(Z)$ *is naturally isomorphic to the algebra* $R(K_{\mathbb{C}} \cdot e)$ *of regular functions on the orbit.*

Sketch of proof. That π is proper is a consequence of the properness of the flag variety $K_{\mathbb{C}}/Q$ and the properness of the inclusion $\mathfrak{v} \hookrightarrow \mathcal{N}_\theta$ ([VDix], Proposition A.2). Consequently the image is closed. By Proposition 3.2.4, it contains $K_{\mathbb{C}} \cdot e$ as a dense subvariety. That π is birational (more precisely, bijective over $K_{\mathbb{C}} \cdot e$) is equivalent (by [VDix], ($A.3$)) to the fact that $K_{\mathbb{C}}(e) \subset Q$, proved in Proposition 3.2.3. The remaining statements are proved exactly as in [McGmult], Theorem 3.1. □

Next, we want to understand how Theorem 3.4 might allow us to compute rings of functions. The answer requires a little notation. It is convenient to change the meaning of G for a moment here.

Definition 3.5. Suppose $G \supset Q$ are algebraic groups, and (δ, E) is an algebraic representation of Q. (This means that each element of E belongs to a finite-dimensional Q-invariant subspace E_0, and that the restriction $\delta_0 \colon Q \to GL(E_0)$ is a morphism of algebraic groups.) Form the fiber product $\mathcal{E} = G \times_Q E$, a vector bundle over G/Q. We may identify $\mathcal{E}$ with the corresponding quasicoherent sheaf on G/Q. Here is a description of the sheaf. Suppose U is an open set in G/Q; identify U with an open set $\widetilde{U}$ in G closed under right multiplication by Q. Then the space of sections of $\mathcal{E}$ over U is

$$\mathcal{E}(U) = \{f \colon \widetilde{U} \to E \mid f \text{ is algebraic, and } f(xh) = \delta^{-1}(h) f(x)\}.$$

We therefore have sheaf cohomology groups $H^p(G/Q, \mathcal{E})$; these carry natural algebraic representations of G, which we say are *cohomologically induced* from Q to G. We write

$$(\mathrm{Ind}_Q^G)^p(E) = H^p(G/Q, \mathcal{E}).$$

When $p = 0$ we may omit the superscript. In particular

$$\mathrm{Ind}_Q^G(E) = \{f \colon G \to E \mid f \text{ is algebraic, and } f(xh) = \delta^{-1}(h) f(x)\}.$$

Proposition 3.6. *Suppose* $G \supset Q$ *are algebraic groups, and that* Z_Q *is an algebraic variety with an action of* Q. *Define* $Z = G \times_Q Z_Q$, *an algebraic variety with an action of* G. *Then there is a natural isomorphism of representations of* G

$$R(Z) \simeq \mathrm{Ind}_Q^G R(Z_Q).$$

This is elementary; some additional discussion of it may be found in [VDix], Proposition A.9.

Corollary 3.7. *In the setting of Theorem 3.4,*

$$R(Z) \simeq \mathrm{Ind}_Q^{K_{\mathbb{C}}}(R(\mathfrak{v})) = \sum_{k=0}^{\infty} \mathrm{Ind}_Q^{K_{\mathbb{C}}}(S^k(\mathfrak{v}^*)).$$

In the presence of the codimension condition of Theorem 3.4, this result begins to look like a computation of the ring of regular functions on an orbit. Roughly speaking, we need two more ingredients: an understanding of the induction functor from Q to $K_{\mathbb{C}}$, and an understanding of $S^k(\mathfrak{v}^*)$ as a representation of Q. The first is provided by the Bott-Borel-Weil theorem (Theorem 3.10 and Corollary 3.11 below). For simplicity we confine our discussion to the case of connected $K_{\mathbb{C}}$. As in Definition 3.5, it is convenient to formulate the general results with different notation; in the application G will be replaced by $K_{\mathbb{C}}$.

Definition 3.8. Suppose G is a connected reductive complex algebraic group, $Q \subset G$ is a parabolic subgroup, and $H \subset Q$ is a maximal torus. Write U for the unipotent radical of Q and L for the Levi subgroup containing H. Choose a system of positive roots $\Delta^+(\mathfrak{l},\mathfrak{h})$ for $\mathfrak{h}$ in $\mathfrak{l}$, so that $\Delta^+(\mathfrak{g},\mathfrak{h}) = \Delta^+(\mathfrak{l},\mathfrak{h}) \cup \Delta(\mathfrak{u},\mathfrak{h})$ is a positive system for $\mathfrak{h}$ in $\mathfrak{g}$. Write $X^*(H) \supset \Delta(\mathfrak{g},\mathfrak{h})$ for the lattice of *weights*, or rational characters of H (morphisms from H to $\mathbb{C}^\times$). Similarly, we write $X_*(H) \supset \check{\Delta}(\mathfrak{g},\mathfrak{h})$ for the dual lattice of *coweights*, or one-parameter subgroups (morphisms from $\mathbb{C}^\times$ to H). If we fix an identification of the Lie algebra of $\mathbb{C}^\times$ with $\mathbb{C}$, then

$$X^*(H) \subset \mathfrak{h}^*, \qquad X_*(H) \subset \mathfrak{h},$$

and these inclusions are compatible with the natural dualities

$$X^*(H) \times X_*(H) \to \mathbb{Z}, \qquad \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}.$$

If we identify coweights with elements of $\mathfrak{h}$ as above, then the coroot $\check{\alpha}$ corresponding to a root α is the element often called h_α, part of a Chevalley basis of the Lie algebra. That is, we can choose root vectors e_α for α and f_α for $-\alpha$ so that

$$[h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = -2f_\alpha, [e_\alpha, f_\alpha] = h_\alpha.$$

The statement that e_α and f_α are root vectors may be expressed as

$$[h, e_\alpha] = \alpha(h)e_\alpha, \quad [h, f_\alpha] = -\alpha(h)f_\alpha \qquad (h \in \mathfrak{h}).$$

A weight $\lambda \in X^*(H)$ is called *G-dominant* (or *$\Delta^+(\mathfrak{g},\mathfrak{h})$-dominant*) if for every positive root $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$, $\lambda(\check{\alpha}) \geq 0$.

In order to state the Bott-Borel-Weil theorem, we need a little notation related to the Weyl group.

Definition 3.9. In the setting of Definition 3.8, define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} \alpha.$$

For $w \in W(\mathfrak{g},\mathfrak{h})$, define

$$\Delta^+(w) = \{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}) \mid w^{-1}\alpha \notin \Delta^+\}.$$

Then the length of w (the length of a minimal expression of w as a product of simple roots) is $l(w) = |\Delta^+(w)|$ ([Hump], Lemma 10.3A). The sign of w (the determinant of the action of w on $\mathfrak{h}$) is $\mathrm{sgn}(w) = (-1)^{l(w)}$. Finally, recall that

$$\rho - w \cdot \rho = \sum_{\alpha \in \Delta^+(w)} \alpha \in X^*(H).$$

Theorem 3.10 ([Bott], page 228, or [Kost], Theorem 6.4). *Suppose we are in the setting of Definition 3.8; use also the notation of Definition 3.9. Suppose $\lambda \in X^*(H)$ is G-dominant, and V_λ is the irreducible algebraic representation of G of highest weight λ. Similarly, suppose $\mu \in X^*(H)$ is L-dominant, and E_μ is the irreducible algebraic representation of L of highest weight μ. Then V_λ^* appears in $(\mathrm{Ind}_Q^G)^p(E_\mu^*)$ if and only if $\mu + \rho = w(\lambda + \rho)$, for some $w \in W(\mathfrak{g}, \mathfrak{h})$ with $l(w) = p$. In this case V_λ^* appears with multiplicity one.*

To apply this result in Corollary 3.7, a slight reformulation is helpful.

Corollary 3.11. *Suppose we are in the setting of Definitions 3.8 and 3.9. Suppose $\lambda \in X^*(H)$ is G-dominant, and V_λ is the irreducible algebraic representation of G of highest weight λ. Suppose E is a finite-dimensional algebraic representation of Q. Then the multiplicity of V_λ^* in the virtual representation $\sum_p (-1)^p (\mathrm{Ind}_Q^G)^p(E^*)$ is equal to the sum over w of*

$$\mathrm{sgn}(w) \cdot (\textit{multiplicity of } E_{w(\lambda+\rho)-\rho} \textit{ in } E).$$

Here the sum runs over w such that $w(\lambda + \rho)$ is L-dominant; E_μ is the irreducible representation of L of highest weight μ; and the multiplicity is as representations of L.

Proof. We proceed by induction on the length of E as a representation of L. If E is an irreducible representation of L, then the corollary is just a weakened restatement of Theorem 3.10. If not, then there is an exact sequence of Q-modules

$$0 \to E_1 \to E \to E_2 \to 0,$$

with E_i non-zero. Consequently $E = E_1 + E_2$ as representations of L. By inductive hypothesis, the multiplicity formula we want is true for E_1 and E_2. The long exact sequence in sheaf cohomology on G/Q gives a long exact sequence of G-modules

$$\cdots \to (\mathrm{Ind}_Q^G)^p(E_2^*) \to (\mathrm{Ind}_Q^G)^p(E^*) \to (\mathrm{Ind}_Q^G)^p(E_1^*) \to (\mathrm{Ind}_Q^G)^{p+1}(E_2^*) \to \cdots$$

By the Euler-Poincaré principle, it follows that

$$\sum_p (-1)^p (\mathrm{Ind}_Q^G)^p(E^*) = \sum_p (-1)^p (\mathrm{Ind}_Q^G)^p(E_1^*) + \sum_p (-1)^p (\mathrm{Ind}_Q^G)^p(E_2^*)$$

as virtual representations of G. The multiplicity formula for E is therefore the sum of the formulas for E_1 and E_2. □

Even though the virtual multiplicity formula in Corollary 3.11 refers only to the structure of E as an L-module, the Q-module structure is very important. It will control such questions as the vanishing of $(\mathrm{Ind}_Q^G)^p(E^*)$ for $p > 0$.

Corollary 3.7 asks us to compute Ind^0, and Corollary 3.11 computes instead an Euler characteristic. To use them together, we would like a vanishing theorem. Here is a special case.

Theorem 3.12 ([McGmult], page 212 and [Hin], Lemma 2.3). *Suppose we are in the setting of* (3.1) *and* (3.3), *and suppose in addition that G is complex. Then*

$$(\mathrm{Ind}_Q^{K_\mathbb{C}})^p(S^k(\mathfrak{v}^*)) = 0, \qquad (p > 0).$$

McGovern proves only that the alternating sum of the higher induced representations is zero; the conclusion that they vanish is due to Hinich (and apparently independently to Panyushev).

It is natural to hope that the vanishing result in Theorem 3.12 extends to the case of real groups; so we make

Conjecture 3.13. Suppose we are in the setting of (3.1) and (3.3). Then

$$(\mathrm{Ind}_Q^{K_\mathbb{C}})^p(S^k(\mathfrak{v}^*)) = 0, \qquad (p > 0).$$

We can make a few remarks about what makes this conjecture difficult. The argument used by McGovern in the complex case, as refined by Hinich, comes down to Lemma 2.3 of [Hin]. The idea is to find an approximate identification of the sheaf of functions (on a nilpotent orbit) with the sheaf of top degree differential forms. Such an identification exists because a coadjoint orbit is symplectic; the top power of the symplectic form trivializes the sheaf of differential forms. An orbit of $K_\mathbb{C}$ on $\mathcal{N}_\theta^*$ need not be symplectic, and the sheaf of differential forms need not be trivial. In fact it is exactly the sheaf of differential forms that enters the definition of admissible orbit datum (Definition 2.6). We will see in (8.5)(b) that this sheaf is non-trivial in the case of the model orbit for $E_8(\mathbb{R})$ considered in sections 7 and 8. Consequently Lemma 2.3 of [Hin] cannot be directly applied to prove Conjecture 3.13. In fact the vanishing theorem of Grauert and Riemenschneider (which is at the heart of Hinich's argument) seems not to offer any help toward Conjecture 3.13.

Combining this vanishing conjecture with Corollary 3.11 leads immediately to the following more explicit version of Corollary 3.7.

Corollary 3.14. *Suppose we are in the setting of* (3.1) *and* (3.3), *and suppose that Conjecture 3.13 holds (for example if G is complex). Fix a maximal torus $H_\mathbb{C} \subset L_\mathbb{C}$ and a system of positive roots $\Delta^+(\mathfrak{l}, \mathfrak{h})$. Extend this to a system of positive roots for $H_\mathbb{C}$ in $K_\mathbb{C}$ as in Definition 3.8, and write ρ_c for half the sum of the positive roots. Fix a $K_\mathbb{C}$-dominant weight $\lambda \in X^*(H_\mathbb{C})$, and write V_λ for the corresponding irreducible representation of $K_\mathbb{C}$. Then the multiplicity of V_λ^* in the ring of regular functions on the normalization of $\overline{K_\mathbb{C} \cdot e}$ is equal to the sum over w of*

$$\mathrm{sgn}(w) \cdot (\textit{multiplicity of } E_{w(\lambda+\rho_c)-\rho_c} \textit{ in } S(\mathfrak{v})).$$

Here the sum is over $w \in W(\mathfrak{k}, \mathfrak{h})$ such that $w(\lambda + \rho_c)$ is $L_\mathbb{C}$-dominant.

Conjecture 2.9 suggests that one ought to study not the ring of functions on $K_\mathbb{C} \cdot e$, but rather certain spaces of sections of line bundles. McGovern's approach applies to these as well ([McGmult], §4). Here is the idea.

Proposition 3.15. *In the setting of* (3.1) *and* (3.3), *suppose χ is an algebraic representation of $K_\mathbb{C}(e) = Q(e)$; write $\mathcal{V}_\chi$ for the corresponding vector bundle on*

$K_{\mathbb{C}} \cdot e$ (compare Definition 2.6). Assume that there is an algebraic representation δ of Q such that $\delta|_{Q(e)} \simeq \chi$. Then there is a natural inclusion

$$\sum_{k=0}^{\infty} \operatorname{Ind}_Q^{K_{\mathbb{C}}}(S^k(\mathfrak{v}^*) \otimes V_\delta) \hookrightarrow \textit{ sections of } \mathcal{V}_\chi \textit{ over } K_{\mathbb{C}} \cdot e.$$

The proof is parallel to that of Corollary 3.7; we omit the argument (and the precise formulation of an analogue of Corollary 3.14.) Of course the more difficult and interesting questions are left unresolved. (For which χ do representations δ exist? Is it possible to choose δ so that the inclusions of Proposition 3.15 are isomorphisms? So that the higher cohomologically induced representations vanish?)

4. The model orbit in $E_8(\mathbb{C})$.

In this section we will give an explicit description of the orbit we wish to study. In this section G will always be a complex algebraic group of type E_8. Instead of looking at the action of $K_{\mathbb{C}}$ on $\mathfrak{p}$, we will look at the action of G on $\mathfrak{g}_0$; this is equivalent by (2.10). Otherwise our notation is more or less consistent with that of (3.1). We fix a maximal torus H of G, with Lie algebra $\mathfrak{h}_0$. (Recall from the beginning of section 2 that the subscript zero means we have not complexified the Lie algebra. Of course it is a complex Lie algebra, because of the complex structure on G.) There is a standard identification (as in [Hump], page 65, or [Bour], chapter IV, §4.10)

$$\mathfrak{h}_0^* \simeq \mathbb{C}^8 \qquad (4.1)(a)$$

with the property that the root system of $\mathfrak{h}_0$ in $\mathfrak{g}_0$ is

$$\Delta(\mathfrak{g}_0, \mathfrak{h}_0) = \{\pm e_i \pm e_j \mid 1 \le i \ne j \le 8\} \cup \{\frac{1}{2}(\epsilon_1, \cdots, \epsilon_8) \mid \epsilon_i = \pm 1, \prod_i \epsilon_i = 1\}. \qquad (4.1)(b)$$

Here $\{e_i\}$ is the standard basis of $\mathbb{C}^8$. The first collection of roots described therefore has $4\binom{8}{2} = 112$ elements, and the second has $2^8/2 = 128$. Altogether there are $112 + 128 = 240$ roots, so that G has complex dimension $8 + 240 = 248$. For the calculations we wish to do, these coordinates are slightly inconvenient. The reason is this. By inspection of the Dynkin diagram, it is clear that $\mathfrak{g}_0$ has a Levi subalgebra of type A_7; that is, isomorphic to $\mathfrak{gl}(8, \mathbb{C})$. We will make extensive use of this subalgebra, and so we will need to identify its roots in our chosen coordinates. Of course the root system of $\mathfrak{gl}(8, \mathbb{C})$ has a standard presentation, and it is natural to look for this standard form inside the root system for E_8. It is there: the 56 roots $e_i - e_j$. These roots do provide a subalgebra of $\mathfrak{g}_0$ isomorphic to $\mathfrak{gl}(8, \mathbb{C})$, but it is *not* the Levi subalgebra we want. (Here is one way to see this. Any root in the $\mathbb{R}$-span of the roots of a Levi subalgebra must actually be a root for that Levi subalgebra. The root $\frac{1}{2}(1, 1, 1, 1, -1, -1, -1, -1)$ belongs to the $\mathbb{R}$-span of the $e_i - e_j$, but is not one of them.)

We therefore modify the isomorphism $(4.1)(a)$ by composing it with the automorphism of $\mathbb{C}^8$ that changes the sign of the first coordinate. In this new identification

$$\mathfrak{h}_0^* \simeq \mathbb{C}^8, \qquad (4.2)(a)$$

the root system of $\mathfrak{h}_0$ in $\mathfrak{g}_0$ is

$$\Delta(\mathfrak{g}_0,\mathfrak{h}_0) = \{\pm e_i \pm e_j \mid 1 \le i \ne j \le 8\} \cup \{\frac{1}{2}(\epsilon_1,\cdots,\epsilon_8) \mid \epsilon_i = \pm 1, \prod_i \epsilon_i = -1\}. \tag{4.2)(b}$$

The standard bilinear form on $\mathbb{C}^8$ is invariant by the Weyl group $W = W(\mathfrak{g}_0,\mathfrak{h}_0)$. It therefore provides an identification $\mathfrak{h}_0 \simeq \mathbb{C}^8$. Using this identification, we define

$$h = (1,1,1,1,1,1,1,1) \in \mathfrak{h}_0. \tag{4.2)(c}$$

Write L for the subgroup of G fixing h in the adjoint action. Then L is a Levi subgroup of G, with root system

$$\Delta(\mathfrak{l}_0,\mathfrak{h}_0) = \{\alpha \in \Delta(\mathfrak{g}_0,\mathfrak{h}_0) \mid \alpha(H) = 0\} = \{e_i - e_j \mid 1 \le i \ne j \le 8\}. \tag{4.2)(d}$$

Each root α gives rise to an algebraic homomorphism

$$\phi_\alpha \colon SL(2,\mathbb{C}) \to G, \tag{4.3)(a}$$

with the property that ϕ_α carries diagonal matrices into H and upper triangular unipotent matrices into the root subgroup for α. These requirements characterize ϕ_α up to conjugation by a diagonal element in $SL(2,\mathbb{C})$. In particular, the coroot

$$h_\alpha = d\phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.3)(b}$$

is uniquely defined. In our coordinate system, the identification of both $\mathfrak{h}_0^*$ and $\mathfrak{h}_0$ with $\mathbb{C}^8$ makes $\alpha = h_\alpha$. We are interested in the four orthogonal roots

$$\alpha_1 = e_1 + e_2, \quad \alpha_2 = e_3 + e_4, \quad \alpha_3 = e_5 + e_6, \quad \alpha_4 = e_7 + e_8. \tag{4.3)(c}$$

The corresponding root $SL(2)$ subgroups commute with each other, and so define a homomorphism of $SL(2,\mathbb{C})^4$ into G. We are interested in the restriction of this homomorphism to the diagonal:

$$\phi \colon SL(2,\mathbb{C}) \to G, \qquad \phi(x) = \phi_{\alpha_1}(x)\phi_{\alpha_2}(x)\phi_{\alpha_3}(x)\phi_{\alpha_4}(x) \tag{4.3)(d}$$

By the remarks after (4.3)(b),

$$d\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h_{\alpha_1} + h_{\alpha_2} + h_{\alpha_3} + h_{\alpha_4} = h, \tag{4.3)(e}$$

the element considered in (4.2)(c). We can now write down the nilpotent element we want to study:

$$e = d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.3)(f}$$

the sum of root vectors for the four orthogonal roots α_i.

In order to calculate the Dynkin diagram of the nilpotent element e, we need to choose a system of positive roots making h dominant. An obvious choice is

$$\Delta^+(\mathfrak{g}_0,\mathfrak{h}_0) = \{e_i - e_j \mid i < j\} \cup \{\alpha \mid \alpha(h) > 0\}. \tag{4.4)(a}$$

It is easy to determine the simple roots for this positive system: they are

$$\Pi(\mathfrak{g}_0, \mathfrak{h}_0) = \{e_i - e_{i+1} \mid 1 \leq i \leq 7\} \cup \{\frac{1}{2}(-1,-1,-1,1,1,1,1,1)\}. \qquad (4.4)(b)$$

We label these simple roots in the order listed as $\{\beta_1, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_2\}$. (This is the numbering used in [Bour] and [Hump].) Then the Dynkin diagram of G is

$$\begin{array}{ccccccccccccc} & & & & \beta_2 & & & & & & & & \\ & & & & | & & & & & & & & \\ \beta_1 & — & \beta_3 & — & \beta_4 & — & \beta_5 & — & \beta_6 & — & \beta_7 & — & \beta_8 \end{array} \qquad (4.4)(c)$$

To get the weighted Dynkin diagram of the nilpotent element e, we replace each vertex β by the non-negative integer $\beta(h)$. These integers may be calculated immediately from $(4.2)(c)$ and $(4.4)(c)$; the result is

$$\begin{array}{ccccccccccccc} & & & & 1 & & & & & & & & \\ & & & & | & & & & & & & & \\ 0 & — & 0 & — & 0 & — & 0 & — & 0 & — & 0 & — & 0 \end{array} \qquad (4.4)(d)$$

Proposition 4.5. *In the setting of* (4.2)–(4.4)*, let* $X = G \cdot e$ *be the adjoint orbit of the element* e*. Then* X *is the unique nilpotent orbit of complex dimension* 128 *in* $\mathfrak{g}_0$*, and its weighted Dynkin diagram is given by* $(4.4)(d)$*. The stabilizer* $L(e)$ *of* e *in* L *is isomorphic to* $Sp(8, \mathbb{C})$*. In particular,* $L(e)$ *and* $G(e)$ *are connected.*

Proof. Because e belongs to the 2-eigenspace of $\operatorname{ad} h$, it is certainly nilpotent. We have already computed the Dynkin diagram of e. Once the diagram is known, that X is the unique nilpotent orbit of dimension 128 may be read off from published tables (for example [Coll], page 132; [Cart], 13.1; or [Dynk], Table 20). (We will see in (4.6) how to calculate that X has dimension 128.) Here we will only outline the calculation of $L(e)$. We will see in Lemma 5.2 that the representation of L on $\mathfrak{g}_0(2)$ is $\wedge^2(\tau)$, the second exterior power of the standard representation. The element e has an open L orbit in this representation (see for example [Coll], page 42); so we want to calculate the stabilizer in L of a generic element of $\wedge^2(\tau)$. In $GL(8)$ this stabilizer is $Sp(8)$ more or less by definition (since $\wedge^2(V)$ may be identified with skew-symmetric bilinear forms on V^*). The group L differs slightly from $GL(8)$ (see the remarks at (5.1)) but nevertheless $L(e)$ is still isomorphic to $Sp(8)$. We will carry out the analogous calculation in the real case in detail in section 8. Now the connectedness of $G(e)$ follows from Proposition 3.2.3. □

We will study the group G using a parabolic subgroup with Levi factor L. The operator $\operatorname{ad}(h)$ has integer eigenvalues on $\mathfrak{g}_0$, and we define

$$\mathfrak{g}_0(m) = \{z \in \mathfrak{g}_0 \mid [h, z] = mz\}. \qquad (4.6)(a)$$

Each of these spaces is a representation of the group L, whose Lie algebra is precisely $\mathfrak{g}_0(0)$. For $m \neq 0$, $\mathfrak{g}_0(m)$ is a sum of root spaces, for the roots α such that $\alpha(h) = m$. These sets of roots are easily calculated. First,

$$\Delta(\mathfrak{g}_0(1), \mathfrak{h}_0) = \{\frac{1}{2}(\epsilon_1, \ldots, \epsilon_8) \mid \epsilon_i = \pm 1, \epsilon_i = 1 \text{ for exactly 5 values of } i\}. \quad (4.6)(b)$$

This set has $\binom{8}{3} = 56$ elements, which is therefore the complex dimension of $\mathfrak{g}_0(1)$. It is a general fact about nilpotent elements (see [Coll], Lemma 4.1.3, or [Cart]) that $\dim \mathfrak{g}_0(e) = \dim \mathfrak{g}_0(0) + \dim \mathfrak{g}_0(1)$, and therefore that $\dim G \cdot e = \dim G - \dim \mathfrak{g}_0(0) - \dim \mathfrak{g}_0(1)$. In our case, $\dim X = 248 - 64 - 56 = 128$, as we had already claimed in Proposition 4.5. Next,

$$\Delta(\mathfrak{g}_0(2), \mathfrak{h}_0) = \{e_i + e_j \mid i \neq j\}. \tag{4.6)(c}$$

This set has cardinality $\binom{8}{2} = 28$. Finally,

$$\Delta(\mathfrak{g}_0(3), \mathfrak{h}_0) = \{\frac{1}{2}(\epsilon_1, \dots, \epsilon_8) \mid \epsilon_i = \pm 1, \epsilon_i = 1 \text{ for exactly 7 values of } i\}. \tag{4.6)(d}$$

This set has cardinality $\binom{8}{1} = 8$. Now define

$$\mathfrak{u}_0 = \sum_{m \geq 1} \mathfrak{g}_0(m), \qquad \mathfrak{q} = \sum_{m \geq 0} \mathfrak{g}_0(m) = \mathfrak{l}_0 + \mathfrak{u}_0. \tag{4.6)(e}$$

Then $\mathfrak{q}_0$ is a parabolic subalgebra of $\mathfrak{g}_0$, corresponding to a parabolic subgroup with Levi decomposition $Q = LU$. We will also need the subalgebra

$$\mathfrak{v}_0 = \sum_{m \geq 2} \mathfrak{g}_0(m) = \mathfrak{g}_0(2) + \mathfrak{g}_0(3). \tag{4.6)(f}$$

According to Theorem 3.4, the vector bundle

$$Z = G \times_Q \mathfrak{v}_0 \tag{4.6)(g}$$

provides a resolution of singularities for the orbit closure $\overline{X} \subset \mathfrak{g}_0$. Applying McGovern's multiplicity formula (Corollary 3.14) to this case, we find

Theorem 4.7. *Suppose we are in the setting of* (4.2)–(4.4) *and* (4.6). *Fix a G-dominant weight $\lambda \in X^*(H)$, and write V_λ for the corresponding irreducible representation of G. Then the multiplicity of V_λ^* in the ring of regular functions on $G \cdot e$ is equal to the sum over w of*

$$\operatorname{sgn}(w) \cdot (\text{multiplicity of } E_{w(\lambda+\rho)-\rho} \text{ in } S(\mathfrak{v}_0)).$$

Here ρ is half the sum of the positive roots, and the sum is over $w \in W(\mathfrak{g}, \mathfrak{h})$ such that $w(\lambda + \rho)$ is L-dominant.

In order to make this formula more explicit, we need to calculate $S(\mathfrak{v}_0)$ as a representation of L. This we will do in the following section.

5. Decomposition of $S(\mathfrak{v}_0)$.

In this section we will work in the setting of (4.2)–(4.4) and (4.6), and calculate $S(\mathfrak{v}_0)$ as a representation of L. Because $\mathfrak{l}_0 \simeq \mathfrak{gl}(8)$, this is entirely a classical invariant theory problem, and it can be solved by standard methods. We fix an isomorphism

$$j \colon \mathfrak{gl}(8) \to \mathfrak{l}_0 \tag{5.1)(a}$$

carrying diagonal matrices to $\mathfrak{h}_0 \simeq \mathbb{C}^8$ in the obvious way. It is important to remember that the group L is *not* precisely isomorphic to $GL(8)$. (One can construct L as a two-fold cover of $GL(8)$ modulo a two-element central subgroup; details appear at (8.1) below.) The set of dominant weights for $\mathfrak{gl}(8)$ is

$$\{\lambda = (\lambda_1, \dots, \lambda_8) \in \mathbb{C}^8 \mid \lambda_i - \lambda_j \text{ is a non-negative integer for } i < j\}. \qquad (5.1)(b)$$

We write $(\tau, \mathbb{C}^8_\tau)$ for the standard (tautological) representation of $\mathfrak{l}_0$; its weights are

$$\Delta(\tau, \mathfrak{h}_0) = \{e_i \mid i = 1, \dots, 8\}. \qquad (5.1)(c)$$

The highest weight is $(1, 0, \dots, 0)$. Similarly, the dual representation τ^* has weights

$$\Delta(\tau^*, \mathfrak{h}_0) = \{-e_i \mid i = 1, \dots, 8\}. \qquad (5.1)(d)$$

The one-dimensional determinant character $(\det, \mathbb{C}_{\det})$ has weight

$$\Delta(\det, \mathfrak{h}_0) = \{(1, \dots, 1)\}. \qquad (5.1)(e)$$

(We write det to suggest the group representation; on the Lie algebra the definition is $\det(x) = \text{trace of } x$.) Even though we cannot extract the square root of the determinant on $GL(8)$, there is a well-defined Lie algebra representation $(\det^{1/2}, \mathbb{C}_{\det^{1/2}})$, with weight

$$\Delta(\det^{1/2}, \mathfrak{h}_0) = \{(1/2, \dots, 1/2)\}. \qquad (5.1)(f)$$

With these calculations in hand, we can easily determine the weights of other simple representations. For example, the second exterior power of the standard representation has weights

$$\Delta(\wedge^2(\tau), \mathfrak{h}_0) = \{e_i + e_j \mid 1 \leq i \neq j \leq 8\}. \qquad (5.1)(g)$$

The highest weight is $(1, 1, 0 \dots, 0)$. Similarly, τ^* twisted by $\det^{1/2}$ has weights

$$\Delta(\tau^* \otimes \det^{1/2}, \mathfrak{h}_0) = \{\frac{1}{2}(\epsilon_1, \dots, \epsilon_8) \mid \epsilon_i = \pm 1, \epsilon_i = 1 \text{ for exactly 7 values of } i\}. \qquad (5.1)(h)$$

Now an algebraic representation of a reductive group is determined up to equivalence by its set of weights with multiplicities. Comparing (5.1) with (4.6), we deduce

Lemma 5.2. *Suppose we are in the setting of* (4.6). *In terms of the isomorphism* (5.1)(a), *the representation of* $\mathfrak{l}_0$ *on* $\mathfrak{v}_0$ *is* $\wedge^2(\tau) \oplus \tau^* \otimes \det^{1/2}$. *Consequently*

$$S(\mathfrak{v}_0) \simeq S(\wedge^2 \mathbb{C}^8_\tau) \otimes \left(\sum_{k=0}^{\infty} S^k(\mathbb{C}^8_{\tau^*}) \otimes \det^{k/2} \right).$$

In light of Lemma 5.2, we are left with two tasks: decomposing the symmetric algebra of $\wedge^2(\tau)$ (as a representation of $\mathfrak{gl}(8)$), and decomposing the tensor product of an arbitrary representation with the symmetric algebra of τ^*. (The twist by the one-dimensional character $\det^{k/2}$ is easy to handle.) Here are the results.

Proposition 5.3 ([Sch], page 63, Behauptung c)). *In the setting* (5.1), *the symmetric algebra* $S(\wedge^2\tau)$ *is the direct sum of the representations of* $GL(8)$ *of highest weights* (a, a, b, b, c, c, d, d), *with* $a \geq b \geq c \geq d \geq 0$ *integers.*

Proof. Consider the Hermitian symmetric space $SO^*(2n)/U(n)$ (see for example [Helg], page 445, or [Knapp], page 6). The holomorphic tangent space $\mathfrak{p}_+$ at the identity coset is isomorphic to the second exterior power $\wedge^2(\tau_n)$ of the standard representation of $U(n)$. (This may be seen by explicit calculation with matrices, or by inspection of the root systems of $SO(2n)$ and $U(n)$.) The content of the result of Schmid cited in the statement is a decomposition of $S(\mathfrak{p}_+)$ as a representation of K for any Hermitian symmetric space G/K. In the case of $SO^*(4m)/U(2m)$, Schmid's result is that the representations of $U(2m)$ on $S(\wedge^2\tau_{2m})$ are those of highest weights $(a_1, a_1, \ldots, a_m, a_m)$, with (a_i) a decreasing sequence of non-negative integers. Taking $m = 4$ gives the proposition. (We use also the equivalence of categories between finite-dimensional representations of $U(n)$ and algebraic representations of $GL(n, \mathbb{C})$.) □

Proposition 5.4 (Pieri's formula; see for example [Mac], Chapter I, (5.12) (page 73)). *Suppose* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *is a dominant weight for* $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$; *that is, that* $\lambda_i - \lambda_j$ *is a non-negative integer whenever* $i < j$. *Let* V_λ *be the irreducible representation of* $\mathfrak{g}$ *of highest weight* λ, *and let* τ *be the* n*-dimensional standard representation. Then the tensor product* $V_\lambda \otimes S^k(\tau_n)$ *is the direct sum of the representations* V_μ, *where* $\mu \in \mathbb{R}^n$ *satisfies*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n, \qquad \mu - \lambda \in \mathbb{Z}^n, \qquad \sum_i \mu_i - \lambda_i = k.$$

Each of these constituents V_μ *appears with multiplicity one.*

The formula stated in Macdonald's book is about multiplying Schur functions. Because Schur functions are characters for $GL(n)$ ([Mac], page 163), it amounts to a statement about the tensor product of two representations. To make this translation, one needs also the formula (3.9) on page 42 of [Mac], and the discussion on the top of page 5 there. What emerges when the dust has settled is Proposition 5.4 in the case that λ_n is a non-negative integer. The general case follows at once by tensoring with powers of the determinant character.

(We are grateful to Chris Woodward for showing us a beautiful geometric proof of Proposition 5.4. He later found his proof in a paper of Brion, who in turn refers to Macdonald.)

The setting of Lemma 5.2 actually involves the symmetric algebra of the dual of the standard representation. It is a simple matter to take duals of everything in Proposition 5.4; the result is

Proposition 5.5 (Pieri's formula). *Suppose* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *is a dominant weight for* $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. *Let* V_λ *be the irreducible representation of* $\mathfrak{g}$ *of highest weight* λ, *and let* τ *be the* n*-dimensional standard representation. Then the tensor product* $V_\lambda \otimes S^k(\tau_n^*)$ *is the direct sum of the representations* V_μ, *where* $\mu \in \mathbb{R}^n$ *satisfies*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq \mu_n, \qquad \mu - \lambda \in \mathbb{Z}^n, \qquad \sum_i \lambda_i - \mu_i = k.$$

Each of these constituents V_μ *appears with multiplicity one.*

The decomposition of $S(\mathfrak{v}_0)$ is now straightforward. Here is the result.

Proposition 5.6. *Suppose we are in the setting of* (4.6)*; use the notation of* (5.1)*. Suppose* $\lambda = (\lambda_1, \dots, \lambda_8) \in \mathbb{C}^8$ *is a dominant weight for* $\mathfrak{l}_0 \simeq \mathfrak{gl}(8)$*; write* E_λ *for the irreducible representation of highest weight* λ*. Then the multiplicity of* E_λ *in* $S(\mathfrak{v}_0)$ *is equal to* 1 *if*

$$\lambda_i - \lambda_{i+1} \text{ is a non-negative integer for } i = 1, \dots, 7, \text{ and}$$

$$\frac{1}{2}(-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) \text{ is a non-negative integer.}$$

Otherwise the multiplicity is zero. In terms of the eight simple roots β_i *of* (4.4)(c) *and the corresponding coroots* h_i*, the conditions for multiplicity one may be expressed as*

$$\lambda(h_i) \in \mathbb{N} \qquad (i = 1, 3, 4, 5, 6, 7, 8), \qquad \lambda(h_2 + h_3 + h_4 + h_5) \in \mathbb{N}.$$

Proof. We use Lemma 5.2 to describe $S(\mathfrak{v}_0)$. According to Proposition 5.3, the highest weights of $S(\wedge^2\tau)$ are those of the form (a, a, b, b, c, c, d, d), with (a, b, c, d) a decreasing sequence of non-negative integers. According to Proposition 5.5, the highest weights of $S(\wedge^2\tau) \otimes S(\tau^*)$ are those $\mu \in \mathbb{Z}^8$ satisfying

$$a \geq \mu_1 \geq a \geq \mu_2 \geq b \geq \cdots \geq d \geq \mu_8. \tag{5.7)(a}$$

These conditions force $\mu_1 = a$, $\mu_3 = b$, $\mu_5 = c$, and $\mu_7 = d$. This shows that μ determines (a, b, c, d); and any dominant μ can occur, as long as $\mu_7 \geq 0$. The integer k of Proposition 5.5 can also be read off from μ: it is

$$k = \mu_1 - \mu_2 + \mu_3 - \mu_4 + \mu_5 - \mu_6 + \mu_7 - \mu_8. \tag{5.7)(b}$$

According to Lemma 5.2, the highest weights λ that we want are obtained by twisting these μ by the weight of $\det^{k/2}$. That is,

$$\lambda = \mu + \frac{k}{2}(1, 1, 1, 1, 1, 1, 1, 1). \tag{5.7)(c}$$

Comparing (5.7)(b) and (5.7)(c), we see that

$$k = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8. \tag{5.7)(d}$$

We can interpret (5.7)(c) and (5.7)(d) as formulas for μ in terms of λ. In particular, they give

$$\mu_7 = \lambda_7 - \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8),$$

or

$$\mu_7 = \frac{1}{2}(-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) \tag{5.7)(e}$$

We know that μ occurs in $S(\wedge^2\tau) \otimes S(\tau^*)$ if and only if μ_7 is a non-negative integer; so it follows that λ appears in $S(\mathfrak{v}_0)$ if and only if the expression in (5.7)(e) is a non-negative integer. This is the first formula in the proposition. The second follows from the formulas (4.4)(b) for the simple positive roots. $\square$

Corollary 5.8. *Suppose we are in the setting of* (4.2)–(4.4) *and* (4.6). *Let* α *be the positive root*

$$\alpha = \beta_2 + \beta_3 + \beta_4 + \beta_5 = \frac{1}{2}(-1,1,-1,1,-1,1,1,1).$$

Write h_α *for the corresponding coroot. Define a subset* $S \subset W$ *of the Weyl group by*

$$S = \{w \in W \mid w^{-1}(\beta_i) > 0 \quad (i = 1,3,4,\dots,8), \quad w^{-1}(\alpha) > 0\}.$$

Fix a G*-dominant weight* $\lambda \in X^*(H)$, *and write* V_λ *for the corresponding irreducible representation of* G. *Define a subset* $S(\lambda) \subset W$ *by*

$$S(\lambda) = \{w \in S \mid (\lambda+\rho)(w^{-1}(h_\alpha)) \geq 4\}.$$

Then the multiplicity of V_λ^* *in the ring of regular functions on* $G \cdot e$ *is equal to*

$$\sum_{w \in S(\lambda)} \operatorname{sgn}(w).$$

Proof. We combine the formula of Theorem 4.7 with Proposition 5.6. Define $S(\lambda)$ to be the set of $w \in W$ for which $w(\lambda+\rho) - \rho$ is the highest weight (for $\mathfrak{l}_0$) of a representation in $S(\mathfrak{v}_0)$. (We will see that this is the set defined in the Corollary.) Theorem 4.7 says that we must sum $\operatorname{sgn} w$ times a multiplicity over $S(\lambda)$. Proposition 4.6 says that all the multiplicities are one, and that these w are characterized by the conditions

$$[w(\lambda+\rho) - \rho](h_i) \geq 0 \quad (i = 1,3,4,\dots,8),$$

and

$$[w(\lambda+\rho) - \rho](h_\alpha) \geq 0.$$

Now $\rho(h_i) = 1$ (see for example [Hump], Lemma 13.3A). These two conditions may therefore be written as

$$w(\lambda+\rho)(h_i) \geq 1 \quad (i = 1,3,4,\dots,8),$$

and

$$w(\lambda+\rho)(h_\alpha) \geq 4.$$

Because $\lambda+\rho$ is dominant, integral, and regular, the first condition is equivalent to $w^{-1}(\beta_i)$ being a positive root. The second implies that $w^{-1}(\alpha)$ is a positive root. The two together may be written as

$$w^{-1}(\beta_i) > 0 \quad (i = 1,3,4,\dots,8), \qquad w^{-1}(\alpha) > 0$$

and

$$(\lambda+\rho)(w^{-1}(h_\alpha)) \geq 4.$$

The first condition is just the definition of S in the Corollary, and the addition of the second gives $S(\lambda)$ as in the Corollary. □

What remains is to calculate S and to understand the possibilities for $S(\lambda)$. This we will do in the next section.

6. Calculations in the Weyl group of E_8.

In this section we will calculate the subsets of the Weyl group of E_8 described in Corollary 5.8. We begin with some general discussion of the nature of the subsets. The conditions $w^{-1}(\beta_i) > 0$ for $i \in I$ are very familiar. They define natural coset representatives for the subgroup W_I of W generated by simple reflections in the roots β_i (for $i \in I$). In our case, I consists of all the simple roots except β_2. The other seven simple roots span a system of type A_7, so W_I is S_8, the symmetric group on 8 elements. The number of Weyl group elements satisfying $w^{-1}(\beta_i) > 0$ for $i = 1, 3, 4, \ldots, 8$ is therefore equal to

$$|W(E_8)|/8! = 17280.$$

This is a discouragingly large number for hand calculations. It is not inaccessible to a computer, however. Our original determination of S essentially listed these elements (by computer) and calculated $w^{-1}(\alpha)$ for each of them. We discovered in this way that S had only seventeen elements. Once we knew that S was small, we were able to prove that without computer assistance.

First we write the definition of S (Corollary 5.8) in the language of Definition 3.9. This is

$$S = \{w \in W \mid \ \Delta^+(w) \text{ does not contain } \alpha \text{ or any } \beta_i, \quad i = 1, 3, 4, \ldots, 8\}. \qquad (6.1)$$

(The equivalence with the definition in Corollary 5.8 is immediate.) We propose to find all the elements w satisfying (6.1) by induction on the length $l(w)$. Longer elements of W are produced from shorter ones by multiplication by simple reflections; so we need to see what effect that multiplication has on $\Delta^+(w)$.

Lemma 6.2. *In the setting of Definition 3.9, suppose $w \in W$ and β is a simple root. Then exactly one of the following possibilities holds:*

1. $l(ws_\beta) = l(w) + 1$, $w\beta > 0$, *and* $\Delta^+(ws_\beta) = \Delta^+(w) \cup \{w\beta\}$.
2. $l(ws_\beta) = l(w) - 1$, $w\beta < 0$, *and* $\Delta^+(ws_\beta) = \Delta^+(w) - \{-w\beta\}$.

Similarly, exactly one of the following possibilities holds:

1. $l(s_\beta w) = l(w) + 1$, $w^{-1}\beta > 0$, *and* $\Delta^+(s_\beta w) = s_\beta\Delta^+(w) \cup \{\beta\}$.
2. $l(s_\beta w) = l(w) - 1$, $w^{-1}\beta < 0$, *and* $\Delta^+(s_\beta w) = s_\beta\Delta^+(w) - \{-\beta\}$.

This is elementary and well-known (compare [Hump], proof of Lemma 10.3A).

Corollary 6.3. *In the setting of (6.1), suppose $w \in S$ has length m. Let β be a simple root such that*

$$w\beta > 0, w\beta \notin \{\alpha, \beta_1, \beta_3, \beta_4, \ldots, \beta_8\}.$$

Then $ws_\beta \in S$ has length $m+1$. Every element of S of length $m+1$ arises by this construction.

Using this corollary, we can construct a list of elements of S. The results appear in the following table. The first column gives a reduced expression $w = s_{i_1} \cdots s_{i_m}$. Here $s_i = s_{\beta_i}$, and as usual the roots are labelled as in (4.4)(c). The next eight columns show the roots $w\beta_i$. In order to keep the table small, we have written here

-2456 instead of $-(\beta_2+\beta_4+\beta_5+\beta_6)$. The elements are grouped according to their lengths.

w	$w\beta_1$	$w\beta_2$	$w\beta_3$	$w\beta_4$	$w\beta_5$	$w\beta_6$	$w\beta_7$	$w\beta_8$
1	1	2	3	4	5	6	7	8
s_2	1	-2	3	24	5	6	7	8
s_2s_4	1	4	234	-24	245	6	7	8
$s_2s_4s_3$	1234	4	-234	3	245	6	7	8
$s_2s_4s_5$	1	4	234	5	-245	2456	7	8
$s_2s_4s_3s_1$	-1234	4	1	3	245	6	7	8
$s_2s_4s_3s_5$	1234	4	-234	2345	-245	2456	7	8
$s_2s_4s_5s_6$	1	4	234	5	6	-2456	24567	8
$s_2s_4s_3s_1s_5$	-1234	4	1	2345	-245	2456	7	8
$s_2s_4s_3s_5s_6$	1234	4	-234	2345	6	-2456	24567	8
$s_2s_4s_5s_6s_7$	1	4	234	5	6	7	-24567	245678
$s_2s_4s_3s_1s_5s_6$	-1234	4	1	2345	6	-2456	24567	8
$s_2s_4s_3s_5s_6s_7$	1234	4	-234	2345	6	7	-24567	245678
$s_2s_4s_5s_6s_7s_8$	1	4	234	5	6	7	8	-245678
$s_2s_4s_3s_1s_5s_6s_7$	-1234	4	1	2345	6	7	-24567	245678
$s_2s_4s_3s_5s_6s_7s_8$	1234	4	-234	2345	6	7	8	-245678
$s_2s_4s_3s_1s_5s_6s_7s_8$	-1234	4	1	2345	6	7	8	-245678

Table 6.4. Elements of S.

We now explain more carefully how Table 6.4 was constructed. Suppose we have constructed the lines for elements of S of length m; we want to find the lines for elements of length $m+1$. According to Corollary 6.3, each new element will be of the form ws_β, where $w \in S$ has length m, β is a simple root, and $w\beta$ is a positive root not in $\{\alpha, \beta_1, \beta_3, \beta_4, \dots, \beta_8\}$. Our table through length m contains all of this information. Suppose for example that $m = 5$, so that we are trying to construct elements of S of length 6. Our table lists three elements $s_2s_4s_3s_1s_5$, $s_2s_4s_3s_5s_6$, and $s_2s_4s_5s_6s_7$ of length 5; let us call them x, y, and z. Looking at the table for $x\beta_i$, we see that $x\beta_i$ is positive only for $i = 2, 3, 4, 6, 7$, and 8. Of these possibilities, 2, 3, 7, and 8 are excluded because $x\beta_i$ is one of the forbidden simple roots $\beta_1, \beta_3, \dots, \beta_8$. Similarly 4 is excluded because $x\beta_4 = \beta_2+\beta_3+\beta_4+\beta_5 = \alpha$. So the only candidate for an element of length 6 coming from x is $xs_6 = s_2s_4s_3s_1s_5s_6$, and this is the first on the list in Table 6.4. In the same way we see that y gives rise to ys_1 and ys_7, and z to zs_3 and zs_8. Table 6.4 shows ys_7 and zs_8, but seems to omit ys_1 and zs_3. The reason is that $ys_1 = xs_6$, and $zs_3 = ys_7$. There are many ways to detect these equalities. For us the easiest is in the rest of the table: two Weyl group elements agree exactly when they have the same effect on all simple roots. So we need to know how to calculate these remaining columns of the table; that is, how to calculate $ws_\beta(\beta')$ given knowledge of all $w\beta'$. (Here β and β' are simple roots.)

Because all the simple roots for E_8 have the same length, we have

$$s_\beta(\beta') = \begin{cases} \beta', & \text{if } \beta \text{ is not adjacent to } \beta' \\ \beta+\beta', & \text{if } \beta \text{ is adjacent to } \beta' \\ -\beta, & \text{if } \beta \text{ is equal to } \beta'. \end{cases} \tag{6.5}$$

Therefore each entry $ws_\beta(\beta_i)$ in the row for ws_β is either equal to the corresponding entry $w\beta_i$ for w (if β is not adjacent to β_i), or it is the sum of the entries $w\beta_i$ and $w\beta$ (if β is adjacent to β_i), or it is the negative of the $w\beta$ (if $\beta = \beta_i$). In the example considered above, the entry $ys_7(\beta_6)$ is computed as the sum of $y\beta_6$ and $y\beta_7$; that is, as $-2456 + 24567 = 7$.

We leave to the reader the rest of the task of verifying Table 6.4, which can be done more or less by visual inspection. Next we want to understand the subsets $S(\lambda) \subset S$ defined in Corollary 5.8. Their definition involves the positive root $w^{-1}\alpha$. This is easy to calculate in our inductive calculation of S, since $(ws_\beta)^{-1}(\alpha) = s_\beta(w^{-1}\alpha)$. At each stage we will express $w^{-1}\alpha$ in terms of simple roots, and then (6.5) tells us how to apply s_β to this expression. Here is the result. The last four columns of the table are computed in the same way; they will be needed in section 7. We use the same shorthand as in Table 6.4 for roots.

w	$l(w)$	$w^{-1}\alpha$	$w^{-1}\beta_1$	$w^{-1}\beta_4$	$w^{-1}\beta_6$	$w^{-1}\beta_8$
1	0	2345	1	4	6	8
s_2	1	345	1	24	6	8
s_2s_4	2	345	1	2	6	8
$s_2s_4s_3$	3	45	13	2	6	8
$s_2s_4s_5$	3	34	1	2	56	8
$s_2s_4s_3s_1$	4	45	3	2	6	8
$s_2s_4s_3s_5$	4	4	13	2	56	8
$s_2s_4s_5s_6$	4	34	1	2	5	8
$s_2s_4s_3s_1s_5$	5	4	3	2	56	8
$s_2s_4s_3s_5s_6$	5	4	13	2	5	8
$s_2s_4s_5s_6s_7$	5	34	1	2	5	78
$s_2s_4s_3s_1s_5s_6$	6	4	3	2	5	8
$s_2s_4s_3s_5s_6s_7$	6	4	13	2	5	78
$s_2s_4s_5s_6s_7s_8$	6	34	1	2	5	7
$s_2s_4s_3s_1s_5s_6s_7$	7	4	3	2	5	78
$s_2s_4s_3s_5s_6s_7s_8$	7	4	13	2	5	7
$s_2s_4s_3s_1s_5s_6s_7s_8$	8	4	3	2	5	7

Table 6.6. Data for $S(\lambda)$.

Inspection of this table establishes

Lemma 6.7. *The non-identity elements of S may be partitioned into pairs (u, v) in such a way that*

$$u^{-1}(\alpha) = v^{-1}(\alpha), \qquad \operatorname{sgn}(u) = -\operatorname{sgn}(v).$$

Every set $S(\lambda)$ (Corollary 5.8) consists of the identity element and some of these pairs. Consequently

$$\sum_{w\in S(\lambda)} \mathrm{sgn}(w) = 1.$$

Proof. The partition into pairs is clear from Table 6.6; we can even arrange $v = us_\beta$ for a simple root β, and then the partition is unique. The statement about $S(\lambda)$ is clear from Corollary 5.8, and the sum formula follows. □

Proof of Theorem 1.1. We use the notation (4.2)–(4.4) and (4.6). Every irreducible (algebraic) representation of G is of the form V_λ for a dominant weight $\lambda \in X^*(H)$. The multiplicity of V_λ in the ring of functions on $X = G\cdot e$ is computed by Corollary 5.8 as a sum over $S(\lambda)$; and this sum is computed by Lemma 6.7 to be 1. □

7. The model orbit in $E_8(\mathbb{R})$.

In this section we consider real group analogues of Theorem 1.1. In the notation of section 2, we therefore want to take G to be a real form of E_8, and X_θ to be an orbit of $K_{\mathbb{C}}$ on $\mathcal{N}_\theta^*$, whose complexification is the model orbit in E_8. Real forms of complex orbits have been tabulated in [Djok], and those tables are reproduced in chapter 9 of [Coll]. What one finds from those tables is that the model orbit has a real form only in the split real form of E_8 (sometimes denoted $E_{8(8)}$); and in that case there is only one real form. In this section we will therefore take for G a simply connected split real form of E_8. The maximal compact subgroup K is then isomorphic to $Spin(16)$, the compact simply connected double cover of $SO(16)$. In accordance with (4.2), we arrange this isomorphism so that the representation of K on $\mathfrak{p}$ is the negative half spin representation of dimension $2^{8-1} = 128$. A little more precisely, we let H be a compact Cartan subgroup of K, and choose an isomorphism

$$\mathfrak{h}^* \simeq \mathbb{C}^8 \tag{7.1)(a}$$

in such a way that

$$\Delta(\mathfrak{k},\mathfrak{h}) = \{(\pm e_i \pm e_j) \mid 1 \le i \ne j \le 8\} \tag{7.1)(b}$$

and

$$\Delta(\mathfrak{p},\mathfrak{h}) = \{\frac{1}{2}(\epsilon_1,\cdots,\epsilon_8) \mid \epsilon_i = \pm 1, \prod_i \epsilon_i = -1\}. \tag{7.1)(c}$$

It will be convenient to notice that

$$\Delta(\mathfrak{k},\mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \mid \alpha(1,1,1,1,1,1,1,1) \in 2\mathbb{Z}\}. \tag{7.1)(b'}$$

We now choose e, h, and f as in (3.1), in such a way that $e \in \mathcal{N}_\theta^*$ belongs to the model orbit; equivalently, so that

$$\dim K_{\mathbb{C}}\cdot e = 64. \tag{7.1)(d}$$

After conjugation by $K_{\mathbb{C}}$, we may assume that $h \in \mathfrak{h}$ is dominant with respect to a standard positive root system for $\mathfrak{k}$. The tables of Djoković allow us to compute

h (see [Djok], entry 6 of Table XIV, or [Coll], entry 6 on page 161). The result (determined up to the Weyl group of $Spin(16)$) is

$$h = (1,1,1,1,2,0,0,0). \tag{7.1)(e}$$

Just as in section 4, we will be interested in the parabolic subalgebra defined by h, and it will again be convenient to have coordinates in which h takes the simpler form of (4.2)(c). For this purpose we will twist the identifications in (7.1)(a) by an element of the Weyl group. Specifically, consider the two roots

$$\phi = \frac{1}{2}(1,-1,-1,1,1,-1,-1,-1), \qquad \psi = \frac{1}{2}(-1,1,1,-1,1,-1,-1,-1).$$

These roots are orthogonal, so the product $w = s_\phi s_\psi$ is an element of order two in the Weyl group. We compute

$$w(1,1,1,1,2,0,0,0) = (1,1,1,1,1,1,1,1),$$

which is equivalent to

$$w(1,1,1,1,1,1,1,1) = (1,1,1,1,2,0,0,0). \tag{7.2)(a}$$

Our identification of $\mathfrak{h}^*$ with $\mathbb{C}^8$ is that of (7.1)(a) twisted by w. The description of the compact roots in (7.1)(b') translates to

$$\Delta(\mathfrak{k},\mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \mid \alpha(1,1,1,1,2,0,0,0) \in 2\mathbb{Z}\}. \tag{7.2)(b'}$$

From this one calculates easily

$$\begin{gathered}\Delta(\mathfrak{k},\mathfrak{h}) = A \cup B,\\ A = \{\pm e_i \pm e_j \mid 1 \le i,j \le 4 \quad \text{or} \quad 5 \le i,j \le 8\} \qquad (7.2)(b)\\ B = \{\frac{1}{2}(\epsilon_1,\dots,\epsilon_8) \mid \epsilon_i = \pm 1, \quad \epsilon_1\cdots\epsilon_4 = -1, \quad \epsilon_5\cdots\epsilon_8 = 1\}.\end{gathered}$$

The remaining roots are

$$\begin{gathered}\Delta(\mathfrak{p},\mathfrak{h}) = C \cup D,\\ C = \{\pm e_i \pm e_j \mid 1 \le i \le 4, \quad 5 \le j \le 8\}, \qquad (7.2)(c)\\ D = \{\frac{1}{2}(\epsilon_1,\dots,\epsilon_8) \mid \epsilon_i = \pm 1, \quad \epsilon_1\cdots\epsilon_4 = 1, \quad \epsilon_5\cdots\epsilon_8 = -1\}.\end{gathered}$$

Roughly speaking, this says that $\mathfrak{k}$ is built from a subalgebra $\mathfrak{so}(8) \times \mathfrak{so}(8)$ (the roots in A) by adjoining a copy of the representation $spin^- \otimes spin^+$. Similarly, $\mathfrak{p}_0$ is the sum of the two representations $\mathbb{R}^8 \otimes \mathbb{R}^8$ and $spin^+ \otimes spin^-$. (Our passage from the standard realization of $\mathfrak{so}(16)$ in (7.1) is an incarnation of the "triality" linking the three eight-dimensional representations $\mathbb{R}^8$, $spin^+$, and $spin^-$ of $\mathfrak{so}(8)$.) Now (7.2)(a) ensures that in our new coordinates the element h is

$$h = (1,1,1,1,1,1,1,1). \tag{7.2)(d}$$

By analogy with (4.6), we find

$$\Delta(\mathfrak{p}(2),\mathfrak{h}) = \{e_i + e_j \mid 1 \le i \le 4, \quad 5 \le j \le 8\}, \tag{7.2)(e}$$

$$\Delta(\mathfrak{p}(3),\mathfrak{h}) = \{\frac{1}{2}(1,\dots,1,\epsilon_5,\dots,\epsilon_8) \mid \epsilon_i = 1 \text{ for exactly 3 values of } i\}. \tag{7.2)(f}$$

We will need a set of positive roots making h dominant; but the one chosen in (4.4) is not so convenient here. We modify it by permuting the coordinates, sending

$$(1,2,3,4,5,6,7,8) \to (1,5,2,6,3,7,4,8).$$

The corresponding simple roots are

$$\{(e_1-e_5, e_5-e_2, e_2-e_6, e_6-e_3, e_3-e_7, e_7-e_4, e_4-e_8, \frac{1}{2}(-1,-1,1,1,-1,1,1,1)\}. \tag{7.2)(g}$$

Just as in (4.4), we label these roots as $\{\beta_1,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\beta_8,\beta_2\}$. One interesting property (following from (7.2)(b) and (7.2)(c)) is that all of these simple roots are noncompact.

We want to calculate the various objects introduced in (3.1)–(3.3). We imitate the ideas in section 5. First, we fix an isomorphism

$$j\colon \mathfrak{gl}(4)_L \times \mathfrak{gl}(4)_R \to \mathfrak{l} \tag{7.2)(h}$$

carrying pairs of diagonal matrices to $\mathfrak{h} \simeq \mathbb{C}^8$ in the obvious way. (The labels L and R, standing for "left" and "right," are intended simply to distinguish the two factors of $\mathfrak{l}$.)

Lemma 7.3. *Suppose we are in the setting of* (7.2). *In terms of the isomorphism* (7.2)(g), *the representation of* $\mathfrak{l}$ *on* $\mathfrak{p}(2)$ *is* $(\tau_{4,L}) \otimes (\tau_{4,R})$, *and that on* $\mathfrak{p}(3)$ *is* $(\det_L^{1/2}) \otimes (\tau^*_{4,R} \otimes \det_R^{1/2})$. *In each case the factors enclosed in parentheses are representations of* $\mathfrak{gl}(4)$. *Consequently*

$$S(\mathfrak{v}) \simeq [S(\tau_{4,L}\otimes\tau_{4,R})] \otimes \sum_{k=0}^{\infty} (\det_L^{k/2}) \otimes \left(S^k(\tau^*_{4,R}) \otimes \det_R^{k/2}\right)$$

This is clear from (7.2). Of course it is precisely analogous to Lemma 5.2. In order to proceed, we need an analogue of Proposition 5.3 to calculate the term in square brackets. This is a standard fact in invariant theory, but we will again deduce it from Schmid's result.

Proposition 7.4 ([Sch], page 63, Behauptung c)). *In the setting* (7.2), *the symmetric algebra* $[S(\tau_{n,L}\otimes\tau_{n,R})]$ *is the direct sum of the representations of* $GL(n)\times GL(n)$ *of highest weights* $[(a_1,\dots,a_n),(a_1,\dots,a_n)]$, *with* $a_1 \ge \dots \ge a_n \ge 0$ *integers.*

Proof. Consider the Hermitian symmetric space $U(n,n)/U(n)\times U(n)$ (see for example [Helg], page 444, or [Knapp], page 6). The holomorphic tangent space $\mathfrak{p}_+$ at the identity coset is isomorphic to the outer tensor product $\tau_n \otimes \tau_n^*$ of the standard representations of $U(n)$. (This may be seen by explicit calculation with matrices, or by inspection of the root systems.) If we now twist the identification of the second factor of K by the outer automorphism inverse transpose of $U(n)$, we get instead $\tau_n \otimes \tau_n$. In these twisted coordinates, Schmid's result becomes precisely what is stated here. (Just as in Proposition 5.3, we use the equivalence between representations of $U(n)$ and $GL(n,\mathbb{C})$.) $\square$

Proposition 7.5. *Suppose we are in the setting (7.2). Suppose* $\lambda = (\lambda_1, \dots, \lambda_8) \in \mathbb{C}^8$ *is a dominant weight for* $\mathfrak{l} \simeq \mathfrak{gl}(4) \times \mathfrak{gl}(4)$*; write* E_λ *for the irreducible representation of highest weight* λ*. Then the multiplicity of* E_λ *in* $S(\mathfrak{v})$ *is equal to* 1 *if*

$$\lambda_1 \geq \lambda_5 \geq \lambda_2 \geq \lambda_6 \geq \lambda_3 \geq \lambda_7 \geq \lambda_4 \geq \lambda_8,$$

with all differences integers, and

$$\frac{1}{2}(-\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8)$$

is a non-negative integer. Otherwise the multiplicity is zero. In terms of the eight simple roots β_i *of* (7.2)(g) *and the corresponding coroots* h_i*, the conditions for multiplicity one may be written as*

$$\lambda(h_i) \in \mathbb{N} \qquad (i = 1, 3, 4, 5, 6, 7, 8), \qquad \lambda(h_2 + h_3 + h_4 + h_5) \in \mathbb{N}.$$

Proof. The argument is exactly like the one for Proposition 5.6, using Lemma 7.3 and Proposition 7.4 in place of Lemma 5.2 and Proposition 5.3. We leave the details to the reader. □

We are going to use Corollary 3.14 to translate this analogue of Proposition 5.6 into a multiplicity formula analogous to Corollary 5.8. To do that, we need to calculate the weight ρ_c, half the sum of the positive compact roots. Because all of the simple roots β_i are noncompact, the compact roots are precisely the sums of even numbers of simple roots. If β_i and β_j are adjacent in the Dynkin diagram, it follows that $\beta_i + \beta_j$ is a simple compact root. This accounts for seven of the eight simple compact roots. The eighth is $\beta_2 + \beta_3 + \beta_4 + \beta_5$. Computing inner products of these roots, we find that the Dynkin diagram of K is

$$\begin{array}{ccccccccccccc} & & & & & & & & & & \beta_{78} & & \\ & & & & & & & & & & | & & \\ \beta_{2345} & - & \beta_{67} & - & \beta_{45} & - & \beta_{13} & - & \beta_{24} & - & \beta_{56} & - & \beta_{34} \end{array} \tag{7.6)(a}$$

Here we have written β_{ij} for $\beta_i + \beta_j$. Now the value of ρ_c on a coroot for a compact simple root is one. This provides eight equations for the eight values $\rho_c(h_i)$, which may be solved to give

$$\rho_c(h_i) = \begin{cases} 1, & \text{if } i = 1, 4, 6, \text{ or } 8 \\ 0, & \text{if } i = 2, 3, 5, \text{ or } 7. \end{cases} \tag{7.6)(b}$$

Corollary 7.7. *Suppose we are in the setting (7.2). Write* $\alpha = \beta_2 + \beta_3 + \beta_4 + \beta_5$*, and* h_α *for the corresponding coroot. Fix a* K*-dominant weight* $\mu \in X^*(H)$*, and write* V_μ *for the corresponding irreducible representation of* K*. Define a subset* $S_0(\mu) \subset W_K$ *by*

$$\begin{aligned} S_0(\mu) = \{x \in W_K \mid & x(\mu + \rho_c)(h_\alpha) \in \mathbb{N} + 1, \quad x(\mu + \rho_c)(h_i) \in \mathbb{N} \quad (i = 3, 5, 7), \\ & x(\mu + \rho_c)(h_j) \in \mathbb{N} + 1 \quad (j = 1, 4, 6, 8)\}. \end{aligned}$$

Assume that Conjecture 3.13 holds. Then the multiplicity of V_μ^* *in the ring of regular functions on the normalization of* $\overline{K_{\mathbb{C}} \cdot e}$ *is*

$$\sum_{x \in S_0(\mu)} \operatorname{sgn}(x).$$

Proof. The argument is identical to the one for Corollary 5.8, with the formula in (7.6) for ρ_c replacing the fact that $\rho(h_i) = 1$. □

The conditions defining $S_0(\mu)$ involve mostly noncompact roots, so it is not so clear how to control $S_0(\mu)$ as a subset of the Weyl group of K. On the other hand, the conditions are very close to those defining the set S of Corollary 5.8. In order to continue, we need some elementary information about how W_K is included in the full Weyl group. For this we temporarily weaken the hypotheses of (7.2).

Proposition 7.8. *Suppose $\Delta \subset \mathfrak{h}^*$ is a root system, and $\Delta_c \subset \Delta$ is a subroot system. Write $W_K \subset W$ for the corresponding Weyl groups. Fix a positive system $\Delta^+ \subset \Delta$, and write Δ_c^+ for the corresponding positive system in Δ_c. Define*

$$W^1 = \{t \in W \mid t\Delta^+ \supset \Delta_c^+\}.$$

In the notation of Definition 3.9, this amounts to

$$W^1 = \{t \in W \mid \Delta^+(t) \cap \Delta_c^+ = \emptyset\}.$$

Then W^1 is a set of coset representatives for W_K in W. More precisely, every element w of W has a unique decomposition

$$w = xt, \qquad x \in W_K, \quad t \in W^1.$$

The element x is characterized by the property that

$$\Delta^+(w) \cap \Delta_c = \Delta_c^+(x)$$

(notation as in Definition 3.9). That is, if α is any root in Δ_K, then $w^{-1}\alpha$ is positive if and only if $x^{-1}\alpha$ is positive.

The special case when Δ_c is a Levi subsystem appears in [Kost], Proposition 5.13. The proof in general is identical.

We turn now to the computation of the set $S_0(\mu)$ in Corollary 7.7.

Lemma 7.9. *Fix a K-dominant weight μ, and let λ be the unique G-dominant weight conjugate by W to $\mu + \rho_c$. Then the set $S_0(\mu)$ (Corollary 7.7) is non-empty only if λ is integral for G. Assume therefore that λ is integral. For each $x \in S_0(\mu)$, let w be the unique shortest element of W such that*

$$x(\mu + \rho_c) = w\lambda. \tag{7.9}$$

Then $w \in S$ (Corollary 5.8). The element x may be computed from w by Proposition 7.8, as the first term in the factorization $w = xt$. Furthermore $\mu + \rho_c = t\lambda$. Finally, we have

$$S_0(\mu) = \{x' \in W_K \mid w' = x't \in S\}.$$

(Here the factorization of w' is the one in Proposition 7.8.)

Proof. We first record the condition on w for it to be the shortest element solving the equation (7.9). This equation puts w in a certain coset of the stabilizer of λ, which is generated by the reflections fixing λ. By Lemma 6.2, the condition is therefore

$$\lambda(h_i) = 0 \Rightarrow w\beta_i > 0. \tag{7.10(a)}$$

It will be convenient to rewrite this condition as

$$w\beta_i < 0 \Rightarrow \lambda(h_i) > 0. \tag{7.10)(a'}$$

Suppose $S_0(\mu)$ contains an element x. The conditions in Corollary 7.7 imply that $x(\mu + \rho_c)$ takes integer values on all the h_i except perhaps for $i = 2$; and an integer value on $h_\alpha = h_2 + h_3 + h_4 + h_5$. From this it follows that the value on h_2 is an integer as well, so $x(\mu + \rho_c)$ is integral. Because the Weyl group preserves integrality, λ must be integral as well.

Next, the condition for x to belong to $S_0(\mu)$ may be written using (7.9) as

$$w\lambda(h_\alpha) \in \mathbb{N} + 1, \quad w\lambda(h_i) \in \mathbb{N} \quad (i = 3, 5, 7), \quad w\lambda(h_j) \in \mathbb{N} + 1 \quad (j = 1, 4, 6, 8). \tag{7.10)(b}$$

The first condition implies that $\lambda(w^{-1}h_\alpha) > 0$. Because λ is dominant, we conclude that $w^{-1}\alpha$ is a positive root. Similarly $w^{-1}\beta_j > 0 \quad (j = 1, 4, 6, 8)$. To prove that $w \in S$, we only need to show that $w^{-1}\beta_i > 0 \quad (i = 3, 5, 7)$. Suppose not. We conclude from (7.10)(b) that β_i must be orthogonal to $w\lambda$, and therefore that $s_i w\lambda = w\lambda$. We may replace w by $w' = s_i w$ without affecting $w\lambda$, but Lemma 6.2 says that w' is shorter than w. This contradicts the choice of w, completing the proof that $w \in S$.

Because $\mu + \rho_c$ is dominant and regular for K, (7.9) guarantees that $\Delta^+(w) \cap \Delta_c = \Delta_c^+(x)$. Proposition 7.8 therefore implies that x is the first term in the factorization of w, and the formula for $\mu + \rho_c$ follows immediately.

The description of $S_0(\mu)$ appears at first glance to be immediate, but there are two non-trivial points to check. First, $S_0(\mu)$ could have an element x' for which the corresponding $w' = x't'$ had a different factor t', still satisfying $t\lambda = t'\lambda$. Second, an element $w' = x't$ in S might fail to satisfy the conditions (7.10)(a) and (b). Both of these possibilities are best addressed by explicit calculation, so we postpone the end of the proof until we have some explicit information. □

Corollary 7.7 and Lemma 7.9 allow us to compute K multiplicities in the ring of functions on $\overline{K_{\mathbb{C}} \cdot e}$ (always assuming Conjecture 3.13). In order to make the result explicit, we need to compute the factorization of Proposition 7.8 for the elements of S. The result is in Table 7.11. The first column lists the elements of S (as for example in Table 6.4). The second lists the elements x of W_K, written as products of simple reflections for the positive system shown in (7.6)(a). (Here for example s_{34} denotes reflection in the root $\beta_{34} = \beta_3 + \beta_4$.) The third column lists the coset representatives $t \in W^1$. The fourth column is the set $\Delta^+(w)$ of positive roots that change sign under w^{-1} (Definition 3.9). These are written using the shorthand of Table 6.4: 2456 denotes $\beta_2 + \beta_4 + \beta_5 + \beta_6$. Finally, the compact roots (roots of H in $\mathfrak{k}$) are underlined.

Here is how Table 7.11 was computed. The first column comes from Table 6.4. The fourth column may be computed by induction on the length of w, using Lemma 6.2 and the information in Table 6.4. Because every simple root is noncompact, the compact roots are exactly those involving an even number of simple roots. (More precisely, a root $\sum m_i\beta_i$ is compact if and only if $\sum m_i$ is even.) The second column is computed as follows. According to Proposition 7.8, the element $x \in W_K$ is characterized by the property that $\Delta_c^+(x)$ consists of the compact roots in $\Delta^+(w)$. That is, $\Delta_c^+(x)$ consists of the underlined roots in the fourth column. Lemma 6.2 may be used in reverse to compute x from $\Delta_c^+(x)$; this gives the second column.

Finally, the element t is $x^{-1}w$. This is computed from the first two columns using facts like $s_{24} = s_2 s_4 s_2$.

w	x	t	$\Delta^+(w)$
1	1	1	$\emptyset$
s_2	1	s_2	2
s_2s_4	s_{24}	s_2	$2, \underline{24}$
$s_2s_4s_3$	s_{24}	s_2s_3	$2, \underline{24}, 234$
$s_2s_4s_5$	s_{24}	s_2s_5	$2, \underline{24}, 245$
$s_2s_4s_3s_1$	$s_{24}s_{13}$	s_2s_3	$2, \underline{24}, 234, \underline{1234}$
$s_2s_4s_3s_5$	s_{24}	$s_2s_3s_5$	$2, \underline{24}, 234, 245$
$s_2s_4s_5s_6$	$s_{24}s_{56}$	s_2s_5	$2, \underline{24}, 245, \underline{2456}$
$s_2s_4s_3s_1s_5$	$s_{24}s_{13}$	$s_2s_3s_5$	$2, \underline{24}, 234, \underline{1234}, 245$
$s_2s_4s_3s_5s_6$	$s_{24}s_{56}$	$s_2s_3s_5$	$2, \underline{24}, 234, 245, \underline{2456}$
$s_2s_4s_5s_6s_7$	$s_{24}s_{56}$	$s_2s_5s_7$	$2, \underline{24}, 245, \underline{2456}, 24567$
$s_2s_4s_3s_1s_5s_6$	$s_{24}s_{13}s_{56}$	$s_2s_3s_5$	$2, \underline{24}, 234, \underline{1234}, 245, \underline{2456}$
$s_2s_4s_3s_5s_6s_7$	$s_{24}s_{56}$	$s_2s_3s_5s_7$	$2, \underline{24}, 234, 245, \underline{2456}, 24567$
$s_2s_4s_5s_6s_7s_8$	$s_{24}s_{56}s_{78}$	$s_2s_5s_7$	$2, \underline{24}, 245, \underline{2456}, 24567, \underline{245678}$
$s_2s_4s_3s_1s_5s_6s_7$	$s_{24}s_{13}s_{56}$	$s_2s_3s_5s_7$	$2, \underline{24}, 234, \underline{1234}, 245, \underline{2456}, 24567$
$s_2s_4s_3s_5s_6s_7s_8$	$s_{24}s_{56}s_{78}$	$s_2s_3s_5s_7$	$2, \underline{24}, 234, 245, \underline{2456}, 24567, \underline{245678}$
$s_2s_4s_3s_1s_5s_6s_7s_8$	$s_{24}s_{56}s_{78}s_{13}$	$s_2s_3s_5s_7$	$2, \underline{24}, 234, \underline{1234}, 245, \underline{2456},$ $24567, \underline{245678}$

Table 7.11. Factorization of elements of S.

With this information in hand, we can complete the proof of Lemma 7.9. Recall that we have a dominant integral weight λ, assumed conjugate by W to some $\mu+\rho_c$; say $z\lambda = \mu + \rho_c$. If α and β are adjacent simple roots, then at least one of the three roots $z\alpha$, $z\beta$, and $z(\alpha+\beta)$ must be compact. A compact root has non-zero inner product with $\mu+\rho_c$, so it follows that $z\lambda$ cannot be orthogonal to both $z\alpha$ and $z\beta$. Therefore λ can be orthogonal only to a collection of mutually orthogonal simple roots.

Suppose now that $w' = x't' \in S$ is factored as in Table 7.11. We want to understand when $\mu + \rho_c = t'\lambda$, and x' gives a solution to the equations (7.9) and (7.10). It will turn out that this happens only when λ is strictly positive on certain simple coroots, indicated in Table 7.12 below. The first requirement is specified in (7.10)(a'): whenever $w'\beta_i < 0$, $\lambda(h_i)$ must be strictly positive. The simple roots with $w'\beta_i$ negative are listed in Table 6.4; they appear in the third column of Table 7.12. Second, (7.10)(b) implies that λ must be strictly positive on $w^{-1}h_\alpha$ and on $w^{-1}h_j$ for $j = 1,4,6,8$. These elements are computed in Table 6.6: each is a sum of one, two, or (in the case of α) three or four simple coroots. Because λ is dominant and singular only on certain mutually orthogonal roots, this positivity is automatic on the sums of two or more simple coroots. The simple coroots appearing among $w^{-1}h_\alpha$ and $w^{-1}h_j$ (for $j = 1,4,6,8$) may be found in Table 6.6; they are listed in the fourth column of Table 7.12.

By inspection of Table 7.12, we find first of all that the element t is always a product of a set of orthogonal simple reflections from the set $\{s_2, s_3, s_5, s_7\}$. Write $I(w)$ for the corresponding subset of $\{2, 3, 5, 7\}$, so that $t = \prod_{i \in I(w)} s_i$. We observe that all the elements of $I(w)$ appear in the third and fourth columns of Table 7.12, and therefore that λ takes strictly positive values on the corresponding h_i. In the setting of (7.9)–(7.10), we therefore have

$$\mu + \rho_c = t\lambda = \lambda - \sum_{i \in I(w)} m_i \beta_i,$$

with $m_i > 0$. It follows from this equation that $I(w)$, and therefore t, is determined uniquely by μ.

w	t	$w\beta_i < 0$	$w^{-1}h_\alpha$, $w^{-1}h_j$ $(j = 1, 4, 6, 8)$	$w\beta$ compact
1	1		$1, 4, 6, 8$	
s_2	s_2	2	$1, 6, 8$	4
s_2s_4	s_2	4	$1, 2, 6, 8$	4
$s_2s_4s_3$	s_2s_3	3	$2, 6, 8$	1
$s_2s_4s_5$	s_2s_5	5	$1, 2, 8$	6
$s_2s_4s_3s_1$	s_2s_3	1	$2, 3, 6, 8$	1
$s_2s_4s_3s_5$	$s_2s_3s_5$	$3, 5$	$2, 4, 8$	$1, 4, 6$
$s_2s_4s_5s_6$	s_2s_5	6	$1, 2, 5, 8$	6
$s_2s_4s_3s_1s_5$	$s_2s_3s_5$	$1, 5$	$2, 3, 4, 8$	$1, 4, 6$
$s_2s_4s_3s_5s_6$	$s_2s_3s_5$	$3, 6$	$2, 4, 5, 8$	$1, 4, 6$
$s_2s_4s_5s_6s_7$	$s_2s_5s_7$	7	$1, 2, 5$	8
$s_2s_4s_3s_1s_5s_6$	$s_2s_3s_5$	$1, 6$	$2, 3, 4, 5, 8$	$1, 4, 6$
$s_2s_4s_3s_5s_6s_7$	$s_2s_3s_5s_7$	$3, 7$	$2, 4, 5$	$1, 4, 8$
$s_2s_4s_5s_6s_7s_8$	$s_2s_5s_7$	8	$1, 2, 5, 7$	8
$s_2s_4s_3s_1s_5s_6s_7$	$s_2s_3s_5s_7$	$1, 7$	$2, 3, 4, 5$	$1, 4, 8$
$s_2s_4s_3s_5s_6s_7s_8$	$s_2s_3s_5s_7$	$3, 8$	$2, 4, 5, 7$	$1, 4, 8$
$s_2s_4s_3s_1s_5s_6s_7s_8$	$s_2s_3s_5s_7$	$1, 8$	$2, 3, 4, 5, 7$	$1, 4, 8$

Table 7.12. Roots on which λ must be positive to satisfy (7.9)–(7.10).

This resolves the first of the two problems mentioned at the end of the argument for Lemma 7.9: since t is determined by μ, $S_0(\mu)$ is clearly a subset of the set in Lemma 7.9. What remains is to show that every element of this set—that is, every $w' = x't$ in S—actually contributes a solution to the equations (7.9)–(7.10). The nature of the difficulty is apparent for example when $w = s_2$ and $w' = s_2s_4$. We are given μ, and so we know that the corresponding λ solves the equations with w. From Table 7.12 we deduce that λ must be strictly positive on the roots $\beta_1, \beta_2, \beta_6$, and β_8. In order for w' to give a solution as well (as Lemma 7.9 says it should) Table 7.12 says that we also need λ to be strictly positive on β_4. But we compute

$$\lambda(h_4) = w'\lambda(w'h_4) = x'(\mu + \rho_c)(-h_{24}).$$

Here in the last equality we use (7.9) and the calculation in Table 6.4. Since h_{24} is a compact coroot (because it involves an even number of simple coroots) the K-regular weight $x'(\mu+\rho_c)$ must take a non-zero value on it. We conclude that $\lambda(h_4) \neq 0$, as we wished to show. More generally, we find that if λ and w satisfy (7.9)–(7.10), β is a simple root, and $w\beta$ is compact, then $\lambda(h_\beta)$ is automatically positive. These roots are listed in the last column of Table 7.12. They are the ones for which $w\beta$ is a sum of an even number of simple roots, so they may be read off from Table 6.4.

We now find that for a fixed value of t, the same simple roots appear in the last three columns of Table 7.12 for all elements w with this fixed t. This completes the verification of the formula for $S_0(\mu)$, and finishes the omitted part of the proof of Lemma 7.9.

Theorem 7.13. *Suppose we are in the setting (7.2). Fix a K-dominant weight $\mu \in X^*(H)$, and write V_μ for the corresponding irreducible representation of K. Assume that Conjecture 3.13 holds. Then the multiplicity of V_μ^* in the ring of regular functions on the normalization of $\overline{K_{\mathbb{C}} \cdot e}$ is equal to 1 if μ is dominant and integral for G (and the positive system (7.2)(g)) and zero otherwise.*

Proof. The multiplicity is calculated by Corollary 7.7 as a sum of signs of certain Weyl group elements $S_0(\mu)$. This set is empty unless $\mu+\rho_c$ is integral for G (Lemma 7.9). Because ρ_c is integral for G (by (7.6)(b)), this is equivalent to the integrality of μ. Now assume μ is integral, and define λ to be the dominant weight conjugate to $\mu+\rho_c$. Lemma 7.9 calculates $S_0(\mu)$ whenever it is non-empty; it consists of all elements x in W_K so that xt belongs to S, with t a certain fixed element of W^1. Table 7.11 shows that $S_0(\mu)$ (when it is non-empty) has 1, 2, or 4 elements; and that the sum of the signs of these elements is zero except in the first case. So the multiplicity is zero unless the elements t and w are both trivial; that is, unless $\mu+\rho_c$ is G-dominant. Table 7.12 shows that in this case $\lambda = \mu+\rho_c$ must actually have strictly positive inner product with the roots β_1, β_4, β_6, and β_8; so by (7.6)(b), μ is also G-dominant.

We have shown that the multiplicity is zero unless μ is integral and dominant. Conversely, if μ is integral and dominant, then $x = 1$ belongs to $S_0(\mu)$ (by inspection of (7.6) and Corollary 7.7), and the corresponding element w from Lemma 7.9 is 1. Therefore Lemma 7.9 shows that $S_0(\mu) = \{1\}$, so the multiplicity is one. □

8. Representations of $E_8(\mathbb{R})$

In this section we want to consider the representation-theoretic consequences of the calculations in the last section. As explained in section 2, this means that we want to consider not just functions on $K_{\mathbb{C}} \cdot e$, but sections of certain bundles over the orbit. In order to describe the bundles, we need to understand more precisely the isotropy group $K_{\mathbb{C}}(e)$. Recall that K is $Spin(16)$. We will work mostly inside the Levi subgroup $L_{\mathbb{C}}$ of $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{l} = \mathfrak{gl}(4)_L \times \mathfrak{gl}(4)_R$ (see (7.2)(h)); recall from (3.1) that $L_{\mathbb{C}}$ is the subgroup fixing h. We need to understand the group $L_{\mathbb{C}}$. The torus $H_{\mathbb{C}}$ is a maximal torus of $L_{\mathbb{C}}$. We will first compute the lattice of rational characters $X^*(H)$ for this torus, and then use that to deduce the structure of $L_{\mathbb{C}}$. Recall that we have identified $\mathfrak{h}^*$ with $\mathbb{C}^8$ in (7.1)(a). It will be convenient to write this as

$$\mathfrak{h}^* = \{\lambda = (\lambda_L, \lambda_R) \mid \lambda_L, \lambda_R \in \mathbb{C}^4\}. \qquad (8.1)(a)$$

It will be convenient to understand at the same time structures related to the adjoint group

$$\overline{G} = G/Z(G); \tag{8.1)(b}$$

we write $\overline{H}$, $\overline{K}$, $\overline{L}$, and so on for the corresponding subgroups. The character lattice of $\overline{H}_{\mathbb{C}}$ is just the root lattice of E_8. This is easily computed from (4.2) to be

$$X^*(\overline{H}) = \{\lambda \in \mathbb{Z}^8 \mid \sum \lambda_i \in 2\mathbb{Z}\} \cup \{\lambda \in (\mathbb{Z} + \frac{1}{2})^8 \mid \sum \lambda_i \in 2\mathbb{Z} + 1\}. \tag{8.1)(c}$$

As an introduction to the calculations to come, we will use this description to describe the Levi subgroup of $E_8(\mathbb{C})$ appearing in (4.2). We will call this group F instead of L to avoid confusion with the Levi subgroup L of K. The Lie algebra $\mathfrak{f}$ is $\mathfrak{gl}(8,\mathbb{C})$ by (5.1). If Λ is a lattice containing the root lattice of $GL(8,\mathbb{C})$ and satisfying appropriate integrality conditions, then there is a complex group $F(\Lambda)$ locally isomorphic to $GL(8)$, with weight lattice Λ. It is some of these groups $F(\Lambda)$ that we wish to understand. Notice first that $X^*(\overline{H})$ has index two in the lattice

$$\Lambda^8_{1/2} = \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8. \tag{8.2)(a}$$

It follows that the group $F = F(X^*(\overline{H}))$ is a quotient of $F(\Lambda^8_{1/2})$ by a two-element central subgroup. On the other hand, $\Lambda^8_{1/2}$ contains $\mathbb{Z}^8$ as a sublattice of index 2. It follows that $F(\Lambda^8_{1/2})$ is a two-fold cover of $F(\mathbb{Z}^8)$. This last group is just $GL(8,\mathbb{C})$. The two-fold cover may be constructed by the "square root of the determinant" construction:

$$F(\Lambda^8_{1/2}) = \{(g,z) \mid g \in GL(8,\mathbb{C}),\ z \in \mathbb{C}^\times,\ \det(g) = z^2\}. \tag{8.2)(b}$$

The covering map from $F(\Lambda^8_{1/2})$ to $GL(8)$ is projection on the first factor. Projection on the second factor defines a character of $F(\Lambda^8_{1/2})$, which is a square root of the determinant character of $GL(8)$. The differential of this character is therefore half the differential of the determinant character, or $(1/2,\dots,1/2)$. Using these ideas, it is easy to verify that $F(\Lambda^8_{1/2})$ really has the weight lattice we want. Finally, we must divide $F(\Lambda^8_{1/2})$ by a two-element central subgroup on which the weights of $X^*(\overline{H})$ act trivially. This is easy to compute. Define

$$\epsilon = (-I_8, -1) \in F(\Lambda^8_{1/2}). \tag{8.2)(c}$$

A character $\lambda \in \mathbb{Z}^8$ sends ϵ to $(-1)^{\sum \lambda_i}$, which is one exactly when $\lambda \in X^*(\overline{H})$. To calculate the action of $\lambda \in (\mathbb{Z} + \frac{1}{2})^8$, we write it as a sum of $(1/2,\dots,1/2)$ and an element of $\mathbb{Z}^8$. The first summand acts on ϵ by projection on the second factor (hence by -1); so altogether we find

$$\lambda(\epsilon) = -(-1)^{\sum(\lambda_i - 1/2)} = -(-1)^{\sum \lambda_i}. \tag{8.2)(d}$$

This is one exactly when $\lambda \in X^*(\overline{H})$. Consequently

$$F(X^*(\overline{H})) = F(\Lambda^8_{1/2})/\{1,\epsilon\}. \tag{8.2)(e}$$

This is the Levi subgroup considered in (4.2).

We return now to the Levi subgroup $L_{\mathbb{C}} \subset K_{\mathbb{C}}$ considered in (7.2). Recall that $L_{\mathbb{C}}$ is locally isomorphic to $GL(4) \times GL(4)$. Just as above, if Λ is an appropriately integral lattice containing the root lattice of $GL(4) \times GL(4)$, we will write $L_{\mathbb{C}}(\Lambda)$ for a locally isomorphic group with weight lattice Λ. The image of L in the adjoint group is easily computed from (8.2). It is a subgroup of F, namely

$$\overline{L_{\mathbb{C}}} = L_{\mathbb{C}}(X^*(\overline{H})) = \{(g,h,z) \mid g,h \in GL(4),\ z \in \mathbb{C}^\times,\ \det(gh) = z^2\}/\{1,\epsilon\}. \tag{8.3)(a}$$

Here $\epsilon = (-I_4, -I_4, -1)$ is the element from (8.2)(c).

Because the maximal compact subgroup of a simply connected group is simply connected, the lattice $X^*(H)$ consists of the weights $\lambda \in \mathfrak{h}^*$ integral with respect to the compact roots. Because the roots all have length two in the standard inner product, these are the weights having integer inner product with all compact roots. The compact roots are listed in (7.2)(b). Integrality with respect to the roots A leads to the lattice

$$\Lambda^4_{1/2} = \mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4. \tag{8.3)(b}$$

Imposing integrality with respect to the roots B gives

$$X^*(H) = \{(\lambda_L, \lambda_R) \in \Lambda^4_{1/2} \times \Lambda^4_{1/2} \mid \sum \lambda_i \equiv 2\lambda_8 \pmod{2\mathbb{Z}}\} \tag{8.3)(c}$$

The group with weight lattice $\Lambda^4_{1/2}$ may be computed as a double cover of $GL(4)$ just as in (8.2)(b); $L_{\mathbb{C}}(\Lambda^4_{1/2} \times \Lambda^4_{1/2})$ is a product of two copies of this double cover. Because $X^*(H)$ is a sublattice of index 2 in the product lattice, the group $L_{\mathbb{C}}$ is a quotient of the product group by a subgroup of order two. Explicitly,

$$L_{\mathbb{C}} = L_{\mathbb{C}}(X^*(H)) = \{(g_L, z_L, g_R, z_R) \in (GL(4) \times \mathbb{C}^\times)^2 \mid z_x^2 = \det(g_x)\}/\{1,\delta\}. \tag{8.3)(d}$$

Here $\delta = (-I_4, 1, -I_4, -1)$. The verification that this group has the correct weight lattice is parallel to the argument for (8.2)(e), and we omit it. The two-to-one covering map from $L_{\mathbb{C}}$ to $\overline{L_{\mathbb{C}}}$ may be defined before the quotients as

$$(g_L, z_L, g_R, z_R) \mapsto (g_L, g_R, z_L z_R) \tag{8.3)(e}$$

Since this map carries δ to ϵ, it passes to the quotients.

The representation of $L_{\mathbb{C}}$ on $\mathfrak{p}(2)$ has been identified in Lemma 7.3. Identifying $\mathbb{C}^4 \otimes \mathbb{C}^4$ with 4 by 4 matrices, we find that the class of (g_L, z_L, g_R, z_R) acts by

$$(g_L, z_L, g_R, z_R) \cdot A = g_L A\, {}^t g_R. \tag{8.3)(f}$$

In this identification the matrix unit e_{ij} has weight $(e_i, e_j) \in \mathbb{Z}^4 \times \mathbb{Z}^4$. Using (4.3), we deduce that the element e may be taken to be the identity matrix I_4. Now (8.3)(f) and (8.3)(d) allow us to compute the stabilizer of e in $L_{\mathbb{C}}$: it is

$$L_{\mathbb{C}}(e) = \{(g, z, {}^t g^{-1}, w) \mid z^2 = \det(g) = w^{-2}\}/\{1,\delta\}. \tag{8.3)(g}$$

Before passage to the quotient, this is easily seen to be a direct product

$$\{(g, z, {}^t g^{-1}, z^{-1}) \mid z^2 = \det(g)\} \times \{1, \delta\}.$$

Consequently $L_{\mathbb{C}}(e)$ is isomorphic to the first factor. This in turn is the connected double cover of $GL(4)$:

$$L_{\mathbb{C}}(e) \simeq \{(g,z) \mid g \in GL(4),\ z \in \mathbb{C}^\times,\ z^2 = \det(g)\}. \tag{8.3)(h}$$

Now Proposition 3.2 implies that $K_{\mathbb{C}}(e)$ is connected.

We turn next to the determination of the admissible orbit data. According to Definition 2.6, we must begin by computing the character γ by which $K_{\mathbb{C}}(e)$ acts on the top exterior power of $(\mathfrak{k}/\mathfrak{k}(e))^*$. This is an algebraic character, so the unipotent radical $U(e)$ acts trivially. It is therefore enough to compute the action of $L_{\mathbb{C}}(e)$ (Proposition 3.2). Using the representation theory of $SL(2)$, one can reorganize this problem in various ways. Here is one that will suffice for our purposes.

Lemma 8.4. *Suppose we are in the setting (3.1). Define $\tau(m)$ to be the character of $L_{\mathbb{C}}$ given by the determinant of the adjoint action on $\mathfrak{k}(m)$. That is, the weight of $\tau(m)$ (with respect to a Cartan subalgebra of $\mathfrak{k}$ containing h) is the sum of the compact roots α such that $\alpha(h) = m$. Consider the character*

$$\tau = \tau(1) \otimes \tau(3)^2 \otimes \tau(5)^2 \otimes \cdots$$

of $L_{\mathbb{C}}$. Then the restriction of τ to $L_{\mathbb{C}}(e)$ agrees with $\gamma(e)$ (Definition 2.6) up to sign. In particular, τ and $\gamma(e)$ agree on the identity component of $L_{\mathbb{C}}(e)$.

This result refines Corollary 7.27 of [Vunip]. We omit the proof; the necessary ideas (taken from [Schwartz]) may be found in section 7 of [Vunip] (particularly (7.26)(a) and (7.24)(f)). In our case the weights $\tau(m)$ may be computed from (7.2)(b). We find

$$\tau(1) = (5,5,5,5,2,2,2,2), \quad \tau(3) = (1,1,1,1,2,2,2,2),$$

and therefore

$$\tau = \tau(1) + 2\tau(3) = (7,7,7,7,6,6,6,6). \tag{8.5)(a}$$

The restriction to $L_{\mathbb{C}}(e)$ of any weight (λ_L, λ_R) for $L_{\mathbb{C}}$ is $\lambda_L - \lambda_R$ by (8.3)(g). Lemma 8.4 now gives the character $\gamma(e)$ of Definition 2.6 as

$$\gamma(e) = (1,1,1,1), \tag{8.5)(b}$$

the ordinary determinant character of $GL(4)$; of course this is using the identification (8.3)(h) of $L_{\mathbb{C}}(e)$. This character has a unique square root $\chi(e)$, which is the unique admissible orbit datum for $K_{\mathbb{C}} \cdot e$ (Definition 2.6). Explicitly,

$$\chi(e) = (1/2, 1/2, 1/2, 1/2). \tag{8.5)(c}$$

Conjecture 2.9 says that the K-types of a representation associated to this nilpotent orbit should be given by the sections of the line bundle $\mathcal{V}_{\chi(e)}$ over $K_{\mathbb{C}} \cdot e$. Proposition 3.15 suggests a way to compute these sections. We first need a character $\delta(e)$ of $L_{\mathbb{C}}$ whose restriction to $L_{\mathbb{C}}(e)$ is $\chi(e)$. A natural choice is

$$\delta(e) = (1/2, 1/2, 1/2, 1/2, 0, 0, 0, 0). \tag{8.5)(d}$$

(Many other characters, such as $(1/2+m, 1/2+m, 1/2+m, 1/2+m, m, m, m, m)$, also restrict to $\chi(e)$. Our choice is the smallest one possible, and the weight $\delta(e)$ is dominant for K. The first property of $\delta(e)$ should make it possible to prove a cohomology vanishing theorem in the setting of Proposition 3.15; and the second should make the inclusion of Proposition 3.15 more nearly an isomorphism. We will not prove any results in this direction, but an analogous argument appears in [Gra] (Proposition 1.4 and Theorem 1.3).) At any rate, we make

Conjecture 8.6. Suppose G is the simply connected split real group of type E_8, $K \simeq Spin(16)$ is a maximal compact subgroup, and $e \in \mathfrak{p}$ is a representative of the nilpotent $K_{\mathbb{C}}$ orbit of complex dimension 64. In the notation of (3.1) and (3.3), let $\delta(e)$ be the character of Q (a parabolic subgroup of $K_{\mathbb{C}}$) be the character introduced in (8.5).

1. The inclusion
$$\sum_{k=0}^{\infty} \mathrm{Ind}_Q^{K_{\mathbb{C}}}(S^k(\mathfrak{v}^*) \otimes V_{\delta(e)}) \hookrightarrow \text{ sections of } \mathcal{V}_\chi \text{ over } K_{\mathbb{C}} \cdot e$$
of Proposition 3.15 is an isomorphism.
2. For all $p > 0$, the higher cohomology spaces $(\mathrm{Ind}_Q^{K_{\mathbb{C}}})^p(S^k(\mathfrak{v}^*) \otimes V_{\delta(e)})$ are equal to zero.

We can now imitate the calculation of section 7 with the twist by $\delta(e)$. In the setting of Proposition 7.5, E_λ occurs in $S(\mathfrak{v}) \otimes V^*_{\delta(e)}$ if and only if $E_\lambda \otimes V_{\delta(e)}$ occurs in $S(\mathfrak{v})$. The highest weight of $E_\lambda \otimes V_{\delta(e)}$ is
$$(\lambda_1 + 1/2, \ldots, \lambda_4 + 1/2, \lambda_5, \ldots, \lambda_8).$$

Proposition 8.7. *Suppose we are in the setting (7.2), and $\delta(e)$ is the character of (8.5)(d). Suppose $\lambda = (\lambda_1, \ldots, \lambda_8) \in \mathbb{C}^8$ is a dominant weight for $\mathfrak{l} \simeq \mathfrak{gl}(4) \times \mathfrak{gl}(4)$; write E_λ for the irreducible representation of highest weight λ. Then the multiplicity of E_λ in $S(\mathfrak{v}) \otimes V^*_{\delta(e)}$ is equal to 1 if*
$$\lambda_1 + 1/2 \geq \lambda_5 \geq \lambda_2 + 1/2 \geq \lambda_6 \geq \lambda_3 + 1/2 \geq \lambda_7 \geq \lambda_4 + 1/2 \geq \lambda_8,$$
with all differences integers, and
$$\frac{1}{2}(-\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8) - 1/2$$
is a non-negative integer. Otherwise the multiplicity is zero. In terms of the eight simple roots β_i of (7.2)(g) and the corresponding coroots h_i, the conditions for multiplicity one may be written as
$$\lambda(h_i) \in \mathbb{N} + 1/2 \quad (i = 3, 5, 7), \qquad \lambda(h_j) \in \mathbb{N} - 1/2 \quad (j = 1, 4, 6, 8),$$
$$\lambda(h_2 + h_3 + h_4 + h_5) \in \mathbb{N} + 1/2.$$

This is just a translation of Proposition 7.5.

Corollary 8.8. *Suppose we are in the setting (7.2). Write $\alpha = \beta_2 + \beta_3 + \beta_4 + \beta_5$, and h_α for the corresponding coroot. Fix a K-dominant weight $\mu \in X^*(H)$, and write V_μ for the corresponding irreducible representation of K. Define a subset $\widetilde{S}_0(\mu) \subset W_K$ by*
$$\widetilde{S}_0(\mu) = \{x \in W_K \mid x(\mu + \rho_c)(h_\alpha) \in \mathbb{N} + 3/2, \quad x(\mu + \rho_c)(h_i) \in \mathbb{N} + 1/2 \quad (i \neq 2)\}$$
Assume that Conjecture 8.6 holds. Then the multiplicity of V^*_μ in the space of sections of $\mathcal{V}_\chi$ over $K_{\mathbb{C}} \cdot e$ is
$$\sum_{x \in \widetilde{S}_0(\mu)} \mathrm{sgn}(x).$$

The proof is identical to that of Corollary 7.7.

The calculation of the sets $\widetilde{S}_0(\mu)$ is actually easier than in section 7. The reason is that the conditions in Corollary 8.8 force $\mu + \rho_c$ to be regular. (The reason is that its value on each noncompact coroot must belong to $\mathbb{Z} + 1/2$, and we already know that the values on compact coroots are non-zero.) Here is the main step.

Lemma 8.9. *Fix a K-dominant weight μ, and let λ be the unique G-dominant weight conjugate by W to $\mu + \rho_c$. Then the set $\widetilde{S}_0(\mu)$ (Corollary 7.7) is non-empty only if $\lambda(h_i)$ is regular for G. Assume therefore that λ is regular. For each $x \in \widetilde{S}_0(\mu)$, let w be the unique element of W such that*

$$x(\mu + \rho_c) = w\lambda. \tag{8.9}$$

Then $w \in S$ (Corollary 5.8). The element x may be computed from w by Proposition 7.8, as the first term in the factorization $w = xt$. Furthermore $\mu + \rho_c = t\lambda$. Finally, we have

$$\widetilde{S}_0(\mu) = \{x' \in W_K \mid w' = x't \in S\}.$$

(Here the factorization of w' is the one in Proposition 7.8.)

The proof is a simplified version of that of Lemma 7.9, so we omit it.

Theorem 8.10. *Suppose we are in the setting (7.2). Fix a K-dominant weight $\mu \in X^*(H)$, and write V_μ for the corresponding irreducible representation of K. Assume that Conjecture 8.6 holds. Then the multiplicity of V_μ^* in the space of sections of the line bundle $\mathcal{V}_\chi$ over $K_{\mathbb{C}} \cdot e$ is equal to 1 if*

$$(\mu + \rho_c)(h_i) \in \mathbb{N} + 1/2, \quad (i = 1, 2, \ldots, 8).$$

Otherwise the multiplicity is equal to zero.

The proof is a simplified version of that of Theorem 7.13. The main point is that in Table 7.11, the elements x appearing in factorizations xt (for fixed non-identity t) have signs summing to zero. We omit the details.

Recall that Conjecture 2.9 says that the K-types computed in Theorem 8.10 should be those of a unitary unipotent representation. It is not completely trivial to list the representations of $Spin(16)$ given by Theorem 8.10, but there is certainly an obvious "lowest" one. (In fact it is very easy to see that it is lowest in the technical sense of [Green].) This is the one with highest weight μ_0 defined by

$$(\mu_0 + \rho_c)(h_i) = 1/2, \quad (i = 1, \ldots, 8). \tag{8.11(a)}$$

Using (7.6)(b) we deduce

$$\mu_0(h_i) = \begin{cases} -1/2, & \text{if } i = 1, 4, 6, \text{ or } 8 \\ 1/2, & \text{if } i = 2, 3, 5, \text{ or } 7. \end{cases} \tag{8.11(b)}$$

Using (7.6)(a), we can now compute the value of μ_0 on the compact simple coroots h_β. The result is

$$\mu_0(h_\beta) = \begin{cases} 1, & \text{if } \beta = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ 0, & \text{otherwise.} \end{cases} \tag{8.11(c)}$$

That is, V_{μ_0} is the fundamental representation of $Spin(16)$ attached to the simple root at the unbranched end of the Dynkin diagram. This is the standard 16-dimensional representation:

$$V_{\mu_0} = \mathbb{C}^{16}.$$

We would like to identify the representation of G that (according to Conjecture 2.5) is attached to the real nilpotent orbit of dimension 128. According to Conjecture 2.9 and (8.11), it should have lowest K-type the 16-dimensional representation. We therefore look at the entire family of representations of G having this lowest K-type. Fix a minimal parabolic subgroup $P = MAN$ of G; here A is a vector group with Lie algebra $\mathfrak{a}_0$ a maximal abelian subalgebra of $\mathfrak{p}_0$, and M is the centralizer of A in G. Because $\overline{G}$ is a linear group split over $\mathbb{R}$, its Cartan subgroup $\overline{MA}$ is a product of 8 copies of $\mathbb{R}^\times$. Consequently

$$\overline{M} \simeq (\mathbb{Z}/2\mathbb{Z})^8, \tag{8.12)(a}$$

an abelian group of order 256. The preimage M of $\overline{M}$ in G is therefore a group of order 512. According to the Bernstein-Gelfand-Gelfand theory of fine representations of $\overline{K}$ (see for example [Green], Theorem 4.3.16,)

$$(S^2(\mathbb{C}^{16}) + \wedge^2\mathbb{C}^{16}) \mid_{\overline{M}} \simeq \text{regular representation of } \overline{M}. \tag{8.12)(b}$$

Now the left side here is just the tensor product of the (self-dual) representation $\mathbb{C}^{16}$ with itself. By Schur's lemma, the fact that the trivial representation of M appears just once in the tensor product guarantees that

$$\mathbb{C}^{16} \mid_M \simeq \text{irreducible representation } \delta \text{ of } M. \tag{8.12)(c}$$

Because the sum of the squares of the dimensions of the irreducible representations of the finite group M must be 512, we see that the 256 one-dimensional representations of $\overline{M}$ and the 16-dimensional representation δ of M together exhaust $\widehat{M}$. In fact the theory of fine representations extends to the non-linear group G (see [Alg], section 6); one finds that $\mathbb{C}^{16}$ is a fine representation of K.

Proposition 8.13 ([Alg], Proposition 6.7). *Suppose as above that $P = MAN$ is a minimal parabolic subgroup of a split real group of type E_8, and that δ is the 16-dimensional irreducible representation of M. Write $K \simeq Spin(16)$ for a maximal compact subgroup of G, and $\overline{K}$ for its image in the linear group $\overline{G}$.*

1. *If V is any irreducible representation of K not factoring to $\overline{K}$, then $V \mid_M$ is a multiple of δ. In particular, the dimension of V is divisible by 16.*
2. *If $\nu \in \widehat{A} \simeq \mathfrak{a}^*$ is any character, define*

$$I(\delta \otimes \nu) = \mathrm{Ind}_P^G(\delta \otimes \nu \otimes 1),$$

   a principal series representation of G. The restriction of $I(\delta \otimes \nu)$ to K is the sum of all the irreducible representations of K not factoring to $\overline{K}$; each occurs with multiplicity equal to its dimension divided by 16.
3. *The infinitesimal character of $I(\delta \otimes \nu)$ is given in the Harish-Chandra parametrization by the weight $\nu \in \mathfrak{a}^*$.*
4. *The character of $I(\delta \otimes \nu)$ depends only on the Weyl group orbit of ν.*
5. *Define $J(\delta \otimes \nu)$ to be the unique irreducible subquotient of $I(\delta \otimes \nu)$ containing the K-type $\mathbb{C}^{16}$ (which has multiplicity one). Then the representations $J(\delta \otimes \nu)$ exhaust the irreducible $(\mathfrak{g}, K)$-modules containing the K-type $\mathbb{C}^{16}$. Two of these are equivalent if and only the weights in $\mathfrak{a}^*$ are conjugate by the Weyl group.*

In addition to the results in [Alg], the main observation incorporated here is that the stabilizer in the Weyl group of δ is the entire Weyl group. This is clear from (8.12)(c), or from the fact that δ is the only representation of M of dimension 16.

Proposition 8.13 says that there is exactly one irreducible representation of G containing the K-type $\mathbb{C}^{16}$ for each infinitesimal character. In order to specify the representation attached to our 128-dimensional orbit by Conjecture 2.5, we need only specify the infinitesimal character. Here is a precise statement.

Conjecture 8.14. Suppose G is the simply connected split real Lie group of type E_8, and $\lambda \in i\mathfrak{g}_0^*$ is a representative of the unique nilpotent coadjoint orbit of dimension 128. With notation as in Proposition 8.13, let $\rho \in \mathfrak{a}^*$ be half the sum of the positive roots. Then the irreducible $(\mathfrak{g}, K)$-module $J(\delta \otimes (\rho/2))$ is an irreducible unitary representation of G attached to the orbit $G \cdot \lambda$. In particular, the restriction of $J(\delta \otimes (\rho/2))$ to K is described by Theorem 8.10: the K-types all have multiplicity one, and lie in the translate of the cone of dominant integral weights for E_8 by the highest weight of $\mathbb{C}^{16}$.

The concrete content of the conjecture is in the assertion that $J(\delta \otimes (\rho/2))$ is unitary, and in the description of its K-types. It is not very difficult to show that the annihilator of $J(\delta \otimes (\rho/2))$ in $U(\mathfrak{g})$ is the maximal primitive ideal of infinitesimal character $\rho/2$. The corresponding primitive quotient $U(\mathfrak{g})/I$ is the Harish-Chandra module for complex E_8 studied by McGovern in [McGmodel]. He shows that its K-types are a model representation for the compact form of E_8, in agreement with the prediction given by Theorem 1.1. According to the general philosophy of unipotent representations of real groups, having this kind of "unipotent annihilator" is a necessary condition for being a unipotent representation (see [Vorange], chapter 9). But we do not know how to prove either of the concrete assertions in the conjecture.

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