Reductive Group Representations and Weyl Group Representations

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Representation Theory and Lie Groups—
in celebration of the career of Jeffrey Adams

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Intro

Coherent families

W reps

Computers

About *W* action

KL cells

HC cells

Homework

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Outline

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- Weyl group representations
- Interlude: computers
- Understanding coherent continuation reps
- Kazhdan-Lusztig cells
- Harish-Chandra cells
- What questions can you ask?
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Slides at `http://www-math.mit.edu/~dav/paper.html`

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Where did this come from?

When I was almost thirty and Jeff was even younger, I tried to make him solve my problems.

Pushing people around is always a mistake.

Although you can learn from others here that I didn't give it up.

When I was almost fifty, it was time to pay the piper.

Jeff wanted software to do representation theory.

He could almost create it single-handed. . . but not quite.

In the next two decades, I learned **C++**, **gdb**, **git**, **gzip**, **tar**, **screen**, **top**, **slurm**, and new modes of **emacs**.

I learned to cringe before **segmentation fault** and **Killed**.

I drooled over a laptop with 14 cores and 36 gigs.

Four years ago I retired, in order to play with these toys seven days a week.

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Sorry, wasn't this a **math** conference?

Of course Jeff promised that this was all for math. . .

. . . that we were, as one says, only in it for the articles.

The road was paved with good intentions.

This talk concerns mathematics that we claim to have understood better by pushing it to **github**.

A shorter title might have been *Whited Sepulchres*.

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General setting

G complex connected reductive algebraic group

$B \subset G$ Borel subgroup

$T \subset B \subset G$ maximal torus

$X^*(T)$ lattice of rational characters

$R = R(G,T) \subset X^*(T)$ roots of T in G

$R^+ = R^+(G,T) \subset X^*(T)$ positive roots of T in B

$\Pi \subset R^+$ simple roots of T in B

$(X^*, \Pi, X_*, \Pi^\vee)$ based root datum of $T \subset B \subset G$

$\alpha \in \Pi \rightsquigarrow s_\alpha \in \mathrm{Aut}(X^*), \quad s_\alpha(\gamma) = \gamma - (\gamma(\alpha^\vee)) \cdot \alpha$
simple reflection

$W = W(G,T) = \langle s_\alpha \alpha \in \Pi > \subset \mathrm{Aut}(X^*)$ Weyl group

W is finite; seek to relate reps of G to reps of W.

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General setting, part two

$T \subset G$ max torus, $X^*(T)$ char lattice, $W \subset \mathrm{Aut}(X^*)$ Weyl group.

Tensoring with k makes lattice into k-vector space:

$$\mathfrak{t}^* = X^*_{\mathbb{C}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C},$$

dual of Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$.

$\mathfrak{Z}(\mathfrak{g}) = U(\mathfrak{g})^G$ center of enveloping algebra

Theorem (Harish-Chandra, Chevalley) $\mathrm{Hom}(\mathfrak{Z}(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{t}^*/W$.

$G(\mathbb{R})$ real form of G

Could use nonlinear $\widetilde{G}(\mathbb{R})$ with finite map to open subgp of $G(\mathbb{R})$.

$\theta \in \mathrm{Aut}(G)$ Cartan involution for $G(\mathbb{R})$

$K(\mathbb{R}) = G(\mathbb{R})^\theta$ maximal compact subgroup

reps of $G(\mathbb{R})$: $\mathcal{F}(\mathfrak{g}(\mathbb{R}), K(\mathbb{R})) = \mathcal{F}(\mathfrak{g}, K)$ fin lgth HC modules.

Some actual mathematics

Easy to show that $\mathfrak{Z}(\mathfrak{g})$ acts by scalars on any irr $G(\mathbb{R})$ rep π, defining infinitesimal character $\xi_\pi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$.

HC-Chev theorem says $\xi_\pi \leftrightarrow \gamma_\pi \in \mathfrak{t}^*/W$.

Write $\mathcal{F}(\mathfrak{g}, K)_\gamma =$ reps of (generalized) infl char γ.

Big difference between real and p-adic rep theory is alg (irr) reps F of G restrict to (irr) reps of $G(\mathbb{R})$.

Theorem (Kostant): if $\pi_\gamma \in \mathcal{F}(\mathfrak{g}, K)_\gamma$ $G(\mathbb{R})$ rep of infl char γ, F alg rep of G, then

$$\pi_\gamma \otimes F \in \sum_{\phi \text{ wt of } F} \mathcal{F}(\mathfrak{g}, K)_{\gamma+\phi} :$$

that is, tensoring with F translates infl char by wts of F.

Jantzen, Zuckerman, and Hecht-Schmid made tensoring with alg reps into a systematic construction of new $G(\mathbb{R})$-reps from old. Here's how...

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Coherent families

Recall Kostant: $\pi_\gamma \otimes F \in \sum_{\phi \text{ wt of } F} \mathcal{F}(\mathfrak{g}, K)_{\gamma+\phi}$.

Definition (Hecht-Schmid) Suppose $\gamma \in \mathfrak{t}^*$. A coherent family based on $\gamma + X^*$ is a map $\Theta\colon X^* + \gamma \to$ virtual $G(\mathbb{R})$ reps satisfying (for any algebraic rep F of G)

1. $\Theta(\gamma+\tau)$ has infl char $\gamma+\tau$
2. $\Theta(\gamma+\tau) \otimes F = \sum_{\phi \text{ wt of } F} \Theta(\gamma+\tau+\phi)$.

Def depends only on coset $\gamma + X^*$, **not** on base point γ.

Theorem (Jantzen/Zuckerman/Hecht-Schmid) The map

$$\text{eval}_\gamma\colon \text{coh fams on } \gamma + X^* \to \text{ virt reps of infl char } \gamma$$

is surjective. If γ is regular, it is an isomorphism.

Example: any γ, $\Theta_\gamma(\gamma+\tau) =$ Verma of hwt $\gamma+\tau-\rho$.

Example: $\gamma = \rho$,

$$\Theta_F(\gamma_1) = \begin{cases} \text{sgn}(x) \cdot F_{x\cdot(\gamma_1)-\rho} & (x(\gamma_1) \text{ dom regular}) \\ 0 & (\gamma_1 \text{ singular }) \end{cases}$$

So where is the Weyl group action?

Definition (Hecht-Schmid) Suppose $\gamma \in \mathfrak{t}^*$. A coherent family based on $\gamma + X^*$ is a map $\Theta\colon X^* + \gamma \to$ virtual $G(\mathbb{R})$ reps satisfying (for any algebraic rep F of G)

1. $\Theta(\gamma + \tau)$ has infl char $\gamma + \tau$
2. $\Theta(\gamma + \tau) \otimes F = \sum_{\phi \text{ wt of } F} \Theta(\gamma + \tau + \phi)$.

Given $w \in W$ and any coh fam Θ based on $\gamma + X^*$, get new coh fam $w \cdot \Theta$ based on $w \cdot \gamma + X^*$,

$$(w \cdot \Theta)(w \cdot \gamma + \tau) =_{\text{def}} \Theta(\gamma + w^{-1} \cdot \tau).$$

This is coherent by two elementary facts:

1. infl char ξ = infl char $w\xi$, and
2. set of wts of F is W-invt.

We get in this way an action of W on the set of coherent families (either obviously or I got the inverse wrong).

To get a representation, need something like a vector space inside the set of coherent families.

And finally a representation!

Write $\mathcal{T}(\mathfrak{g}, K)_{\gamma+X^*}$ = coh fams based on $\gamma + X^*$

$\mathcal{V}(\mathfrak{g}, K)$ = virtual reps (free abelian on irreps),

$\mathcal{V}(\mathfrak{g}, K)_\gamma$ = virtual reps of infl char $\gamma \in \mathfrak{t}^*$ (finite rk lattice).

Recall surj (isom for γ regular) $\mathrm{eval}_\gamma : \mathcal{T}(\mathfrak{g}, K)_{\gamma+X^*} \to \mathcal{V}(\mathfrak{g}, K)_\gamma$.

Since $\gamma + X^*$ has a reg elt γ_{reg}, $\mathcal{T}(\mathfrak{g}, K)_{\gamma+X^*}$ has rk $\#(\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_{\mathrm{reg}}}$.)

Finally, the W action is $w\cdot : \mathcal{T}(\mathfrak{g}, K)_{\gamma+X^*} \to \mathcal{T}(\mathfrak{g}, K)_{w\cdot\gamma+X^*}$

$$(w \cdot \Theta)(w \cdot \gamma + \tau) =_{\mathrm{def}} \Theta(\gamma + w^{-1} \cdot \tau).$$

Define $W(\gamma) = \{w \in W \mid w \cdot \gamma - \gamma \in X^*\}$, big integral Weyl gp.

$W_0(\gamma) = \{w \in W \mid w \cdot \gamma - \gamma = \text{sum of roots}\}$, small int Weyl gp.

Thm (Chevalley) $W_0(\gamma)$ = Weyl gp of $R(\gamma) = \{\alpha \in R \mid \gamma(\alpha^\vee) \in \mathbb{Z}\}$.

Theorem W action on coh fams restricts to (integral) reps of $W_0(\gamma) \subset W(\gamma)$ on the fin rk lattice $\mathcal{T}(\mathfrak{g}, K)_{\gamma+X^*} \simeq \mathcal{V}(\mathfrak{g}, K)_{\gamma_{\mathrm{reg}}}$.

Rep of $W_0(\gamma)$ on $\mathcal{T}(\mathfrak{g}, K)_\gamma$ is the one we want to study.

Recall our two examples. . .

. . . of coherent families; see what $W(\gamma)$ reps they give.

$\gamma = \rho$, coh fam on $\rho + X^*$ of alg reps:

$$\Theta_F(\gamma_1) = \begin{cases} \mathrm{sgn}(x) \cdot F_{x\gamma_1 - \rho} & (x\gamma_1 \text{ dom regular}) \\ 0 & (\gamma_1 \text{ singular }) \end{cases}$$

All roots are integral, $W_0(\gamma) = W$, and

$$\begin{aligned} w \cdot \Theta_F(\gamma_1) &= \Theta_F(w^{-1} \cdot \gamma_1) \\ &= \mathrm{sgn}(x) F_{xw^{-1}\gamma_1 - \rho} \qquad (xw^{-1}\gamma_1 \text{ dom regular}) \\ &= \mathrm{sgn}(w)\,\mathrm{sgn}(xw^{-1}) F_{xw^{-1}\gamma_1 - \rho} = \mathrm{sgn}(w)\Theta_F(\gamma_1). \end{aligned}$$

So the coh fam Θ_F of alg reps transforms by $\mathrm{sgn}(W)$.

Similar but simpler calculation shows for coh fam of Verma mods

$$w \cdot \Theta_\gamma = \Theta_{w^{-1}\gamma};$$

so $\{\Theta_{x\gamma} \mid x \in W_0(\gamma)\}$ transforms by regular rep of $W_0(\gamma)$.

Moral of exs: $W(\gamma)$ rep is dull on dull reps (like Verma mods), interesting on interesting reps (like algebraic reps).

Interlude: computers

Lie groups are defined using differential geometry.

That isn't perfectly suited to computer models.

But I talked about G almost entirely in terms of the based root datum $(X^*, \Pi, X_*, \Pi^\vee)$.

X^* and X_* are dual lattices; can take them to be $\mathbb{Z}^n$.

Based root datum $\leftrightsquigarrow$ $n \times r$ int mat Π (cols are the r simple roots), and $r \times n$ int mat $\Pi^\vee$ (rows are the r simple coroots).

$W \subset n \times n$ integer matrices , computable from Π and $\Pi^\vee$.

W reps σ above have $\mathbb{Z}$ bases, so $\sigma(W) \subset GL(n, \mathbb{Z})$.

Hard to imagine math better suited to digitization.

Evidently Chevalley, Harish-Chandra, Jantzen, *et alia* had computers in mind when they created this stuff.

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About the W action

Fix $\gamma \in \mathfrak{t}^*$, $\gamma_0 \in \gamma + X^*$ dominant, $\gamma_1 \in \gamma + X^*$ dom reg.

Write $R(\gamma)$ = integral roots, $\Pi(\gamma)$ simple roots for $R(\gamma)$.

Fix $\Theta \in \mathcal{T}(\mathfrak{g}, K)_\gamma$ coh fam: Θ is irreducible if $\Theta(\gamma_1)$ is irreducible.

Θ is standard if $\Theta(\gamma_1)$ is standard (*i.e.*, induced(rel disc series)).

τ-invariant of Θ is $\tau(\Theta) = \{\alpha \in \Pi(\gamma) \mid s_\alpha \cdot \Theta = -\Theta\} \subset \Pi(\gamma)$.

Theorem Suppose $\Theta \in \mathcal{T}(\mathfrak{g}, K)_\gamma$ irr and $\alpha \in \Pi(\gamma) \backslash \tau(\Theta)$. Then

$$s_\alpha \cdot \Theta = \Theta + \sum_{\substack{\Theta' \in \mathcal{T}(\mathfrak{g},K)_\gamma \text{ irr} \\ \alpha \in \tau(\Theta')}} \mu(\Theta, \Theta')\Theta'.$$

Here $\mu(\Theta, \Theta') = \dim \mathrm{Ext}^1_{\mathfrak{g},K}(\Theta(\gamma_1), \Theta'(\gamma_1)))$, nonneg integer.

Theorem (Langlands, Kazhdan-Lusztig, Hecht-Schmid)

1. Std coh fams and irr coh fams are each a basis of $\mathcal{T}(\mathfrak{g}, K)_\gamma$.
2. Chg of basis upper tri, 1s on diag; entries KL polys at $q = 1$.
3. Action of s_α on std basis is block diag: blocks

$$(1), \quad (-1), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Conclusion: $W_0(\gamma)$ rep in basis of stds is easy $\rightsquigarrow$ info about all reps.

$W_0(\gamma)$ rep in basis of irrs is difficult $\rightsquigarrow$ info about individual reps.

What's a Harish-Chandra cell (A)?

$G(\mathbb{R})$ real reductive $\supset K(\mathbb{R}) = G(\mathbb{R})^\theta$ maxl compact

$G \supset K = G^\theta$ complexifications, $\mathfrak{g} = \mathrm{Lie}(G)$

Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$

$\gamma_1 \in \mathfrak{h}^*$ dom reg int drop general infinitesimal char to simplify

$\mathcal{F}(\mathfrak{g}, K)_{\gamma_1} = (\mathfrak{g}, K)$-modules of infinitesimal char γ_1

$\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$ = irr reps, $\mathcal{V}(\mathfrak{g}, K)_{\gamma_1} = \mathbb{Z} \cdot \mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$ virt reps

Weyl group W acts on $\mathcal{V}(\mathfrak{g}, K)_{\gamma_1}$.

Get natural W representation, basis $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

Goal: divide W rep into Harish-Chandra cells.

Analogous to left reg rep of $W \rightsquigarrow$ left cells (KL).

What's a Harish-Chandra cell (B)?

Have coh cont rep of W on $\mathcal{V}(\mathfrak{g}, K)_{\gamma_1}$, basis $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

Submod $\mathcal{S} \subset \mathcal{V}(\mathfrak{g}, K)_{\gamma_1}$ called basal if spanned by irrs.

HC cone of $\pi \in \mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$ is min basal W-submod containing π:

$$\overline{C}_{\mathbb{Z}}(\pi) = \mathrm{span}\,\{\pi' \in \mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1} \mid \exists w \in W \text{ so } \pi' \text{ appears in } w \cdot \pi\}$$

This set of irrs π' (also called the HC cone of π) is

$$\overline{C}(\pi) = \{\pi' \in \mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1} \mid \exists F \text{ alg for } G^{\mathrm{ad}} \text{ so } \pi' \text{ comp fac of } \pi \otimes F\}$$

What's a Harish-Chandra cell (C)?

Preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$: Kazhdan-Lusztig style definition is

$$Y \underset{LR}{\leq} X \iff \exists w \in W,\ [Y] \text{ appears in } w \cdot X.$$

Representation def is (with F fin-diml rep of G^{ad})

$$Y \underset{LR}{\leq} X \iff \exists F,\ Y \text{ composition factor of } F \otimes X.$$

Equiv rel $Y \underset{LR}{\sim} X$ means $Y \underset{LR}{\leq} X \underset{LR}{\leq} Y$; complement is $Y \underset{LR}{<} X$.

A Harish-Chandra cell is an $\underset{LR}{\sim}$ equiv class in $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

Two irrs are in the same cell iff can get from each to other by tensoring with fin-diml reps of G^{ad}.

$C(X) = \underset{LR}{\sim}$ equiv class of X = HC cell $\subset \mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

$\overline{C}(X) = \underset{LR}{<}$ interval below X = HC cone $\subset \mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

$\partial C(X) = \overline{C}(X) - C(X)$.

What's true about Harish-Chandra cells?

An interesting and difficult-to-compute invariant of a representation X is its associated variety $\mathcal{AV}(X)$.

Theorem. (Consequence of rep theory defn of cells.)

1. $Y \underset{LR}{\leq} X \implies \mathcal{AV}(Y) \subset \mathcal{AV}(X)$.
2. $Y \underset{LR}{<} X \implies \mathcal{AV}(Y) \subsetneq \mathcal{AV}(X)$.
3. $Y \underset{LR}{\sim} X \implies \mathcal{AV}(Y) = \mathcal{AV}(X)$.

Theorem. (Consequence of KL defn of cells.)

1. W acts on $\overline{C}_{\mathbb{Z}}(X) = \Big[\sum_{Y \underset{LR}{\leq} X} \mathbb{Z}Y\Big] \supset \partial C_{\mathbb{Z}}(X)$.
2. W acts on $C_{\mathbb{Z}}(X) \simeq \overline{C}_{\mathbb{Z}}(X)/\partial C_{\mathbb{Z}}(X)$.
3. $C_{\mathbb{Z}}(X)$ contains unique special rep $\sigma(X) \in \widehat{W}$.
4. $\mathcal{AV}(X) =$ union of closures of K-forms of $\mathcal{O}(\sigma(X))$.

Nilpotent orbit $\mathcal{O}(\sigma(X))$ is defined by Springer correspondence.

KL cells

Theorem (Kazhdan-Lusztig)

1. KL relations $\underset{L}{\sim}$ and $\underset{LR}{\sim}$ partition W into left cells and two-sided cells $C_L(w) \subset C_{LR}(w)$ $(w \in W)$.
2. $\mathbb{Z}$-module $C_{\mathbb{Z},L}(w)$ carries a representation of W.
3. $C_{\mathbb{Z},LR}(w)$ carries a representation of $W \times W$.
4. $\sum_{C_{LR}} C_{\mathbb{Z},LR} \simeq \mathbb{Z}W$, regular representation of $W \times W$.
5. Two-sided cells C_{LR} partition $\widehat{W}$ into subsets $\Sigma(C_{LR})$ called families: $C_{\mathbb{Z},LR} \simeq \sum_{\sigma \in \Sigma(C_{LR})} \sigma \otimes \sigma^*$.
6. As a representation of the first W, $C_{\mathbb{Z},LR} \simeq \sum_{C_L \subset C_{LR}} C_{\mathbb{Z},L}$.

Lusztig's description of families

For any finite group F, Lusztig in 1979 defined

$$\mathcal{M}(F) = \{ (x, \xi) \mid x \in F,\ \xi \in \widehat{F^x} \}/(\text{conjugation by } F)$$

The group F acts itself by conjugation;

$$\mathcal{M}(F) \simeq \text{simple } F\text{-equivariant coherent sheaves on } F.$$

Theorem (Lusztig) Suppose that Σ is a family in $\widehat{W}$.

1. Σ has exactly one special representation $\sigma_s(\Sigma) \in \widehat{W}$.
2. $\sigma_s \underset{\text{Springer}}{\longleftrightarrow}$ special nilpotent orbit $\mathcal{O}_s(\Sigma) = \mathcal{O}_s(\sigma_s) \subset \mathcal{N}^*/G$.
3. Put $A(\mathcal{O}_s) = \pi_1^G(\mathcal{O}_s)$ (equivariant fundamental group). Put

   $$\{\sigma_s = \sigma_1, \sigma_2 \ldots, \sigma_r\} = \Sigma \cap (\text{Springer}(\mathcal{O}_s));$$

   this is all W-reps in $\Sigma \underset{\text{Springer}}{\longleftrightarrow} \xi_j \in \widehat{A(\mathcal{O}_s)}$. Define

   $$\overline{A} = \overline{A}(\mathcal{O}_s) = A(\mathcal{O}_s)/[\cap_j \ker \xi_j]$$

4. Have inclusion $\Sigma \hookrightarrow \mathcal{M}(\overline{A})$, $\quad \sigma \mapsto (x(\sigma), \xi(\sigma))$ so

   $$x(\sigma_s) = x(\sigma_j) = 1 \in \overline{A}, \quad \xi(\sigma_j) = \xi_j \in \widehat{\overline{A}}.$$

Why do families matter?

Lusztig's families of W reps control a wide range of problems in representation theory. For example, suppose $\mathcal{O}^\vee$ is an even nilpotent orbit for the Langlands dual group $G^\vee$. Attached to $\mathcal{O}^\vee$ is a weak Arthur packet $\Pi(\mathcal{O}^\vee)$.

Reps in weak Arthur packets are at the heart of Arthur's conjectures about the residual automorphic spectrum, and therefore of the unitary dual of $G(\mathbb{R})$.

Want to know: how big is the weak Arthur packet?

Theorem (Barbasch-V) Write $\overline{A}$ for Lusztig's quotient of the component group of $\mathcal{O}^\vee$, $\Sigma \subset \widehat{W}$ the corr family, and

$$\Sigma_{\text{Lusztig}} = \{\sigma(x, \text{triv}) \mid x \in \overline{A} \text{ mod conjugacy } \} \subset \Sigma$$

for the corresponding Lusztig left cell of W reps. Then $\#\Pi(\mathcal{O}^\vee)$ is the sum of the multiplicities of the reps in Σ_{Lusztig}, in the representation of W on $\mathcal{T}(\mathfrak{g}, K)_{H^\vee/2}$

Where do families come from?

I've said that families in $\widehat{W}$ are interesting: they can answer questions you might ask without knowing about families.

So it's important to understand why some W reps are special, and why they are naturally attached to other W reps.

Meinolf Geck made a conjecture in 2018, (proven by Geck and Dong-Yang), characterizing special nilpotent orbits by a geometric integrality property.

Special W reps, corresponding fams, may be det by algebraic integrality of the coherent continuation representation. Here is an example...

Example of $Sp(4,\mathbb{R})$

Take $G(\mathbb{R}) = Sp(4,\mathbb{R})$; $X^* = \mathbb{Z}^2$. W acts by sgn chg, coord perm.

Suppose $\exists$ HC cell $\mathcal{C} \rightsquigarrow$ 2-diml rep of W on $\mathbb{C}^2$.

By (easy) facts about coh cont above, $\mathbb{C}$ has $\mathbb{Z}$-basis of two irr reps X and Y, W action

$$s_\alpha \cdot X = -X \qquad s_\beta \cdot X = X + b \cdot Y$$
$$s_\alpha \cdot Y = Y + a \cdot X \qquad s_\beta \cdot Y = -Y$$

for nonneg ints a and b.

In standard coords $(1,0)$ and $(0,1)$ on the $\mathbb{C}^2$ rep,

$$s_\alpha \cdot (1,0) = (0,1) \qquad s_\beta \cdot (1,0) = (1,0)$$
$$s_\alpha \cdot (0,1) = (1,0) \qquad s_\beta \cdot (0,1) = (0,-1)$$

Now linear algebra says

$$X = (p,-p), \quad Y = (0,q), \quad b = 2p/q, \quad a = q/p.$$

So the nonneg ints a and b satisfy $ab = 2$; so $\{a,b\} = \{1,2\}$.

So $\dim \mathrm{Ext}^1(X,Y) = 1$ and $\dim \mathrm{Ext}^1(Y,X) = 2$ (or vice versa).

But the two Ext groups are the same, so no such $\mathcal{C}$ can exist: Lusztig family of $\mathbb{C}^2$ must include other W reps.

Classical W reps: special in red

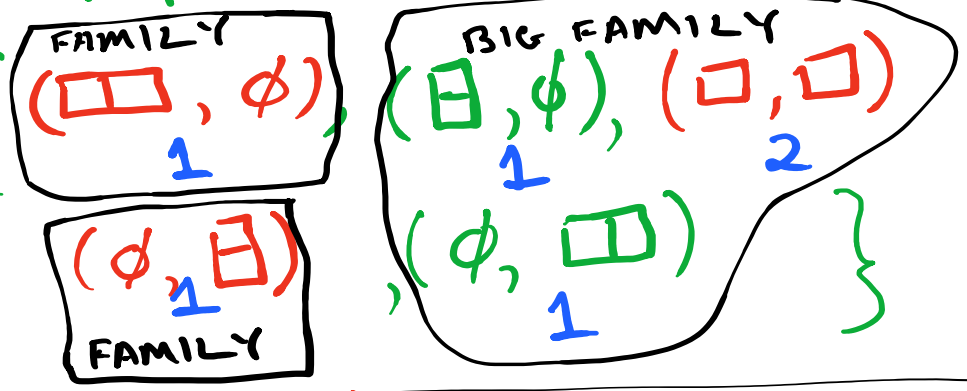
Friday, September 10, 2021 10:18 PM

$\hat{S}_n \leftrightarrow$ partitions of n

$\hat{S}_4 = \{$  $\}$

dim 1 3 2 3 1

$W(BC_n)^\wedge \leftrightarrow$ pairs of partitions (π_1, π_2) $|\pi_1| + |\pi_2| = n$ SPECIAL

$(BC_2)^\wedge = \{$ 

dim

BIG family: $\sigma_S = (\square, \square)$, $\bar{A} = \{1, x\}$

$(\square, \square)$ $(\phi, \square\square)$ $(\square\square, \phi)$ $\leftarrow$ element (x, ξ) of $\mathcal{M}(\bar{A})$

$(1, \text{triv})$ $(1, \varepsilon)$ (x, triv)

Lusztig's description of left cells

Recall that finite group F gives

$$\mathcal{M}(F) = \{(x,\xi) \mid x \in F,\ \xi \in \widehat{F^x}\}/(\text{conj by } F)$$
$$\simeq \text{irr conj-eqvt coherent sheaves } \mathcal{E}(x,\xi) \text{ on } F.$$

Given subgroup $S \subset F$, const sheaf $\mathcal{S}$ on S is S-eqvt for conj.
Push forward to F-eqvt sheaf supp on F-conjugates of S:

$$i_*(\mathcal{S}) = \sum_{s,\xi} m_S(s,\xi)\mathcal{E}(s,\xi), \qquad m_S(s,\xi) = \dim \xi^{S^s}.$$

Sum runs over S conjugacy classes $s \in S$. Can write this as

$$i_*(\mathcal{S}) = \sum_s \mathcal{E}(s, \mathrm{Ind}_{S^s}^{F^s}(\mathrm{triv})).$$

Theorem (Lusztig) $C_L \subset C_{LR} \leftrightarrow \Sigma \subset \widehat{W}$, $\overline{A}$ finite group,

$$\Sigma \hookrightarrow \mathcal{M}(\overline{A}), \quad \sigma \mapsto (x(\sigma), \xi(\sigma)).$$

1. $\exists$ subgp $\Gamma = \Gamma(C_L) \subset \overline{A}$ so $C_{\mathbb{Z},L} \simeq \sum_{x,\xi} m_\Gamma(x,\xi)\sigma(x,\xi)$
2. $m_\Gamma(1,\mathrm{triv}) = 1$, so special rep σ_s appears once in $C_{\mathbb{Z},L}$.
3. $\exists$ Lusztig left cells with $\Gamma = \overline{A}$, so $C_{\mathbb{Z},L} \simeq \sum_x \sigma(x,\mathrm{triv})$.
4. G classical $\implies$ $\exists$ Springer left cells with $\Gamma = \{e\}$, so $C_{\mathbb{Z},L} \simeq \sum_{\xi \in \widehat{\overline{A}}} \dim(\xi)\sigma(1,\xi)$, Springer reps for O_s in Σ.

Classical left cells: special in red

Saturday, September 11, 2021 11:11 AM

$\bar{A} = (\mathbb{Z}/2\mathbb{Z})^m$, $\mathcal{M}(\bar{A}) = \bar{A} \times \bar{A}^\wedge$

C2 family: $\sigma_S = (\square, \square)$, $\bar{A} = \{1, x\}$

$(\square, \square)$ $(\phi, \square\square)$ $(\boxplus, \phi)$ element (x, ξ) $\in$ of $\mathcal{M}(\bar{A})$

$(1, \text{triv})$ $(1, \varepsilon)$ (x, triv)

Left cells $\leftrightarrow$ $S \subset \bar{A}$

$C_S = \{\sigma(x, \xi) \mid x \in S, \xi|_S = 1\}$

$C_1 = \{\sigma(1, \xi) \mid \xi \in \bar{A}^\wedge\}$ SPRINGER

$C_{\bar{A}} = \{\sigma(x, \text{triv}) \mid x \in \bar{A}\}$ LUSZTIG

C2 cells:

$C_1 = \{(\square, \square), (\phi, \square\square)\}$ Springer

$C_{\bar{A}} = \{(\square, \square), (\boxplus, \phi)\}$ Lusztig

Consequences of Lusztig for HC cells

HC world $(\mathfrak{g}, K)$: $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1} \supset C =$ HC cell $\rightsquigarrow$ W rep $C_{\mathbb{Z}}$.

$C_{\mathbb{Z}} \supset \sigma_s(C)$ special in $\widehat{W} \rightsquigarrow O(C)$, $\quad \Sigma(O) \subset \widehat{W}$, $\quad \overline{A}(O)$ finite.

Theorem. Suppose C is a HC cell in $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

$$C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma, \quad m_C(\sigma) \in \mathbb{N}, \quad m_C(\sigma_s) = 1.$$

That is, a HC cell is a sum of certain W reps in a single family, including the special W rep with multiplicity one.

Problem: understand the set of W reps on cells.

How many cells are there for each special rep σ of W?

Which other reps from the family of σ appear in each cell?

Conjecture. For each HC cell C as above, $\exists \ S(C) \subset \overline{A}$ so

$$m_C(\sigma(x, \xi)) = m_{S(C)}(x, \xi).$$

Evidence for conjecture

Theorem (McGovern, Binegar) C a HC cell in $\mathrm{Irr}(\mathfrak{g}, K)_{\gamma_1}$.

1. $C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma, \quad m_C(\sigma) \in \mathbb{N}, \quad m_C(\sigma_s) = 1.$
2. IF $G(\mathbb{R})$ real form of type A, $SO(n)$, $Sp(2n)$, or an exceptional group, THEN $\exists S(C) \subset \overline{A}$ so

$$m_C(\sigma(x,\xi)) = m_{S(C)}(x,\xi).$$

3. The subgroup $S(C) \subset \overline{A}$ in (2) is always one of Lusztig's Γ attached to a left cell; so
4. in (2), each HC cell is isomorphic to a KL left cell.
5. IF $G(\mathbb{R})$ cplx, so $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_1$, $\overline{A}(\mathcal{O}) = \overline{A_1} \times \overline{A_1}$, THEN $S(C) = (\overline{A_1})_\Delta$; not one of Lusztig's Γ unless $\overline{A}_1 = 1$.

Finding the conjectural $S(C) \subset \overline{A}$

Suppose we have a HC cell C of irr HC modules:

$$C_{\mathbb{Z}} = \sum_{\sigma \in \Sigma} m_C(\sigma)\sigma.$$

"We" (that is, the `atlas` software!) can calculate nonnegative integers $m_C(\sigma(x,\xi))$.

How do we recover $S(C)$ from these integers?

Recall pairs $(x,\xi) \in \mathcal{M}(\overline{A})$ defined up to $\overline{A}$ conjugacy;

$x \in \overline{A}$, and $\xi \in \widehat{\overline{A}^x}$ irrep of its centralizer.

The W-reps $\sigma(x, \mathrm{triv})$ actually exist.

$m_S(x, \mathrm{triv}) = \#S$-conjugacy classes in (class of x) $\cap$ S: class function on $\overline{A}$ giving nice statistic about S.

Gives question: given a finite group A, class function m on A, is there at most one subgroup S having this statistic?

Answer to such a question for general A is always NO.

But Lusztig's $\overline{A}$ is a product of copies of S_m for $m \leq 5$...

What questions come next?

Given real reductive G, special orbit $\mathcal{O}$, and corr special rep $\sigma \in \widehat{W}$, would be great to understand

1. how many HC cells for G include σ?
2. Is each cell defined as W rep by some $S \subset \overline{A(\mathcal{O})}$?
3. Which subgroups S appear, and how often?

(1) is a complication of classical question how many $\mathbb{R}$-forms does $\mathcal{O}$ have?

(2) ought to be addressable using ideas from Lusztig's classification of left cells.

(3) is a complication of the classical question how much of $\pi_1(\mathcal{O})$ is defined over $\mathbb{R}$?

Since this is the last slide, you can guess that I have no idea how to answer these questions.

Thanks for the invitation; math with people is the best kind!