Invariant measures on homogeneous spaces

Here are some of the basic facts about invariant measures on homogeneous spaces for locally compact groups. Missing proofs may be found for example in Leopold Nachbin's book *The Haar Integral*.

So suppose G is a locally compact topological group. I will write $C_c(X)$ for the space of compactly supported continuous (complex-valued) functions on a locally compact topological space X, and $C_c^{\geq 0}(X)$ for the collection of non-negative real-valued functions. I am interested in Borel measures on X; equivalently (Riesz representation theorem) in linear functionals on $C_c(X)$ that take non-negative values on $C_c^{\geq 0}(X)$.

If G acts on X, then it acts on functions on X by the formula

$$(\lambda(g)f)(x) = f(g^{-1} \cdot x). \tag{1a}$$

I'll use this notation in particular for the *left* action of G on itself. For the *right* action of G on itself, I'll write

$$(\rho(g)f)(x) = f(xg). \tag{1b}$$

The basic theorem about Haar measure is

Theorem 2. *Suppose G is a locally compact topological group. Then there is a non-zero Borel measure d_G^ℓ on G with the property ("left invariance") that*

$$\int_G f(g^{-1}x)\, d_G^\ell(x) = \int_G f(x)\, d_G^\ell(x) \qquad (g \in G, f \in C_c(G)). \tag{2a}$$

This property characterizes d_G^ℓ up to a positive scalar multiple. Any right translate of d_G^ℓ shares this left-invariance, and therefore is a positive scalar multiple of d_G^ℓ:

$$\int_G f(xg)\, d_G^\ell(x) = \delta_G(g^{-1}) \int_G f(x)\, d_G^\ell(x) \qquad (h \in G, f \in C_c(G)). \tag{2b}$$

Here $\delta_G \colon G \to \mathbb{R}^+$ is a continuous homomorphism, called the "modular character" of G.

Because any compact subgroup of the multiplicative group $\mathbb{R}^+$ is trivial, the modular character must be trivial on any compact subgroup of G.

There is a right Haar measure on G defined by

$$\int_G f(x)\, d_G^r(x) = \int_G f(x)\delta_G(x^{-1}) d_G^\ell(x) \qquad (f \in C_c(G)). \tag{3a}$$

(The right-invariance of this linear functional on $C_c(G)$ is an easy consequence of (2b).)

If G is a Lie group, then an elementary calculation shows that

$$\delta_G(g) = |\det(\mathrm{Ad(g)})|. \tag{3b}$$

This formula makes it easy to construct examples of groups with non-trivial modular function (like the group of upper-triangular invertible matrices).

Suppose now that H is a closed subgroup of G (still assumed locally compact). Fix (left) Haar measures d_G^ℓ on G and d_H^ℓ on H. One nice way to relate integration on G, H, and G/H is using

Theorem 4 (Weil?). *The mapping*

$$C_c(G) \to C_c(G/H), \quad f \mapsto \overline{f}, \quad \overline{f}(xH) = \int_H f(xh)\, d_H^\ell(h) \qquad (x \in G) \tag{4a}$$

is a well-defined surjection, carrying non-negative functions to non-negative functions.

What we would like to do is use this description of functions on G/H to introduce an integral on G/H by requiring

$$\int_{G/H} \overline{f}(xH) d_{G/H}(xH) \stackrel{?}{=} \int_G f(x) d_G^\ell(x) \qquad (f \in C_c(G)). \tag{5a}$$

In order to do this (using Theorem 4 and the Riesz representation theorem) we need to know that the integral on the right must vanish whenever $\overline{f} = 0$.

Here is where matters become a little subtle. Suppose that $f \in C_c(G)$ gives rise by Theorem 4 to the function $\overline{f} \in C_c(G/H)$. For each element $h_0 \in H$, the right translate $\rho(h_0)f$ is another function in $C_c(G)$, and it seems only fair that it should give rise to the same function $\overline{f}$ on G/H. But calculation shows that

$$\overline{\rho(h_0)f} = \delta_H(h_0^{-1})\overline{f}. \tag{5b}$$

Well, that's not bad: we can write it as

$$\overline{\delta_H(h_0)\rho(h_0)f} = \overline{f}. \tag{5c}$$

That is, all functions of the form

$$f - \delta_H(h_0)[\rho(h_0)f] \qquad (f \in C_c(G), h_0 \in H) \tag{5d}$$

belong to the kernel of the "bar" map to $C_c(G/H)$. It's not difficult to prove that their span is dense in the kernel.

On the other hand, we calculate (from $(2b)$)

$$\int_G (\rho(h_0)f)(x)\, d_G^\ell(x) = \delta_G(h_0^{-1}) \int_G f(x)\, d_G^\ell(x) \qquad (f \in C_c(G), h_0 \in H). \tag{5e}$$

That is, functions of the form

$$f - \delta_G(h_0)[\rho(h_0)f] \qquad (f \in C_c(G), h_0 \in H) \tag{5f}$$

have integral zero over G.

Comparing $(5d)$ and $(5f)$, we see that the integral in $(5a)$ cannot vanish on all functions with $\overline{f} = 0$ unless the modular function δ_H is equal to the restriction of the modular function δ_G to H. In light of the claim that the functions in $(5d)$ span a dense subspace of the kernel, the converse holds as well. Summarizing,

Theorem 6. *Suppose G is a locally compact group, H a closed subgroup, and d_G^ℓ and d_H^ℓ are left Haar measures.*

a) There is a non-zero G-invariant Borel measure on G/H if and only if the modular functions of G and H satisfy $\delta_H = (\delta_G)|_H$.

b) Suppose that the modular functions agree. Then the invariant measure on G/H is unique up to a positive scalar multiple. It may be normalized so that

$$\int_G f(x)\, d^\ell_G(x) = \int_{G/H} \left[\int_H f(xh)\, d^\ell_H(h) \right] d_{G/H}(xH) \qquad (f \in C_c(G)).$$

Having come so far, I should say a few more words. If G is a Lie group, then G/H is a smooth manifold. You may know that any smooth manifold M admits a real line bundle called the "density bundle," whose sections are the smooth measures on M (and can therefore be integrated). Having an invariant measure on G/H means that the density bundle is equivariantly trivial.

Now line bundles on G/H correspond to characters of the subgroup H; sections of a line bundle may be identified with functions on G transforming by the character under right translation by H. The conclusion from all this (for Lie groups) is that densities on G/H correspond to functions on G transforming on the right by a certain character of H, and that for such functions there is a natural "integral" over G/H. This is simple enough that something similar might be true for general locally compact groups, and this turns out to be the case.

Definition 7. Suppose G is a locally compact group, H a closed subgroup, and d^ℓ_G and d^ℓ_H are left Haar measures. A *compactly supported continuous density function* for G/H is a continuous function f on G such that

a) $f(xh) = \delta_H(h)\delta_G(h^{-1})f(x)$ for all $x \in G$ and $h \in H$, and
b) there is a compact subset K of G such that f is supported in KH.

Theorem 8. *In the setting of Definition 7, the mapping*

$$f \to \tilde{f}, \quad \tilde{f}(x) = \int_H f(xh)\delta_G(h)\delta_H(h^{-1})\, d^\ell_H(h)$$

is a continuous surjection from $C_c(G)$ to compactly supported continuous density functions for G/H. There is a left-invariant "integral" for such densities given by

$$\int_{G/H} \tilde{f} d_{G/H} = \int_G f(x)\, d^\ell_G(x) \qquad (f \in C_c(G)).$$

That is, the left-invariant integral over G may expressed as an iterated integral

$$\int_G f(x)\, d^\ell_G(x) = \int_{G/H} \left[\int_H f(xh)\delta_G(h)\delta_H(h^{-1})\, d^\ell_H(h) \right] d_{G/H} \qquad (f \in C_c(G)).$$

Here the inner integral represents a compactly supported density function on G/H.

Just as for Theorem 6, the key is to understand the kernel of the map "tilde"; this time it turns out to be generated by elements $f - \delta_G(h_0)[\rho(h_0)f]$.

The integrals over H can be written with respect to right Haar measure $d^r_H(h)$, by dropping the factors $\delta_H(h^{-1})$ (*cf.* (3*a*)).