Branching to maximal compact subgroups

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Helgason Conference, 15 August 2007

Outline

Why restrict to K?

G cplx $\supset G(\mathbb{R})$ real $\supset K(\mathbb{R})$ maxl compact

Want to study representations $(\pi, \mathcal{H}_\pi)$ of $G(\mathbb{R})$, but these are complicated and difficult.

Reps of $K(\mathbb{R})$ are easy, so try two things:

- understand $\pi|_{K(\mathbb{R})}$; and
- use understanding to answer questions about π.

Sample question: how often does trivial representation of $K(\mathbb{R})$ appear in $\pi|_{K(\mathbb{R})}$?

Answer: multiplicity zero unless π is (quotient of) spherical principal series, then one.

Application: π can appear in functions on $G(\mathbb{R})/K(\mathbb{R})$ only if π spherical; then exactly once.

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Minimal parabolic subgroup

Write $\theta =$ Cartan involution of $G(\mathbb{R})$ and G;

$$K(\mathbb{R}) = G(\mathbb{R})^\theta \qquad \text{(real groups)},$$
$$K = G^\theta \qquad \text{(complex algebraic groups)}.$$

Iwasawa decomp $G(\mathbb{R}) = K(\mathbb{R})A(\mathbb{R})_0N(\mathbb{R})$.

Here $A =$ maxl cplx torus where θ acts by inverse.

$$L(\mathbb{R}) = \text{centralizer of } A \text{ in } G(\mathbb{R})$$
$$P(\mathbb{R}) = L(\mathbb{R})N(\mathbb{R}).$$

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Helgason's theorem (classical picture)

Minimal parabolic is $P(\mathbb{R}) = M(\mathbb{R})A(\mathbb{R})_0N(\mathbb{R})$.

Fin-diml of $G(\mathbb{R}) \leftrightsquigarrow$ highest wt = $N(\mathbb{R})$-invts.

highest weight $= \delta \otimes \nu$, $\delta \in \widehat{M(\mathbb{R})}$, $\nu \in \widehat{A(\mathbb{R})_0}$.

Theorem (Helgason)

Says: $K(\mathbb{R})$-fixed vecs $\leftrightsquigarrow$ $M(\mathbb{R})N(\mathbb{R})$-fixed vecs.

Reason: $M(\mathbb{R})N(\mathbb{R})$ = deformation of $K(\mathbb{R})$.

Conjugate $K(\mathbb{R})$ by elts of $A(\mathbb{R})_0$, $\rightsquigarrow$ limiting subgroup.

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Theme: complexify, use algebraic geometry.

Helgason's theorem concerns compact $K(\mathbb{R})$, minimal parabolic $P(\mathbb{R})$.

Theme says complexify, considering algebraic groups $K = G^\theta$ and $P = LN$ parabolic in G.

Continuous reps of $K(\mathbb{R}) \leftrightsquigarrow$ algebraic reps of K.

Theme says consider projective algebraic variety

$$\mathcal{P} = \text{subgps of } G \text{ conjugate to } P,$$

a *partial flag variety*.

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Proposition

$K \cdot P$ is open in $\mathcal{P}$: $K/M \simeq K \cdot P \subset \mathcal{P} \simeq G/P$. Here $M =$ cplx pts of $M(\mathbb{R}) =$ cent in K of A.

Follows immediately from Iwasawa decomposition.

Theorem (Borel-Weil, Helgason)

Picture: $\mathcal{P} = K/M \bigcup \{\text{divisors}\}$.

Section on $K/M \rightsquigarrow$ pole order on each divisor.

Section extends to $\mathcal{P} \iff$ no pole on any divisor.

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Picture: $\mathcal{P} = K/M \bigcup \{\text{divisors}\}$.

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Helgason's theorem (alg geometry picture)

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$K \cdot P$ is open in $\mathcal{P}$: $K/M \simeq K \cdot P \subset \mathcal{P} \simeq G/P$. Here $M =$ cplx pts of $M(\mathbb{R}) =$ cent in K of A.

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Repn of G = alg secs of vector bdle on $\mathcal{P}$

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Divisors correspond to simple restricted roots of A.

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Bdle on $\mathcal{P} \rightsquigarrow I_j(\tau_j)$ = secs with pole along divisor D_j.

Bundle τ_j depends on δ **and** on character ν of A.

If ν large on simple root j, then $I_j(\tau_j)$ is small.

Sections on $\mathcal{P} \approx I_0(\delta) - \sum_{j=1}^r I_j(\tau_j)$.

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restr to $K(\mathbb{R})$ of *principal series rep* of $G(\mathbb{R})$.

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Setting for Zuckerman's theorem

First replace $\mathcal{P}$ by $\mathcal{B} =$ Borel subgroups of G, *complete flag variety for G*.

Proposition (Wolf, Beilinson-Bernstein)

1. *K acts on $\mathcal{B}$ with finitely many orbits.*
2. *Unique open orbit $\leadsto$ Borel subgp of Iwasawa P.*
3. *General orbit $\leadsto$ pair (H, Δ^+) mod $G(\mathbb{R})$ conjugation.*
   $H(\mathbb{R})$ = Cartan in $G(\mathbb{R})$, Δ^+ = pos roots for H in G.
4. *Std rep $I(\gamma)$ restr to $K \leadsto \gamma = (Z, \xi)$*
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Hamingjusamur fæðingardagur!