Signatures of invariant Hermitian forms on finite-dimensional representations

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Outline

Introduction

Classical background

Classical proofs of nonunitarity

Signatures of Hermitian forms

Slides at `http://www-math.mit.edu/~dav/paper.html`

What's the topic?

Compact Lie groups K studied by Weyl, Cartan.

1. Irreducible representations $\phi(\lambda) \leftrightsquigarrow \lambda \in \widehat{T}/W$.
2. T max torus; $\widehat{T} \subset \mathfrak{t}^*$ lattice in complex vector space.
3. Reps all finite-dimensional, all unitary.
4. $\dim \phi(\lambda) =$ polynomial in λ, degree $= \frac{1}{2} \dim K/T$.

Noncompact grps $G(\mathbb{R})$ studied by Harish-Chandra.

1. Irreducible representations $\pi(\xi) \leftrightsquigarrow \xi \in \widehat{H}(\mathbb{R})/W_{H(\mathbb{R})}$.
2. $H(\mathbb{R})$ Cartan subgroup; $\widehat{H}(\mathbb{R}) = \Lambda \times \mathfrak{a}^* \subset \mathfrak{h}^*$;
3. lattice times complex vector space;
   $\operatorname{rk} \Lambda + \dim_{\mathbb{C}} \mathfrak{a} = \dim_{\mathbb{R}} H(\mathbb{R})$.
4. Most $\pi(\xi)$ infinite-dimensional, many non-unitary.

But Weyl's finite-diml $\{\phi_{G(\mathbb{R})}(\lambda)\} \subset \{\pi(\xi)\}$.

Almost all $\phi_{G(\mathbb{R})}(\lambda)$ are non-unitary.

Question today: How non-unitary are they?

Joint work with MIT undergraduate Christopher Xu, MIT grad student Daniil Kalinov.

Where does that problem come from?

Classifying reps is algebraic: use algebraic geometry, etc.

Interesting reps are unitary reps, a subset of all repns.

Identify subset in two steps:

1. Is there invariant Hermitian form? (algebraic)
2. Is the form positive? (analytic)

Cartan: $\exists$ invt form on most $\phi_{G(\mathbb{R})}(\lambda)$, not positive.

General $\pi(\xi)$: Analysis is hard; replace (2) by

2.′ What is signature of form? (algebraic)

Have algorithm (Adams/van Leeuwen/Trapa/V) $\rightsquigarrow$ signature of invt Herm form on any $\pi(\xi)$.

Suggests question: what's signature of form on $\phi_{G(\mathbb{R})}(\lambda)$?

V n-diml with Herm form signature (p, q) $\rightsquigarrow$ $\mathrm{Sig}(V) = |p - q|$.

Solution in an example

$G(\mathbb{C}) = GL_n(\mathbb{C})$, subgps $K = U_n$, $G(\mathbb{R}) = GL_n(\mathbb{R})$.

$\lambda = (\lambda_1, \ldots, \lambda_n)$ decreasing integers $\rightsquigarrow \phi(\lambda) = GL_m(\mathbb{C})$ irr.

$\dim \phi(\lambda) = \prod_{i<j} \frac{(\lambda_i - \lambda_j + i - j)}{i-j}$, poly of degree $\binom{n}{2}$.

Restrictions $\phi_K(\lambda)$, $\phi_{G(\mathbb{R})}(\lambda)$ both irreducible.

$\phi_K(\lambda)$ always has invt Hermitian form, always positive definite: $\mathrm{Sig}(\phi_K(\lambda)) = \dim \phi(\lambda)$.

$\phi_{G(\mathbb{R})}(\lambda)$ has invt form $\iff \lambda_j + \lambda_{n-j+1} = 0$.

If form exists, $\sigma(\lambda) =_{\mathrm{def}} \mathrm{Spin}(n)$ repn $(\lambda_1 + 1/2, \ldots, \lambda_{[n/2]} + 1/2)$.

$$\mathrm{Sig}(\phi_{G(\mathbb{R})}(\lambda)) = \frac{\dim \sigma(\lambda)}{\dim \sigma(0)}$$

$$= \text{poly of deg } \begin{cases} (n/2)(n/2-1) & n \text{ even} \\ (n/2 - 1/2)^2 & n \text{ odd} \end{cases}$$

.

$$\dim \phi = \mathrm{Sig}(\phi)^2 \cdot \underbrace{\prod_{i=1}^{[n/2]} \frac{2\lambda_i + n - 2i + 1}{n - 2i + 1}}_{\text{this term is } \geq 1}$$

Sig small: very indef.

How was the problem solved?

Jeff Adams used (his!) `atlas` software $\rightsquigarrow$ interesting signatures of forms on fin diml reps.

MIT undergraduate Chris Xu used `atlas` to compute many signatures for $GL_n(\mathbb{R})$.

Calculations $\rightsquigarrow$ XU CONJECTURE:

$$\mathrm{Sig}(GL_n(\mathbb{R})\text{ rep }) = c_n \cdot \dim(\mathrm{Spin}_n\text{ rep }).$$

Xu conjecture $\rightsquigarrow$ grad student Daniil Kalinov proved

$$\mathrm{Sig}(GL_n(\mathbb{R})\text{ rep }) \leq c_n \cdot \dim(\mathrm{Spin}_n\text{ rep }).$$

Kalinov + Huang-Pandzic[1] Dirac $\rightsquigarrow$ pf of Xu conjecture.

[1] DV contribution: I'm old enough to remember this work

Structure of GL_n

$\mathrm{Lie}(GL_n(\mathbb{R})) =_{\mathrm{def}} \mathfrak{gl}_n(\mathbb{R}) =$ all real $n \times n$ matrices.
$\mathrm{Lie}(O_n(\mathbb{R})) =_{\mathrm{def}} \mathfrak{o}_n(\mathbb{R})) =$ $n \times n$ skew-symm matrices.
$\mathfrak{p}_n(\mathbb{R}) =$ real $n \times n$ symmetric matrices matrices,
$\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{o}_n(\mathbb{R}) \oplus \mathfrak{p}_n(\mathbb{R})$ Cartan decomposition.

$\mathrm{Lie}(GL_n(\mathbb{C})) =_{\mathrm{def}} \mathfrak{gl}_n(\mathbb{C}) =$ complex $n \times n$ matrices.
$\mathrm{Lie}(U_n) =_{\mathrm{def}} \mathfrak{u}_n =$ $n \times n$ skew-hermitian matrices.
$\mathfrak{h}_n =$ $n \times n$ hermitian matrices matrices,
$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n \oplus \mathfrak{h}_n = \mathfrak{u}_n \oplus i\mathfrak{u}_n$ Cartan decomposition.

Two cases related: $\mathfrak{gl}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, $\quad \mathfrak{u}_n = \mathfrak{o}_n(\mathbb{R}) \oplus i\mathfrak{p}_n(\mathbb{R})$.
$GL_n(\mathbb{R})$ and U_n are two real forms of $GL_n(\mathbb{C})$.

Compact forms of noncompact groups

$$\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{o}_n(\mathbb{R}) \oplus \mathfrak{p}_n(\mathbb{R}), \quad \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n \oplus \mathfrak{h}_n = \mathfrak{u}_n \oplus i\mathfrak{u}_n$$

$$\mathfrak{u}_n = \mathfrak{o}_n(\mathbb{R}) \oplus i\mathfrak{p}_n(\mathbb{R}).$$

$V = n$-dimensional real vector space.

$G(\mathbb{R}) \subset GL(V)$ connected semisimple Lie group.

Theorem (Cartan): can choose basis so that
$G(\mathbb{R}) \subset GL_n(\mathbb{R})$ preserved by $\theta(g) = {}^t g^{-1}$.

$K(\mathbb{R}) =_{\text{def}} G(\mathbb{R}) \cap O_n(\mathbb{R})$ maximal compact subgroup.

$\mathfrak{s}(\mathbb{R}) =_{\text{def}} \mathfrak{g}(\mathbb{R}) \cap \mathfrak{p}(\mathbb{R}), \quad \mathfrak{g}(\mathbb{R}) = \mathfrak{k}(\mathbb{R}) \oplus \mathfrak{s}(\mathbb{R})$

$G(\mathbb{C}) \subset GL_n(\mathbb{C}) \leftrightsquigarrow \mathfrak{g}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ cplx semisimple algebraic.

$U(\mathbb{R}) =_{\text{def}} G(\mathbb{C}) \cap U_n, \qquad \text{Lie}(U(\mathbb{R})) = \mathfrak{k}(\mathbb{R}) \oplus i\mathfrak{s}(\mathbb{R}).$

Noncpt $G(\mathbb{R})$, cpt $U(\mathbb{R})$ are two real forms of same $G(\mathbb{C})$.

1st reason fin diml reps mostly not unitary

Write $\mathfrak{g}(\mathbb{R}) = \sum_j \mathfrak{g}(\mathbb{R})_j$, direct sum of simple.

Irr fin diml ϕ of $\mathfrak{g}(\mathbb{R})$ is $\phi \simeq \bigotimes_j \phi_j$ accordingly.

ϕ Hermitian $\iff$ each ϕ_j Hermitian;

$\mathrm{Sig}(\phi) = \prod_j \mathrm{Sig}(\phi_j)$.

If $G(\mathbb{R})_j$ noncompact,

$$\phi_j \neq \text{triv} \implies \phi_j \text{ faithful} \implies \phi_j(G(\mathbb{R})) \text{ noncompact}$$
$$\implies \phi_j \text{ nonunitary.}$$

ϕ unitary $\iff$ ϕ trivial on each noncpt simple factor.

2nd reason fin diml reps mostly not unitary

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{g}(\mathbb{R})^{\theta} \oplus \mathfrak{g}(\mathbb{R})^{-\theta} = \mathfrak{k}(\mathbb{R}) \oplus \mathfrak{s}(\mathbb{R})$$

Theorem (Cartan). If $\tau\colon G(\mathbb{R}) \to G_1(\mathbb{R})$ homomorphism of semisimple Lie groups, then $G_1(\mathbb{R})$ has Cartan involution θ_1 so $\theta_1(\tau(g)) = \tau(\theta(g))$.

Corollary. If ϕ finite-dimensional rep of $G(\mathbb{R})$, then $d\phi(\mathfrak{s}(\mathbb{R}))$ diagonalizable, real eigenvalues.

Corollary. If ϕ fin-diml unitary of $G(\mathbb{R})$, then

$$d\phi(\mathfrak{s}(\mathbb{R})) = 0, \qquad d\phi([\mathfrak{s}(\mathbb{R}), \mathfrak{s}(\mathbb{R})]) = 0,$$

so ϕ factors to largest compact quotient of $G(\mathbb{R})$.

3rd reason fin diml reps mostly not unitary

$G(\mathbb{R}) = K(\mathbb{R}) \cdot A \cdot N(\mathbb{R})$ Iwasawa decomposition.

$\mathfrak{a}(\mathbb{R}) =_{\text{def}}$ max subalg of $\mathfrak{s}(\mathbb{R})$, $M =_{\text{def}} Z_{K(\mathbb{R})}(A)$,

$P_{\text{min}}(\mathbb{R}) =_{\text{def}} M \cdot A \cdot N(\mathbb{R})$ min parabolic of $G(\mathbb{R})$,

$T_M =_{\text{def}}$ max "torus" in M,

$H_s(\mathbb{R}) =_{\text{def}} T_M A$ max split Cartan in $G(\mathbb{R})$.

$$\Delta_s^+ =_{\text{def}} \text{pos roots in } \Delta(\mathfrak{g}, \mathfrak{h}_s) \text{ consistent with } P_{\text{min}}$$
$$= \Delta(\mathfrak{n}, \mathfrak{h}_s) \cup \Delta^+(\mathfrak{m}, \mathfrak{t}_M) = \text{Iwasawa pos system.}$$

$X^*(A) =_{\text{def}}$ res to A of alg chars of H_s: $\mathbb{R}$-valued chars.

$\lambda \in X^*(H_s)$ hwt of unitary $\phi_{G(\mathbb{R})}(\lambda) \implies \lambda|_A =$ trivial.

Very difficult for a Δ_S^+-dominant wt to vanish on A:

$\mathfrak{g}(\mathbb{R})$ simple noncpt $\implies$ only dom wt triv on A is 0.

Linear algebra and Hermitian forms

V N-dimensional complex vector space

Sesquilinear form on V is pairing

$$\langle , \rangle \colon V \times V \to \mathbb{C}, \ \langle u_1 + z \cdot u_2, v\rangle = \langle u_1, v\rangle + z \cdot \langle u_2, v\rangle,$$
$$\langle u, v_1 + z \cdot v_2\rangle = \langle u, v_1\rangle + \overline{z} \cdot \langle u, v_2\rangle.$$

Hermitian form on V is sesq $\langle , \rangle$ with $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

$$\text{Herm dual } = V^h$$
$$=_{\text{def}} \{f \colon V \to \mathbb{C}, \ f(v_1 + z \cdot v_2) = f(v_1) + \overline{z} \cdot f(v_2)\}.$$

Sesquilinear form on $V \leftrightsquigarrow$ linear map $T \colon V \to V^h$,

$$\langle u, v\rangle_T = (Tu)(v).$$

Herm transpose: $A \colon V \to W \rightsquigarrow A^h \colon W^h \to V^h, \ A^h(f)(v) =_{\text{def}} f(Av)$.

$A \colon V \to V$ is Hermitian for sesquilinear $\langle , \rangle_T \iff TA = A^h T^h$.

Everything you know is wrong

Lin alg: Hermitian ops diagonalizable, real eigvals.

Wrong: depends on positive definite.

Proposition. V cplx $\langle , \rangle$ nondeg Herm, signature = (P, Q), $A \in \mathrm{End}(V)$ hermitian operator.

1. Write $V_\lambda =$ generalized eigenspace for A ($\lambda \in \mathbb{C}$), $m(\lambda) = \dim V_\lambda$. Then $\langle V_\lambda, V_\mu \rangle = 0$ unless $\lambda - \overline{\mu} = 0$.
2. $\langle , \rangle$ identifies $V_\kappa^h \simeq V_{\overline{\kappa}}$.
3. $\kappa \neq \overline{\kappa}$ not real $\implies$ $\langle , \rangle$ has signature (m_κ, m_κ) on $V_\kappa + V_{\overline{\kappa}}$.
4. $\rho = \overline{\rho}$ real $\implies$ $\langle , \rangle|_{V_\rho} =$ nondeg, signature $(p(\rho), q(\rho))$.
5. $P - Q = (\sum_{\rho \text{ real}} (p(\rho) - q(\rho))$.

Conclusion: sig computed on real eigspaces of A.

$B = -B^h \implies$ sig computed on imag eigspaces of B.

Computing repn signature with weights

Sig computed on real eigspaces of $A = A^h$.

Sig computed on imag eigspaces of $B = -B^h$.

In Hermitian rep of $\mathfrak{g}(R)$, Lie algebra acts by skew-hermitian operators.

Recall Iwasawa Cartan $H_s(\mathbb{R}) = T_M A$; $\mathfrak{a}(\mathbb{R})$ acts by skew-Herm ops with real eigvals in fin diml rep.

Theorem. Suppose (ϕ, E) fin-diml Hermitian rep of $G(\mathbb{R})$, signature (P, Q). Define $E_0 = E^A$ zero weight space, and (P_0, Q_0) signature of form on E_0. Then $P - Q = P_0 - Q_0$.

$G(\mathbb{R}) = SL(2, \mathbb{R})$ or $SL(3, \mathbb{R})$: form is definite on zero weight space, so $|P - Q| = \dim(\text{zero weight space})$.

$G(\mathbb{R}) = SL(4, \mathbb{R})$, $E =$ irr of hwt $(2, 1, -1, -2)$; $\dim E = 175$, signature $= (90, 85)$, $\dim E_0 = 7$, signature on $E_0 = (6, 1)$: indefinite.

Conclusion: isn't easy to calculate sig using weights.