Local Langlands conjecture for finite groups of Lie type

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Outline

What Langlands can do for you

What's a Weil group?

Root data and reductive groups

Shape of the Langlands classification

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Deligne-Langlands conjecture

Relation to p-adic local Langlands

Old reasons for listening to Langlands

This is introduction number one.

GL_n is everybody's favorite reductive group/local F.

Want to understand $\widehat{GL_n(F)} =$ set of irr repns (hard).

Classical approach (Harish-Chandra *et alia* 1950s):

1. find big compact subgp $K \subset GL_n(F)$;
2. understand $\widehat{K}$ (supposed to be easy?)
3. understand reps of $GL_n(F)$ restricted to K.

Langlands (1960s) studies $\widehat{GL_n(F)}$ (global reasons).

Global theory suggests: $\widehat{GL_n(F)} \overset{\approx}{\leftrightsquigarrow}$ n-diml reps of $\mathrm{Gal}(\overline{F}/F)$.

Harris and Taylor prove Langlands conjecture:

$$\widehat{GL_n(F)} \overset{\text{bij}}{\leftrightsquigarrow} n\text{-diml reps of } W'(F).$$

Here Weil-Deligne group $W'(F)$ is an improvement of $\mathrm{Gal}(\overline{F}/F)$.

Langlands conjecture for other G largely still open.

Representations of finite Chevalley groups

This is introduction number two.

Suppose G is a reductive group defined over $\mathbb{F}_q$.

Deligne-Lusztig and Lusztig described irr reps of $G(\mathbb{F}_q)$.

Can their results be formulated in spirit of Langlands?

Deligne-Lusztig use ratl max torus $T \subset G$, character

$$\theta\colon\ T(\mathbb{F}_q) \to \mathbb{C}^\times.$$

Lusztig: $(T, \theta) \rightsquigarrow$ semisimple conj class $x \in {}^\vee G(\mathbb{F}_q)$.

This is a step in the right direction, but not quite a Langlands classification.

Question today: what's Langlands say about $\widehat{G(\mathbb{F}_q)}$?

Local p-adic Weil group

Fix p-adic $F \supset \mathcal{O} \supset \mathcal{P}$, $\quad \mathcal{O}/\mathcal{P} \simeq k = \mathbb{F}_q$.

Define Galois groups $\Gamma_F = \mathrm{Gal}\, \overline{F}/F$, $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$.

Naturality of integral structures $\rightsquigarrow$

$$1 \to I_F \to \Gamma_F \to \Gamma_k \to 1.$$

Extensions of $k = \mathbb{F}_q$ are $\mathbb{F}_{q^m}$, so $\Gamma_k = \varprojlim_m \mathbb{Z}/m\mathbb{Z}$.

Generator is arithm Frobenius $\mathrm{Frob}\colon \overline{k} \to k$, $\;\mathrm{Frob}(x) = x^q$.

Weil group of F is preimage of $\mathbb{Z} = \langle \mathrm{Frob} \rangle$, so

$$1 \to I_F \to W_F \to \langle \mathrm{Frob} \rangle \to 1.$$

Weil group of $\mathbb{F}_q$ (MacDonald)

$k = \mathbb{F}_q$ finite field; $\Gamma_k = \mathrm{Gal}(\overline{k}/k) = \varprojlim_m \mathbb{Z}/m\mathbb{Z}$.

Definition (MacDonald) Weil grp $W_k = \varprojlim_m \mathbb{F}_{q^m}^\times$.

$\mathbb{F}_{q^m}^\times \simeq \mathbb{Z}/(q^m - 1)\mathbb{Z}$ but not canonically.

Inverse limit is taken using norm maps $N_{md,m} \colon \mathbb{F}_{q^{md}}^\times \to \mathbb{F}_{q^m}^\times$.

$\mathrm{Hom}(\mathbb{F}_{q^m}^\times, H) \simeq \{h \in H \mid h^{q^m - 1} = 1\}$ but not canonically.

$\mathrm{Hom}_{\mathrm{cont}}(W_k, H) \simeq \{h \in H \text{ fin ord prime to } q\}$ but not canonically.

Γ_k acts on W_k, $\mathrm{Frob} \cdot w = w^q$.

Definition (MacDonald) Continuous $\phi \colon W_k \to H$ is Frob-equivariant if $\exists f$ so that $[\mathrm{Ad}(f)\phi](w) = \phi(\mathrm{Frob} \cdot w)$

Equivalently: $\phi \leftrightsquigarrow h \in H$ finite order, h conjugate to h^q.

Example: $q = 2$, $H = SL(2, \mathbb{C})$, $\omega = \exp(2\pi i/3)$,

$$h = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Relating p-adic F and finite k

Recall p-adic $F \supset \mathcal{O} \supset \mathcal{P}$, $\quad \mathcal{O}/\mathcal{P} \simeq k = \mathbb{F}_q$.

Galois groups $\Gamma_F = \mathrm{Gal}\,\overline{F}/F$, $\quad \Gamma_k = \mathrm{Gal}(\overline{k}/k)$.

p-adic Weil $\quad 1 \to I_F \to W_F \to \langle \mathrm{Frob} \rangle \to 1$.

$P_F =$ wild ramif $=$ (normal) p-Sylow subgp $\subset I_F$;

$$I_F/P_F \simeq W_k \qquad \text{(as } \Gamma_k\text{-modules.)}$$

Conclusion:

$$[\text{triv on } P_F \;\; \phi_F\colon W_F \to H] \;\approx [\text{Frob-eqvt } \;\phi_k\colon W_k \to H].$$

Desideratum: Tamely ramified Langlands parameter for nice $G/F \rightsquigarrow$ Langlands parameter for G/k.

$G(k) =$ quotient of a maximal compact $G(\mathcal{O}) \subset G(F)$; $G(k)$ rep $\pi(\phi_k)$ meant to be lowest $G(\mathcal{O})$-type of $\pi(\phi_F)$.

To make this precise, need Langlands classification.

What's a root datum?

G connected reductive alg$/\overline{K}$ alg. closed.

$T \subset G$ maximal torus $\rightsquigarrow$ root datum of T in G

$$\mathcal{R}(G,T) = (X^*, R, X_*, R^\vee), \quad X^* = X^*(T) = \mathrm{Hom}(T, K^\times)$$
$$\supset R = R(G,T) \quad \text{roots of } T \text{ in } G$$
$$X_* = X_*(T) = \mathrm{Hom}(K^\times, T)$$
$$\supset R^\vee = R^\vee(G,T) \quad \text{coroots of } T \text{ in } G.$$

R and $R^\vee$ in bijection, $\alpha \leftrightsquigarrow \alpha^\vee$. Lattices X^*, X^* dual. Pair $(\alpha, \alpha^\vee) \rightsquigarrow$

$$s_\alpha \colon X^* \to X^*, \quad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha, \quad s_{\alpha^\vee} = {}^t s_\alpha^{-1} \colon X_* \to X_*.$$

PROPERTIES: for all $\alpha \in R$

1. RD1: $\langle \alpha, \alpha^\vee \rangle = 2$ (so $s_\alpha^2 = \mathrm{Id}$)
2. RD2: $s_\alpha R = R$, $s_\alpha^\vee R^\vee = R^\vee$, $(s_\alpha \beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$
3. RDreduced: $2\alpha \notin R$, $2\alpha^\vee \notin R^\vee$.

Weyl group of root datum is

$$\mathrm{Aut}(X^*) \supset \langle s_\alpha \mid \alpha \in R \rangle =_{\mathrm{def}} W \overset{w \mapsto {}^t w^{-1}}{\simeq} \langle s_{\alpha^\vee} \mid \alpha \in R \rangle \subset \mathrm{Aut}(X_*).$$

Root datum axioms symmetric in (X^*, R) and $(X_*, R^\vee)$.

G determined by root datum and alg closed field $\overline{K}$.

Based root data...

...means fixing positive roots.

$T \subset G \rightsquigarrow (X^*, R, X_*, R^\vee) = \mathcal{R}(G, T)$.

Borel $T \subset B \subset G \leftrightsquigarrow$ simple positive $\Pi \subset R$, $\Pi^\vee \subset R^\vee$.

$\rightsquigarrow$ based root datum $\mathcal{B}(G, B, T) = (X^*, \Pi, X_*, \Pi^\vee)$.

$\mathcal{B}$ is combinatorial object (two dual lattices, two finite subsets in bijection) which (with $\overline{K}$) determines G.

Definition Automorphism of based root datum $\mathcal{B} = (X^*, \Pi, X_*, \Pi^\vee)$ is $\sigma \in \mathrm{Aut}(X^*)$ such that

$$\sigma(\Pi) = \Pi, \quad {}^t\sigma^{-1}(\Pi^\vee) = \Pi^\vee, \quad \sigma(\alpha)^\vee = {}^t\sigma^{-1}(\alpha^\vee)$$

Nota bene: $\mathrm{Aut}(\mathcal{R}) = \mathrm{Aut}(\mathcal{B}) \ltimes W$. Study factors separately.

Define $\mathcal{B}^\vee = (X_*, \Pi^\vee, X^*, \Pi)$; then $\mathrm{Aut}(\mathcal{B}) \overset{\sigma \mapsto {}^t\sigma^{-1}}{\simeq} \mathrm{Aut}(\mathcal{B}^\vee)$.

If K perfect field, $\Gamma = \mathrm{Gal}(\overline{K}/K)$, then

K-rational form of G

$\rightsquigarrow$ quasisplit K-form of G

$\leftrightsquigarrow$ action of Γ on $[\mathcal{B}(G, B, T) = (X^*, \Pi, X_*, \Pi^\vee)]$.

Shape of local Langlands conjecture

G reductive $/F$ loc $\rightsquigarrow \mathcal{B}(G, B, T)$ based root datum.

$\mathrm{Gal}(\overline{F}/F) = \Gamma_F \xrightarrow{G/F} \mathrm{Aut}(\mathcal{B})$.

Fix field $K = \overline{K}$ to study reps$/K$ of $G(F)$. (Often $\mathbb{C}$.)

Form dual group ${}^\vee G/K \leftrightsquigarrow$ dual root datum $\mathcal{B}^\vee$.

L-group of $G(F)$ is ${}^L G = \Gamma_F \ltimes {}^\vee G(K)$.

$W_F =$ Weil group of F equipped with $W_F \to \Gamma_F$.

Definition Langlands parameter is $\phi\colon W_F \to {}^L G$ so

$$\begin{array}{ccc} W_F & \xrightarrow{\phi} & {}^L G \\ & \searrow \quad \swarrow & \\ & \Gamma_F & \end{array}$$

commutes, other nice properties.

Equivalence of parameters is conjugation by ${}^\vee G(K)$.

Conjecture Irreps of $G(F)$ partitioned to finite $\Pi(\phi)$.

Langlands parameters for $k = \mathbb{F}_q$

$k = \mathbb{F}_q$ finite field; $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$.

Generator is arith Frobenius $\mathrm{Frob} = q$th power map on $\overline{k}$.

k-ratl form of conn reductive alg G = action of Γ on based root datum $\mathcal{B}$ = fin order aut of $\mathcal{B}$.

${}^LG =_{\mathrm{def}} {}^\vee G \rtimes \Gamma$; ${}^\vee G$ over $\mathbb{C}$, or $\overline{\mathbb{Q}}_\ell$, or...: field for reps.

Weil grp $W_k = \varprojlim_m \mathbb{F}_{q^m}^\times$; $W_k \to \Gamma_k$ trivial.

Langlands param $= \big(\underbrace{\phi\colon W_k \to {}^\vee G}_{\text{respect Frob}}\big)/{}^\vee G$ conj.

$\phi(W_k) \subset {}^\vee G$ (not ${}^LG$) since $W_k \to 1 \in \Gamma_k$.

Respect Frob $= \exists\, f \in \mathrm{Frob}({}^\vee G) \subset {}^LG$, $\mathrm{Ad}(f)\phi(\gamma) = \phi(\mathrm{Frob}\cdot\gamma)$.

KEEP COSET $f({}^\vee G_0^\phi)$ as part of parameter ϕ.

Equiv of params $(\phi, f({}^\vee G_0^\phi))$, $(\phi', f'({}^\vee G_0^{\phi'}))$ is ${}^\vee G$-conjugacy.

Representations of finite Chevalley groups

$G \supset B \supset T$ conn red alg $/k = \mathbb{F}_q$, Frob: $G \to G$.

Get Γ_k action on W permuting gens $\rightsquigarrow {}^\Gamma W = W \rtimes \Gamma_k$

$\tilde{w} = w\,\mathrm{Frob}$ (another) Frobenius morphism $T \to T$.

Deligne-Lusztig found all irr chars of $G(\mathbb{F}_q)$ inside virtual chars $R_{\theta'}^{T'}$ (T' ratl maxl torus, θ' char of $T'(\mathbb{F}_q)$).

Proposition. For any rational (= Frob-stable) max torus $T' \subset G$, $\exists!$ W-conj class of $\tilde{w}$ so $(T', \mathrm{Frob}) \simeq (T, \tilde{w})$.

Proposition (Macdonald)

$\widehat{T^{\tilde{w}}} \simeq \{\phi\colon W_k \to {}^\vee T \mid (w\,\mathrm{Frob}) \cdot \phi(\gamma) = \phi(\mathrm{Frob} \cdot \gamma)\}$.

Conclusion: L-params ϕ' for G = DL-pairs (T', θ').

$R_{\theta'}^{T'}$ and $R_{\theta''}^{T''}$ overlap $\iff$ ϕ', ϕ'' ${}^\vee G$-conjugate.

Deligne-Lusztig def: (T', θ') and (T'', θ'') are geom conj.

$\widehat{G(\mathbb{F}_q)}$ partitioned by Langlands parameters.

L-packet $\Pi(\phi)$ = all irrs in all $R_{\theta'}^{T'} \leftrightsquigarrow \phi$

Using Deligne-Langlands params to partition L-pkts harder...

Deligne-Langlands conjecture

G reductive $/K \rightsquigarrow {}^L G$.

Langlands conj: reps of $G(K)$ $\leftrightsquigarrow$ params $\phi\colon W_K \to {}^L G$ from (abelian) class field theory: bijectively true for $G =$ torus.

Deligne understood that difference between torus and reductive is unipotent: can sharpen Langlands conjecture by

Definition (K nonarchimedean or finite) Weil-Deligne group of K is $W_K' = W_K \ltimes \mathbb{G}_a$.

Frob acts on $\mathbb{G}_a$ by qth power map mult by q.

Definition Deligne-Langlands parameter $\rho = (\phi_\rho, N_\rho)$ has $\phi_\rho\colon W_K \to {}^L G$ Langlands param, $N_\rho \in {}^\vee\mathfrak{g}$ nilpotent, with

1. Case $K = F$ local: require $\mathrm{Ad}(\phi_\rho(\mathrm{Frob}))(N_\rho) = qN_\rho$.
2. Case $K = k = \mathbb{F}_q$: require KEPT COSET $f({}^\vee G_0^{\phi_\rho})$ to have rep f_{DL} satisfying $\mathrm{Ad}(f_{DL} N_\rho) = qN_\rho$.

For $K = k = \mathbb{F}_q$, KEEP COSET $f_{DL}({}^\vee G_0^\rho)$ as part of ρ.

Lusztig's big orange book

F finite $\rightsquigarrow \mathcal{S}(F) =_{\text{def}} \{(f,\sigma) | f \in F, \sigma \in \widehat{F^f}\}/(\text{conj by } F)$.

$$\mathcal{S}((\mathbb{Z}/2\mathbb{Z})^n) = (\mathbb{Z}/2\mathbb{Z})^n \times (\widehat{\mathbb{Z}/2\mathbb{Z}})^n \qquad 2n\text{-diml sympl } /\mathbb{F}_2.$$

$$\mathcal{S}(S_3) = \left\{(1,\mathbb{C}),(1,\text{rfl}),(1,\text{sgn}),(s_2,\mathbb{C}),(s_2,\text{sgn}),(s_3,\mathbb{C}),(s_3,\omega),(s_3,\omega^2)\right\}.$$

$G \supset B \supset T$ conn red alg $/\mathbb{F}_q$, ${}^L G$ L-group.

Def $\rho = (\phi, N)$ special if $N \in {}^\vee\mathfrak{g}^\phi$ is special nilp.

Recall that ρ remembers coset $f({}^\vee G_0^{\phi,N})$.

Theorem (Lusztig). Irreducible reps of $G(\mathbb{F}_q)$ are partitioned into packets $\Pi(\rho)$ by special DL parameters ρ. The packet $\Pi(\rho)$ is indexed by $\mathcal{S}(F)$ using Lusztig quotient F of ${}^\vee G^\rho/{}^\vee G_0^\rho$.

To make this look like other Langlands classifications, prefer to drop$_1$ requirement N special, replace$_2$ Lusztig quotient by ${}^\vee G^\rho/[(Z({}^\vee G)^\Gamma)({}^\vee G_0^\rho)]$, replace$_3$ $\mathcal{S}(F)$ by subset $\widehat{F}$.

Prefs 1 and 2 $\rightsquigarrow$ more params, Pref 3 $\rightsquigarrow$ fewer params.

Rewriting Lusztig's book á la Langlands

Definition Deligne-Langlands param for $G(\mathbb{F}_q)$ is

$$\rho = (\phi, N, \bar{f}),$$

1. $\phi\colon W_k \to {}^\vee G$ semisimple, $N \in {}^\vee\mathfrak{g}^\phi$ nilpotent,
2. $\bar{f} = f\left({}^\vee G_0^{\phi,N}\right)$, $f \in {}^L G \to \mathrm{Frob}$
3. $\mathrm{Ad}(f)(\rho(w)) = \rho(w^q),\ \mathrm{Ad}(f)(N) = qN.$

Complete geometric Deligne-Langlands param has also

4. $\xi \in \widehat{{}^\vee G^{\phi,N}/{}^\vee G_0^{\rho,N}}$, $\xi|_{Z({}^\vee G)^\Gamma} = 1$.

Conjecture Irreducible reps of $G(\mathbb{F}_q)$ partitioned into packets $\Pi(\rho)$ by all Deligne-Langlands parameters ρ. Packet $\Pi(\rho)$ indexed by irr reps ξ of ${}^\vee G^\rho/{}^\vee G_0^\rho Z({}^\vee G)^{\Gamma_k}$.

Let us pause for a moment

Talk was meant to make me think through next few slides.

This plan was an abject failure.

So perhaps some of you will think through them?

♡ Hopefully yours,♡

David

Lifting finite to p-adic

G connected reductive algebraic $/k = \mathbb{F}_q$.

Fix p-adic $F \supset \mathcal{O} \supset \mathcal{P}$, $\quad \mathcal{O}/\mathcal{P} \simeq k$.

Fix p-adic $\mathbb{G} \leftrightsquigarrow$ based root datum of G/k, Γ_F acts via Γ_k.

$\mathbb{G}/F$ can be any unramified quasisplit group$/F$.

G/k and $\mathbb{G}/F$ have same complex dual group ${}^\vee G$.

${}^L G_F = {}^\vee G \rtimes \Gamma_F$, $\quad {}^L G_k = {}^\vee G \rtimes \Gamma_k = {}^L G_F / I_F$:

note I_F normal since it acts trivially on based root datum.
Set $P_F =$ wild ramif grp $\subset I_F$; recall $I_F/P_F \simeq W_k$.

tamely ram. ρ_F for F:	$\big($ $\phi\colon I_F/P_F \to {}^\vee G$,	$N \in {}^\vee\mathfrak{g}^\phi$,	f $\big)$
param ρ_k for k:	$\big($ $\phi\colon I_F/P_F \to {}^\vee G$,	$N \in {}^\vee\mathfrak{g}^\phi$,	$f({}^\vee G_0^{\phi,N})$ $\big)$

Here $f \mapsto \text{Frob}$, $\; f\phi(i)f^{-1} = i^q$, $\; \mathrm{Ad}(f)(N) = qN$, $\; (i \in I/P)$.

Deduce tamely ramif params for $\mathbb{G}/F \twoheadrightarrow$ params for G/k

Lowest K-types

$\mathbb{G}/F$ p-adic unramified quasisplit $/F \leftrightsquigarrow G/k$ finite.

$G(F)$ has maximal compact $G(\mathcal{O}) \twoheadrightarrow G(k)$.

tamely ram. ρ_F for F: $\left(\phi\colon I_F/P_F \to {}^\vee G, \quad N \in {}^\vee\mathfrak{g}^\phi, \quad f \right)$

param ρ_k for k: $\left(\phi\colon I_F/P_F \to {}^\vee G, \quad N \in {}^\vee\mathfrak{g}^\phi, \quad f({}^\vee G_0^{\phi,N}) \right)$

Definition component group for Deligne-Langlands ρ is

$$A(\rho) = {}^\vee G^\rho / Z({}^\vee G)^\Gamma \, {}^\vee G_0^\rho.$$

p-adic packet $\Pi(\rho_F)$ conjecturally indexed by $\widehat{A(\rho_F)}$, $\pi_F(\xi) \leftrightsquigarrow \xi$.

k packet $\Pi(\rho_k)$ conjecturally indexed by $\widehat{A(\rho_k)}$, $\pi_k(\tau) \leftrightsquigarrow \tau$.

Suggests: tamely ramified $\mathbb{G}(F)$ rep $\pi_F(\xi)$ has $\mathbb{G}(\mathcal{O})$-type factoring to $G(k)$ rep $\pi_k(\tau)$. Which τ?

Cent means stab of $f({}^\vee G_0^{\phi,N}) \supset$ cent of f: $A(\rho_F) \to A(\rho_k)$.

If ξ doesn't factor to image, no τ.

If ξ factors to $\overline{\xi}$ on image, then all τ in $\mathrm{Ind}_{\text{image}}^{A(\rho_k)}(\overline{\xi})$.

Homework: extend to all params, all max cpts, all $\mathbb{G}$.