Extended groups and representation theory

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Outline

Classification problems: old ideas

Root data, compact groups, complex groups

Classifying real groups

Classifying representations with extended groups

Plan of talk

Lie groups, repn theory are continuous, analytic.

But lists of Lie groups, repns can be discrete.

Allows for exact computer calculations.

Idea: conjugation by G reduces probs about $G \rightsquigarrow$ probs about max torus, Weyl group.

Example (Cartan-Weyl): fin-diml reps $\leftrightsquigarrow$ dom wts.

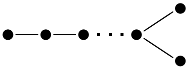
Agenda:

1. Root datum classif of complex reductive groups
2. Extended groups (combinatorial) classif of real forms
3. Extended dual groups = L-groups, classif of repns

Classifying compact groups: take one

Compact conn Lie gps $\rightsquigarrow$ Dynkin diagrams

Dynkin diags for classical compact groups U...

type	diagram	U
A_n		$SU(n+1)$, quotients
B_n		$SO(2n+1)$, $Spin(2n+1)$
C_n		$Sp(n) = U(n, \mathbb{H})$, $PSp(n)$
D_n		$SO(2n)$, $Spin(2n)$, etc.

Missing from pictures: covering groups...

Classifying compact groups: take two

Grothendieck replaced Dynkin diag by root datum.

U compact $\rightsquigarrow$ T maximal torus $\rightsquigarrow$

$X^*(T)$ = lattice of characters

$\supset R(U, T)$ roots of T in U (finite subset),

$X_*(T)$ = lattice of cocharacters

$\supset R^\vee(U, T)$ coroots of T in U (finite subset).

Structure: lattices X^*, X_* dual by $\langle , \rangle \colon X^* \times X_* \to \mathbb{Z}$;

Bijection $\alpha \mapsto \alpha^\vee$ from R to $R^\vee$;

root refl $s_\alpha \colon X^* \to X^*, \quad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$

carrying R to R, $R^\vee$ to $R^\vee$.

Weyl group $W = W(U, T)$ generated by various s_α.

Structure is integer matrices; COMPUTERIZABLE!

Main theorems about root data

Definition. *Root datum* is $(X^*, R, X_*, R^\vee)$ subj to

1. X^* and X_* are dual lattices, by pairing $\langle , \rangle$.
2. $R \subset X^*$, $R^\vee \subset X_*$ finite, in bijection by $\alpha \leftrightarrow \alpha^\vee$.
3. $\langle \alpha, \alpha^\vee \rangle = 2$, all $\alpha \in R$.
4. Aut of X^* $s_\alpha(\lambda) =_{\text{def}} \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ permutes R.
5. Transpose $s_{\alpha^\vee}$ of X_* permutes $R^\vee$.
6. Root datum is *reduced* if $\alpha \in R \implies 2\alpha \notin R$.

Weyl group is $W = \langle s_\alpha \mid \alpha \in R \rangle \subset \text{Aut}(X^*)$.

Theorem.

1. Every reduced root datum arises from a compact connected Lie group.
2. Every isomorphism of root data

$$\Phi\colon (X^*(T), R(U,T), X_*(T), R^\vee(U,T)) \to (X^*(T'), R(U',T'), X_*(T'), R^\vee(U',T'))$$

arises from isomorphism $(U,T) \to (U',T')$ of compact connected Lie groups.

Open problem: describe group maps with root data.

Complexifying U

Root data so good as description of compact groups $\rightsquigarrow$ seek more questions with the same answer...

Cplx alg group is a subgp of $GL(E)$ def by poly eqns.

Cpt Lie gp $U \rightsquigarrow$ faithful rep on complex $E \rightsquigarrow$ embed $U \hookrightarrow GL(E)$.

$$G(\mathbb{C}) =_{\text{1st def}} \text{Zariski closure of } U \text{ in } GL(E)$$

.

$$\begin{aligned} C(U)_U &= \text{alg of } U\text{-finite functions on } U \\ &= \text{matrix coeffs of fin-diml reps} \\ &\simeq \sum_{(\tau, V_\tau) \in \widehat{U}} \mathrm{End}(V_\tau) \quad \text{(Peter-Weyl)}, \end{aligned}$$

finitely-generated commutative algebra $/\mathbb{C}$.

$$G(\mathbb{C}) =_{\text{2nd def}} \mathrm{Spec}(C(U)_U).$$

Theorem. Construction gives *all* cplx reductive alg gps.

Corollary. Root data $\leftrightarrow$ cplx conn reductive alg gps.

Realifying $G(\mathbb{C})$

Real alg gp is cplx alg gp $G(\mathbb{C})$ with $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ action.

Require $(\sigma \cdot f)(g) =_{\text{def}} \sigma[f(\sigma^{-1}g)]$ $(\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}))$ is real algebra aut of reg fns on $G(\mathbb{C})$; $G(\mathbb{R}) =$ gp of fixed pts.

General (separable) $\overline{k}/k$: study rational forms using Galois cohomology, often starting from *split* form.

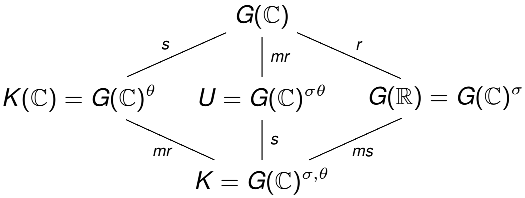
$\mathbb{C}/\mathbb{R}$: study Galois action as *single* automorphism σ (complex conjugation), often relate to *compact form*.

Theorem (Cartan). $G(\mathbb{C})$ cplx conn reductive alg.

1. Given real form form σ of $G(\mathbb{C})$, there is *compact* real form σ_0 s.t. $\sigma\sigma_0 = \sigma_0\sigma$. Therefore $\theta = \sigma\sigma_0$ is alg inv aut of $G(\mathbb{C})$, Cartan involution for $G(\mathbb{R}, \sigma)$.
2. Write $K(\mathbb{C}) = G(\mathbb{C})^\theta$, reductive alg subgp. Real form
$$K =_{\text{def}} K(\mathbb{R}, \sigma) = K(\mathbb{R}, \sigma_0) = G(\mathbb{R}, \sigma)^\theta$$
is *maximal compact subgroup of $G(\mathbb{R})$*.
3. Given alg inv aut θ of $G(\mathbb{C})$, there is *compact* real form σ_0 of $G(\mathbb{C})$, s.t. $\theta\sigma_0 = \sigma_0\theta$. So $\sigma =_{\text{def}} \theta\sigma_0$ is real form, Cartan real form for $G(\mathbb{C})$ and θ.

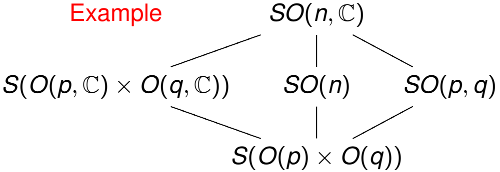
Cartan picture of real reductive groups

Real forms σ / ($G(\mathbb{C})$ conj) $\leftrightsquigarrow$ inv auts θ / ($G(\mathbb{C})$ conj).



Subgp is max cpt (m) or real form (r) or fixed by inv aut (s).

Example



Classify real forms by involutive automorphisms.

Harish-Chandra: (irreps of $G(R)$) $\leftrightsquigarrow$ $(\mathfrak{g}, K(\mathbb{C}))$-mods

Involutive automorphism enough for rep theory!

Root datum automorphisms: inner vs outer

$(X^*, R, X_*, R^\vee)$ root datum; *basing* is choice of basis (*simple roots*) $\Pi \subset R$, $\Pi^\vee \subset R^\vee$. Write $S = \{s_\alpha \mid \alpha \in \Pi\}$ *simple reflections*.

$(X^*, \Pi, X_*, \Pi^\vee)$ called *based root datum*.

(based root datum) $\rightsquigarrow$ Dynkin diagram: nodes $\leftrightsquigarrow$ Π, edges $\leftrightsquigarrow$ $\langle \alpha, \beta^\vee \rangle \neq 0$ $(\alpha, \beta \in \Pi)$.

Theorem.

1. Simple refls S are Coxeter gens for Weyl group W.
2. W is a normal subgroup of $\mathrm{Aut}(X^*, R, X_*, R^\vee)$
3. W acts in simply transitive way on bases.
4. Short exact sequence

$$1 \to W \to \mathrm{Aut}(X^*, R, X_*, R^\vee) \to \mathrm{Out}(X^*, R, X_*, R^\vee) \to 1$$

(defining Out) is split by subgroup $\mathrm{Aut}(X^*, \Pi, X_*, \Pi^\vee)$

Conclude: outer automorphisms of root datum correspond to Dynkin diagram automorphisms.

Group automorphisms: inner vs outer

$G \supset T$ cplx reduc $\supset$ max tor $\rightsquigarrow$ $(X^*(T), R(G,T), X_*(T), R^\vee(G,T))$ root datum; basing is choice of *Borel subgroup* $B \supset T$.

Pinning is choice of *root SL(2)s* $\phi_\alpha\colon SL(2) \to G$ for each $\alpha \in \Pi$; pinning determines $T \subset B$.

Theorem. Suppose that $(G, \{\phi_\alpha\})$ and $(G', \{\phi'_{\alpha'}\})$ are pinnings for reductive groups. Any isom Φ of the based root data lifts to unique isom $\Phi\colon (G, \{\phi_\alpha\}) \to (G', \{\phi'_{\alpha'}\})$ of pinned alg gps.

Corollary. G cplx conn reductive alg.

1. Group $\mathrm{Int}(G) \simeq G/Z(G)$ of inner automorphisms acts simply transitively on pinnings.
2. Short exact sequence
$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$
is split by subgroup $\mathrm{Aut}(G, \{\phi_\alpha\})$.
3. $$\mathrm{Out}(G) \simeq \underbrace{\mathrm{Aut}(G, \{\phi_\alpha\})}_{\textit{distinguished}} \xrightarrow{\sim} \mathrm{Aut}(X^*, \Pi, X_*, \Pi^\vee).$$

$$\mathrm{Aut}(G) \simeq \mathrm{Int}(G) \rtimes \mathrm{Aut}(G, \{\phi_\alpha\}).$$

Classifying real forms: extended groups

$G \supset B \supset T$ cplx conn red alg, Borel, max torus $\rightsquigarrow$ $(X^*, \Pi, X_*, \Pi^\vee)$ based root datum; $\{\phi_\alpha \mid \alpha \in \Pi\}$ pinning.

Recall Cartan: real forms $\sigma \leftrightsquigarrow$ involutive auts θ.

Inv aut $\theta \rightsquigarrow \delta = \delta(\theta) \in \mathrm{Out}(G) \simeq \mathrm{Aut}(X^*, \Pi, X_*, \Pi^\vee)$.

Fix involution $\delta \in \mathrm{Aut}(X^*, \Pi, X_*, \Pi^\vee) \simeq \mathrm{Aut}(G, \{\phi_\alpha\})$.

$\rightsquigarrow$ *extended group* $G^\Gamma =_{\mathrm{def}} G \rtimes \{1, \delta\} = \langle G, \delta \rangle$.

Defining relations:

$$\delta g \delta^{-1} = \delta(g) \qquad \text{(action of automorphism } \delta\text{)},$$
$$\delta^2 = 1 \qquad \text{(or replace by any } z \in Z(G)^\delta\text{)}$$

Definition. *Strong inv* is $\xi \in G^\Gamma \backslash G$ s.t. $\xi^2 \in Z(G)$.

Proposition. $N_G(T)$ orbits on strong invs in $T\delta$

$\xrightarrow{\sim} G$ orbits on strong invs

$\twoheadrightarrow G$ orbits of inv auts θ s.t. $\delta(\theta) = \delta$

FIRST LINE computerizable, LAST LINE interesting.

Secrets of KGB

$G \supset B \supset T$; $G^\Gamma = \langle G, \delta\rangle$ ext grp; θ inv aut, $K = G^\theta$; $G(\mathbb{R})$.

Beilinson-Bernstein (approx): irrs of $G(\mathbb{R}) \leftrightsquigarrow K\backslash G/B$.

Classical: fix K, study orbits on G/B = Borel subgps.

Adams-du Cloux: fix B, study orbits on $K\backslash G$ = strong invs.

Proposition. Given K, any Borel B' contains θ-fixed T', unique up to $B' \cap K$ conjugation.

Proposition. Given $B \supset T$, any strong inv has B-conj ξ' preserving T, unique up to T conj.

Theorem (Adams-du Cloux) There are bijections

$$T \text{ orbits on strong invs in } N_G(T)\delta$$
$$\xrightarrow{\sim} B \text{ orbits on strong invs}$$
$$\xrightarrow{\sim} \coprod_K K \text{ orbits on } G/B$$

FIRST LINE computerizable, LAST LINE interesting.

Cor $N_G(T)$ orbits on strong invs in $N_G(T)\delta \leftrightsquigarrow$ max tori in $G(\mathbb{R})$

Dual *everything*: L-groups

Recall axioms for root datum $(X^*, R, X_*, R^\vee)$:

1. X^* and X_* dual lattices, by pairing $\langle , \rangle$.
2. $R \subset X^*$, $R^\vee \subset X_*$ finite, in bijection by $\alpha \leftrightarrow \alpha^\vee$.
3. $\langle \alpha, \alpha^\vee \rangle = 2$, all $\alpha \in R$.
4. Aut of X^* $s_\alpha(\lambda) =_{\text{def}} \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ permutes R.
5. Transpose $s_{\alpha^\vee}$ of X_* permutes $R^\vee$.

Weyl group is $W = \langle s_\alpha \mid \alpha \in R \rangle \subset \text{Aut}(X^*)$

Dual root datum is $(X_*, R^\vee, X^*, R)$; same Weyl gp.

Similarly $\text{Out}(X^*, R, X_*, R^\vee) \simeq \text{Out}(X_*, R^\vee, X^*, R)$.

Recall (based root datum) $\leftrightsquigarrow$ (cplx reductive G.

Langlands dual group ${}^\vee G$: $\leftrightsquigarrow$ $(X_*, \Pi^\vee, X^*, \Pi)$.

repns of G $\leftrightsquigarrow$ structure of ${}^\vee G$.

Torus: characters of T $\leftrightsquigarrow$ one param subgps of ${}^\vee T$.

$\delta \in \text{Out}(G) \simeq \text{Out}({}^\vee G) \ni {}^\vee\delta \rightsquigarrow {}^\vee G^\Gamma =_{\text{def}} {}^L G$ *L-group of* $G(\mathbb{R})$.

Details about representations: L-groups

G cplx reductive, inner class of real forms $\leftrightsquigarrow$ $\delta \in \mathrm{Out}(G)$ $\leftrightsquigarrow$ $G^\Gamma = \langle G, \delta \rangle$ extended group.

$K\backslash G/B \leftrightsquigarrow \{\xi \in N_G(T)\delta \mid \xi^2 \in Z(G)\}\ /T$.

$\xi \rightsquigarrow$ *twisted inv* $w\delta \in \langle W, \delta\rangle$. (Order 2 elt of $W\delta$)

Reps of $G(R)$ related to *Langlands params* $\phi\colon W_{\mathbb{R}} \to {}^L G$ modulo ${}^\vee G$ conj.

Proposition. Langlands params $\leftrightsquigarrow$ $\{(\eta, \lambda) \in N_{{}^\vee G}({}^\vee T){}^\vee\delta \times {}^\vee\mathfrak{t}^+ \mid \eta^2 = \exp(2\pi i\lambda)\}\ /{}^\vee T$.

$\eta \rightsquigarrow$ *twisted inv* ${}^\vee w {}^\vee\delta \in \langle W, {}^\vee\delta\rangle$.

Definition. ξ and (η, λ) *match* if twisted invs are negative transpose.

Theorem. Matching pairs $(\xi, (\eta, \lambda))\ /\ (T \times {}^\vee T) \leftrightsquigarrow$ (irr reps of real forms in inner class).

FIRST LINE computerizable, LAST LINE interesting.