Generalizing endoscopy

David Vogan

Department of Mathematics
Massachusetts Institute of Technology

Representation Theory XVIII (Lie groups)
Outline

Introduction

L-groups

Langlands parameters

Endoscopic groups

Examples of endoscopic groups

Slides at http://www-math.mit.edu/~dav/paper.html
Joint work with Jeffrey Adams and Lucas Mason-Brown generalizing endoscopic transfer for reductive groups.

Our results concern real reductive groups.

Subject is a morass of technical difficulties, many of which are much worse for $\mathbb{R}$ than for $p$-adic fields.

Example: need to change def of Langlands parameter$/\mathbb{R}$.

I’ll avoid some difficulties by discussing non-archimedean local field $k$, and conn red alg $G/k$.

Avoid remaining difficulties by ignoring them.
What’s the plan?

Study rep theory of reductive algebraic $G$.

Typically $G$ defined over a local field $k$, but details later.

Endoscopic group: smaller reductive $H$, often $H \not\subset G$.

Examples:

- $G = \text{Sp}(2(p + q), \mathbb{R})$, $H = \text{SO}(p, p) \times \text{Sp}(2q, \mathbb{R})$
- $G = \text{Sp}(2(p + r), \mathbb{R})$, $H = \text{GL}(p, \mathbb{R}) \times \text{Sp}(2r, \mathbb{R})$

Endoscopic transfer: (virtual $H$-reps) $\rightarrow$ (virt $G$-reps).

Will define slightly larger class of such $H \not\subset G$.

New examples:

- $G = \text{Sp}(2(p + q + r), \mathbb{R})$, $H = \text{U}(p, q) \times \text{Sp}(2r, \mathbb{R})$
- $G = \text{GL}(2p + q, \mathbb{R})$, $H = \text{GL}(p, \mathbb{C}) \times \text{GL}(q, \mathbb{R})$
How do you name a group? (case of $\bar{k}$)

To ask about a group $G$, you need first to give it a name. Lie, Chevalley and Grothendieck solved this problem:

(reductive algebraic group $G$) / algebraically closed $\bar{k} \leftrightarrow$ based root datum $\mathcal{R}(G) = (X^*, \Pi, X^*, \Pi^\vee)$.

$X^*$ and $X_*$ are dual lattices: chars/ cochars of max torus in $G$.

finite sets $\Pi \subset X^*$ and $\Pi^\vee \subset X_*$: simple roots/simple coroots.

Any lattice is isomorphic to $\mathbb{Z}^n$, so the name $\mathcal{R}(G)$ of $G$ is two finite collections of $n$-tuples of integers.

Two names are the same iff first collections differ by invertible integer matrix $M$, second collections differ by $^t M^{-1}$.

Example: $GL(2)$ is given by $\Pi = \{(1, -1)\}, \quad \Pi^\vee = \{(1, -1)\}$.

Example: the exceptional group $G_2$ is given by

$\Pi = \{(1, 0), (0, 1)\}, \quad \Pi^\vee = \{(2, -1), (-3, 2)\}$. 
How do you name a group? (case of $k$)

A reductive $G/\overline{k}$ named by the (combinatorial) based root datum $\mathcal{R}(G)$: two finite sets of $n$-tuples of integers.

Defining $G/k$ gives action of $\Gamma = \text{Gal}(\overline{k}/k)$ on $\mathcal{R}(G)$.

Concretely: repn of $\Gamma$ by $n \times n$ int matrices $\mu(\sigma)$ so

$$\mu(\sigma) \cdot \Pi = \Pi, \quad ^t\mu(\sigma)^{-1} \cdot \Pi^\vee = \Pi^\vee,$$

respecting axioms for a based root datum.

Shorthand: action of $\Gamma$ on Dynkin diagram of $G$.

$k$-forms of $G$ are inner if $\sim$ same action of $\Gamma$ on $\mathcal{R}(G)$.

Example A rank two unitary group $/k$ starts with a separable quadratic extension of $k$; that is, subgroup $\Gamma_0 \subset \Gamma$ of index two.

Representation of $\Gamma$ on $\mathbb{Z}^2$ is

$$M(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\sigma \in \Gamma_0), \quad M(\sigma) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (\sigma \notin \Gamma_0)$$

Unitary groups w fixed quadratic extension: single inner class.
L-group

Defining \( G / \text{any} \ k \xrightarrow{\sim} \\text{action of} \ \Gamma = \text{Gal}(\overline{k}/k) \text{ on } \mathcal{R}(G) \).

Axioms for root data symmetric in \((X^*, \Pi) \leftrightarrow (X^*, \Pi^\vee)\).

Dual based root datum is \( \mathcal{R}^\vee = (X^*, \Pi^\vee, X^*, \Pi) \).

Gives reductive algebraic dual group \( \overset{\vee}{G} \) and

\[ \mathcal{L}\text{-group } \overset{L}{G} = \overset{\vee}{G} \rtimes \Gamma, \quad \text{(defined over } \mathbb{Z}) \].

Langlands’ insight (local Langlands conjecture):

\((\text{analytic rep theory/} K \text{ of } G(k)) \leftrightarrow (\text{alg geom of } \overset{L}{G}(K))\).

Typically \( K = \mathbb{C} \) and \( k \) is local.

Complex reps of \( G(k) \leftrightarrow \text{complex alg geom of } \overset{L}{G}(\mathbb{C}) \)

Endoscopic (and generalized endoscopic) groups \( H \) correspond to subgroups \( \overset{E}{H} \subset \overset{L}{G} \).

By local Langlands, relating \( \overset{\wedge}{H(k)} \) to \( \overset{\wedge}{G(k)} \) means relating alg geom of \( \overset{L}{G}(\mathbb{C}) \) to alg geom of subgroup \( \overset{E}{H}(\mathbb{C}) \).

Sounds easy and natural!
Weil-Deligne group

A \( p \)-adic field, \( \Gamma = \text{Gal} (\overline{k}/k) \), \( G \) conn reductive alg/\( k \).

\( L \)-group complex: \( 1 \to \check{\wedge} G \to L G \to \Gamma \to 1 \).

Local Langlands explains irr reps \( \hat{G}(k) \) using \( L G \).

Recall: finite residue field \( \mathbb{F}_q \) of \( k \) \( \leadsto \) natural surjection

\[ 1 \to I_k \to \Gamma \to \hat{\mathbb{Z}} = \langle \text{Frob} \rangle \to 1. \]

Inertia subgroup \( I_k \) is profinite compact.

Weil group \( W_k = \) (dense) preimage in \( \Gamma \) of \( \langle \text{Frob} \rangle \):

\[ 1 \to I_k \to W_k \to \mathbb{Z} = \langle \text{Frob} \rangle \to 1. \]

Weil-Deligne group \( W'_k = W_k \rtimes \mathbb{C} \): here \( I_k \) acts trivially on \( \mathbb{C} \), and \( \text{Frob} \) acts by multiplication by \( q \).
Langlands parameters

Recall $\Gamma = \text{Gal}(\overline{k}/k)$, and $W_k \subset \Gamma$ is a dense subgroup.

Have two short exact sequences

$$
1 \rightarrow \vee G \rightarrow L G \rightarrow \Gamma \rightarrow 1
$$

$$
1 \rightarrow C \rightarrow W'_k \rightarrow W_k \rightarrow 1
$$

**Langlands parameter** is a group homomorphism $\phi' : W'_k \rightarrow L G$ compatible with exact sequences.

Means $\phi'|_C : C \rightarrow \vee G$ (one-param nilp alg subgp), and $\phi'$ descends to inclusion $W_k \hookrightarrow \Gamma$.

**Loc Langlands conj:** $\phi' \rightsquigarrow$ finite L-pkt $\Pi(\phi') \subset \hat{G}(k)$.

More conjecture:

1. L-packets partition $\hat{G}(k)$;
2. $\Pi(\phi')$ depends only $\vee G$-conj class of $\phi'$;
3. if $G(k)$ quasisplit, then $\Pi(\phi') \neq \emptyset$. 
Want to translate problems about reps of $G(k)$ to alg geom problems about parameters in $LG$.

Infl char of $\phi'$ is $\phi = \phi'|_{W_k}$. Each infl char $\phi : W_k \rightarrow LG$ extends in finitely many ways to $\phi' : W'_k \rightarrow LG$: the parameters of infl char $\phi$.

Since $I_k$ compact, $\bigvee G^\phi(I_k) = $ centralizer in $\bigvee G$ of $\phi(I_k)$ is reductive algebraic in $\bigvee G$.

Preimage $\widehat{\text{Frob}}$ in $W_k$ defines $\phi(\widehat{\text{Frob}}) \in LG$, so semisimple alg aut (indep of $\widehat{\text{Frob}}$) $\sigma_\phi = \text{Ad}(\phi(\widehat{\text{Frob}})) \in \text{Aut}(\bigvee G^\phi(I_k))$.

$\sigma_\phi$ defines $\bigvee G^\phi = (\bigvee G^\phi(I_k))^{\sigma_\phi}$, twisted pseudolevi of $\bigvee G^\phi(I_k)$.

$\pi(\phi) =_{\text{def}} q$-eigenspace of $\sigma_\phi$ on $\bigvee g^\phi(I_k)$, a vector space of nilpotent Lie algebra elements on which $\bigvee G^\phi$ acts.

The algebraic geom we want is $\bigvee G^\phi$ orbits on $\pi(\phi)$.

$\pi(\phi)$ is prehomogeneous for $\bigvee G^\phi$: finitely many orbits.
Could you repeat that?

Start with a Langlands parameter \( \phi' : W'_k \to {}^L G \).

Restriction \( \phi_I \) of \( \phi' \) to inertia \( I_k \subset \text{Gal}(\overline{k}/k) \) is arithmetic; image is profinite (compact) subgroup of \( {}^L G \):

\[
Z_{\phi}(\phi(I_k)) = {}^\vee G^{\phi(I_k)} \text{ reductive algebraic.}
\]

An extension \( \phi \) of \( \phi_I \) to \( W_k \) (called infinitesimal character) is given by a single element \( \phi(\text{Frob}) \) of \( {}^L G \).

\( \phi(\text{Frob}) \) defines \( \text{aut} \sigma_\phi \) of \( {}^\vee G^{\phi(I_k)} \), fixed points \( {}^\vee G^\phi \).

\( q \)-eigspace of \( d\sigma_\phi = \text{nilp subspace} \pi(\phi) \subset g^{\phi(I_k)} \).

\( \pi(\phi) \) is prehomogeneous for \( {}^\vee G^\phi \).

Parameters \( \phi' \) of infl char \( \phi \leftrightarrow {}^\vee G^\phi \) orbits \( O' \) on \( \pi(\phi) \).

irreps of infl char \( \phi \leftrightarrow {}^\vee G^\phi \)-eqvt perv sheaves on \( \pi(\phi) \).

\( L \)-packet of \( \phi' \leftrightarrow {}^\vee G^\phi \) sheaves with support \( O' \).
What’s the plan?

L-group has short exact seq $1 \to \check{\gamma}G \to {}^L G \to \Gamma \to 1$.

L-subgroup is ${}^L G \supset {^E H} \to \Gamma$, kernel $\check{\gamma}H$ reductive:

\[
\begin{array}{cccccc}
1 & \to & \check{\gamma}G & \to & {^L G} & \to & \Gamma & \to & 1 \\
\cup & & \cup & & \| & & \\
1 & \to & \check{\gamma}H & \to & {^E H} & \to & \Gamma & \to & 1
\end{array}
\]

In this setting param $\phi'_H$ for ${^E H} \rightsquigarrow$ param $\phi$ for $^L G$;

\[\pi(\phi_H) \subset \pi(\phi), \quad \check{\gamma}H^\phi \subset \check{\gamma}G^\phi.\]

This is the geometric part of local Langlands functoriality.

So relating reps of $G$ to reps of $H$ amounts to relating perv sheaves on $\pi(\phi)$ to perv sheaves on $\pi(\phi_H)$.

To get strong theorems relating perverse sheaves to a subvariety, need strong hypotheses on the subvariety.

Example is Goresky-MacPherson Lefschetz formula.

Need subvariety = fixed points of an automorphism.
What’s an endoscopic group?

Langlands params are $^\vee G$ orbits on (algebraic variety).

So action of $s \in ^\vee G \leadsto$ automorphism of params.

Endoscopic datum is

1. $s \in ^\vee G$ semisimple;
2. L-subgroup $^E H \subset (^L G)_s \subset ^L G$, with
3. $^\vee H =$ identity component of $^\vee G_s$ reductive in $^\vee G$.

Root datum $\mathcal{R}(^\vee H)$ has dual root datum $\leadsto H/\bar{k}$.

$^E H \leadsto$ action of $\Gamma = \text{Gal}(\bar{k}/k)$ on root data,

$\leadsto$ inner class of $k$-forms of $H$.

Endoscopic group for $G = H/k$, any form in inner class.
Where’s the fixed point formula?

$s \in V G$ semisimple, L-subgroup $E H \subset (L G)^s \subset L G$, $V H = (G^s)_0$.

Hypotheses imply $E H$ open in $(L G)^s$.

Simplify by assuming $E H = (L G)^s$. Then

(fixed pts of $Ad(s)$ on params) = (params for $E H$).

This equality allows application of a Lefschetz formula.

More precisely:

$$\text{tr}(s \text{ action on perv cohom for } L G) = \text{tr}(s \text{ action on perv cohom for } E H).$$

Since $s$ central in $E H$, right side is easy.

Equality seems to require $s$ to centralize $E H$.

Generalization seems impossible...
Here’s how to generalize

Generalized endoscopic datum is

1. \( s \in ^\vee G \) semisimple;
2. L-subgroup \(^L H \subset ^L G\) normalized by \( s\);
3. \(^\vee H = \) identity component of \(^\vee G^s\) reductive in \(^\vee G\);
4. quotient action of \( \text{Ad}(s) \) on \( \Gamma = ^E H/^\vee H\) is trivial.

As for endoscopic groups,
\[ ^E H \xrightarrow{\sim} \text{Galois action on root datum for } ^\vee H \]
\[ \xrightarrow{\sim} \text{inner class of } k\text{-forms of } H. \]

These \( k \) forms are generalized endoscopic groups.

Define \( \xi : ^E H \to ^\vee H \) by \( \xi(m) = sms^{-1}m^{-1} \) \( (m \in ^E H)\).

Equivalently: \( \text{Ad}(s)(m) = \xi(m)m. \)

\( \xi \) measures failure of \( s \) to commute with \(^E H\), or equivalently failure of \(^E H\) to be endoscopic.

Then \( \xi \) factors to \( \Gamma = ^E H/^\vee H\), values in \( Z(^\vee H) \).

Precisely: \( \xi \) is 1-cocycle of \( \Gamma \) with values in \( Z(^\vee H) \).
How do you generalize endoscopic transfer?

**Endoscopic transfer:** should correspond to map sheaves on $L^G$ params $\leadsto$ sheaves on $E^H$ params.

**Classical endoscopy:** $s$ acts by conjugation on $L^G$ params; fixed points are $E^H$ params.

Only $L^G$-params in image are $^\vee G$-conj to $E^H$-params.

**Generalized endoscopy:** $s$ still acts on $L^G$-params, but does not fix $E^H$ params: $\text{Ad}(s)(\phi_H(\gamma)) = \xi(\gamma)\phi_H(\gamma)$.

Try modify $\text{Ad}(s)$ by $\xi^{-1}$: $(s \ast_\xi \phi)(\gamma) = \xi^{-1}(\gamma)\text{Ad}(s)(\phi(\gamma))$.

But this is not an action except on $E^H$ params.

**Solution:** look only at params conjugate to $E^H$ params:

$^\vee G \times_{^\vee H} (E^H \text{ params}) \to (L^G \text{ params}), \quad (g, \phi'_H) \mapsto \text{Ad}(g)\phi'_H$.

$s$ acts on left space by $s \ast_\xi (g, \phi'_H) = \text{Ad}(g)(\xi^{-1}\phi'_H)$.

Fixed points of $\ast_\xi$ are $E^H$ params.
Suppose $G/k$ reductive and $P = MN$ parabolic over $k$. Put $X^*(M) = \text{ratl chars of } M$, a $\Gamma$-fixed sublattice of $X^*(G)$. 

$\implies \Gamma$-fixed sub $\subset X^*(\check{G}) \implies \Gamma$-fixed torus $\check{A} \subset \check{G}$. 

$\check{M} = \text{def } \check{G}^{\check{A}}$ is $\Gamma$-stable, dual to $M$: $L^M \simeq \check{M} \rtimes \Gamma$. 

Generic $s \in \check{A} \implies (L^G)^s = L^M \implies$ endoscopic group $M$. 

Endoscopic transfer (reps of $M$) $\implies$ (reps of $G$) is $\text{Ind}_{MN}^G$. 

Endoscopy is more powerful than parabolic induction. 

Allows $Z_{L^G}(\Gamma$-fixed element), not just $\Gamma$-fixed torus. 

But endoscopy also misses a lot of interesting subgroups. 

Rational Cartan subgroup of $G$ is almost never endoscopic.
Generalized endoscopic groups

Suppose $L$ any rational Levi subgroup of $G \twoheadrightarrow \Gamma$ action on root datum of $L$.

If $G$ simply connected, easy to find $^L L \subset ^L G$.

In general, get extended group $^E L \subset ^L G$.

$s \in Z(\check{V} L)$ generic $\rightsquigarrow (s, ^E L)$ gen endoscopic datum $\rightsquigarrow L$ generalized endoscopic for $G$.

Endoscopic transfer from general ratl Levi $L$ should be important generalization of parabolic induction.

Over $\mathbb{R}$, this is Zuckerman’s cohomological induction.

Over $p$-adic field, this is still a mystery.