Deforming subgroups

This is a sketch of a solution of the homework assigned for March 17, to deform a subgroup H to another H_0 of the same dimension, such that H_0 is normal in a parabolic subgroup P.

Write $I_H \subset k[G]$ for the ideal of the subgroup H. I'll write $\langle E\rangle$ for the ideal generated by a set E of functions. Arguing as in the construction of the quotient variety G/H (text, 5.5) we can find a finite-dimensional subspace $V \subset k[G]$ with the following properties.

1) The space V is stable under left translations λ and right translations ρ:

$$[\lambda(g)f](x) = f(g^{-1}x), \qquad [\rho(g)f](x) = f(xg).$$

2) The ideal I_H is generated by its intersection with V:

$$I_H(1) = I_H \cap V, \qquad I_H = \langle I_H(1)\rangle.$$

Here is the approximate idea. The ideal of the subgroup $gHg^{-1} = \mathrm{Ad}(g)H$ is

$$I_{gHg^{-1}} = \mathrm{Ad}(g)I_H = \langle \mathrm{Ad}(g)I_H(1)\rangle.$$

The action of $\mathrm{Ad}(g)$ on functions is $\lambda(g)\rho(g)$, so it preserves the finite-dimensional space V. We consider the action of Ad on the (projective) Grassmann variety $\mathrm{Gr}_d(V)$ of d-dimensional subspaces of V (with $d = \dim I_H(1)$). The ideals of the conjugates of H form a single G-orbit. The isotropy group for this action is clearly the normalizer of H:

$$\{gHg^{-1} \mid g \in G\} \simeq \mathrm{Ad}(G)I_H(1) \simeq G/N_G(H) \subset \mathrm{Gr}_d(V).$$

In the closure of this G-orbit there must be a *closed* G orbit $\mathrm{Ad}(g)W(1)$, with $W(1)$ a d-dimensional subspace of V. Since the Grassmannian is projective, this closed orbit is complete, so its isotropy group $P(1)$ is parabolic.

Approximately H_0 *should be the subgroup defined by the ideal* $\langle W(1)\rangle$ *generated by* $W(1)$.

The difficulty is that this ideal may be much smaller than you expect (so that the dimension of the corresponding variety is larger). Even though $W(1)$ is a limit of the corresponding part of the ideals for conjugates of H, it may not be true that the entire ideals (for conjugates of H) converge to $\langle W(1)\rangle$.

I don't know a really simple way to fix matters, but here is something that seems to work. We may (after enlarging V) assume that it generates $k[G]$. We may then filter $k[G]$ by defining

$$k[G]_m = \text{span of products of at most } m \text{ factors in } V,$$

so that

$$k[G]_0 = k \subset k[G]_1 = k + V \subset k[G]_2 \subset \cdots$$

This is an exhaustive increasing filtration which respects multiplication:

$$k[G]_m \cdot k[G]_n \subset k[G]_{m+n},$$

and the associated graded ring is a finitely generated commutative algebra over k (the quotient of a polynomial ring in $\dim V$ variables by a homogeneous ideal).

Now we can define

$$I_H(m) = I_H \cap k[G]_m, \qquad \dim I_H(m) = d_m.$$

Define a "partial flag variety"

$$X(m) = \{\text{chains of subspaces } W = \{W(1) \subset W(2) \subset \cdots \subset W(m) \subset k_m[G]\}\}$$

subject to the requirements

$$\dim W(j) = d_j, \qquad W(j) \subset k[G]_j.$$

This is a projective algebraic variety. The choice of V and the construction of the filtration makes $k[G]_m$ stable by ρ, λ, and Ad, so these actions apply to $X(m)$. Forgetting the largest subspace defines a proper morphism

$$\pi(m+1)\colon X(m+1) \to X(m).$$

Inside $X(m)$ is the G-orbit

$$Z(m) = \mathrm{Ad}(G)I_H.$$

Since by construction $I_H(1)$ generates I_H, it is very easy to check that all the isotropy groups

$$\{g \in G \mid \mathrm{Ad}(g)I_H(i) = I_H(i), 1 \leq i \leq m\}$$

are equal to $N_G(H)$; so

$$Z(m) \simeq G/N_G(H), \qquad m \geq 1.$$

Each closure $\overline{Z(m)}$ is $\mathrm{Ad}(G)$-stable and closed in the projective variety $X(m)$, and therefore complete. I am going to choose a compatible family of closed G orbits

$$O(m) = \mathrm{Ad}(G)W_0 \subset \overline{Z(m)}.$$

The notation is a little ambiguous. The flag W_0 in the preceding formula consists of m subspaces

$$W_0(i) \qquad (1 \leq i \leq m);$$

but the m does not appear in the notation. When I choose another flag W_0' in $\overline{Z(m+1)}$, its first m subspaces $W_0'(i)$ will be equal to $W_0(i)$. So calling the new flag W_0 is more or less harmless.

We have already seen that $\pi(m+1)$ maps $Z(m+1)$ isomorphically onto $Z(m)$; so

$$\pi(m+1)\colon \overline{Z(m+1)} \to \overline{Z(m)}$$

is a surjective proper map.

Begin by choosing a closed orbit

$$O(1) = \mathrm{Ad}(G)W_0(1) \subset \overline{Z(1)}.$$

Once the closed orbit

$$O(m) = \mathrm{Ad}(G)W_0 \subset \overline{Z(m)}$$

is chosen, its preimage

$$\pi(m+1)^{-1}(O(m)) \subset \overline{Z(m+1)}$$

is necessarily a complete subvariety, preserved by Ad (since $\pi(m+1)$ is proper and respects all the group actions). Consequently there is a closed orbit

$$O(m+1) = \mathrm{Ad}(G)W_0' \subset \pi(m+1)^{-1}(O(m)).$$

We may choose the orbit representative W_0' to project to W_0, which means exactly that the first m subspaces in the flag W_0' agree with those already chosen.

Because the orbits are closed, the isotropy groups

$$P(m) = \{g \in G \mid \mathrm{Ad}(g)W_0(i) = W_0(i)(1 \leq i \leq m)\}$$

are all parabolic.

We now consider the increasing family of ideals

$$I_0(m) = \langle W_0(m) \rangle \subset k[G].$$

Because $k[G]$ is Noetherian, this family is eventually constant:

$$I_0(m) = I_0(M) \qquad (m \geq M).$$

We call this limiting ideal I_0, and define

$$H_0 = \text{variety of } I_0.$$

Because I_0 was constructed as a limit of ideals, it is easy to check that

$$I_0 \cap k[G]_m = W_0(m) \qquad (1 \leq m < \infty),$$

and in particular that

$$\dim(I_0 \cap k[G]_m) = \dim(I_H \cap k[G]_m) \qquad (1 \leq m < \infty).$$

From this knowledge of Hilbert functions we conclude that H_0 has the same dimension as H.

We want to show that H_0 is a group. Recall the product morphism

$$\mu: G \times G \to G, \qquad \mu(x, y) = xy,$$

and the corresponding algebra homomorphism

$$\mu^*: k[G] \to k[G] \otimes k[G].$$

If A, B, and C are closed subsets of G, with ideals I_A, I_B, and I_C, then the ideal of $A \times B$ is $I_A \otimes k[G] + k[G] \otimes I_B$; so the condition $A \cdot B \subset C$ is equivalent to

$$\mu^*(I_C) \subset I_A \otimes k[G] + k[G] \otimes I_B).$$

In particular, the condition that H is closed under multiplication is

$$\mu^*(I_H) \subset I_H \otimes k[G] + k[G] \otimes I_H).$$

For every positive integer m, the subspace $I_H(m)$ is finite-dimensional; so there must be positive $N(m)$ so that

$$\mu^*(I_H(m)) \subset I_H(N) \otimes k[G]_N + k[G]_N \otimes I_H(N)).$$

Because this condition (concerning behavior of subspaces under the fixed linear map μ^* on fixed finite-dimensional spaces like $k[G]_m$) is satisfied for all the ideals (and corresponding flags) $I_{gHg^{-1}}$, it is satisfied by the limit ideal I_0 as well. It follows that H_0 is closed under multiplication.

Similar arguments show that H_0 is closed under inversion and contains the identity of G, so H_0 is a subgroup.

The ideal I_0 may not be radical, so the ideal of H_0 may be slightly larger than I_0; but at any rate it is clear that $N_G(H_0)$ contains the isotropy group $P(M)$ of the closed orbit $O(M)$. Since $P(M)$ is parabolic, the larger group $N_G(H_0)$ is parabolic as well.