Signatures of Hermitian forms and unitary representations

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VII Workshop Lie theory and its applications
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Outline

Introduction

Character formulas

Hermitian forms

Character formulas for invariant forms

Computing easy Hermitian KL polynomials

Unitarity algorithm

What is representation theory for?

Example. $\int_{-\pi}^{\pi} \sin^5(t)dt = ?$

Generalize: $f = f_{\text{even}} + f_{\text{odd}}, \quad \int_{-a}^{a} f_{\text{odd}}(t)dt = 0.$

Example. Evolution of initial temp distn of hot ring $T(0, \theta) = A + B\cos(m\theta)$?

Generalize: Fourier series expansion of initial temp. . .

Example. Suppose X is a compact (arithmetic) locally symmetric manifold of dimension 128; $H^{28}(X, \mathbb{Q}) = ?$.

Same as H^{28} for compact globally symmetric space.

Generalize: $X = \Gamma\backslash G/K$,

$$\begin{aligned} H^p(X, \mathbb{Q}) &= H^p_{\text{cont}}(G, L^2(\Gamma\backslash G)) \\ &= \sum_{\pi \text{ irr rep of } G} m_\pi(\Gamma) \cdot H^p\text{cont}(G, \pi). \end{aligned}$$

Here $m_\pi(\Gamma)$ = dim of automorphic forms of type π.

General principal: group G acts on vector space V; decompose V; study pieces separately.

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Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$). Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$. Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u$ = all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic today: **Step 3** for Lie group G.

Mackey theory (normal subgps) $\rightsquigarrow$ case G reductive.

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What's a unitary dual look like?

$G(\mathbb{R})$ = real points of complex connected reductive alg G

Problem: find $\widehat{G(\mathbb{R})}_u$ = irr unitary reps of $G(\mathbb{R})$.

Harish-Chandra: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}$ = "all" irr reps.

Unitary reps = "all" reps with pos def invt form.

Example: $G(\mathbb{R})$ compact $\Rightarrow \widehat{G(\mathbb{R})}_u = \widehat{G(\mathbb{R})}$ = discrete set.

Example: $G(\mathbb{R}) = \mathbb{R}$;

$$\widehat{G(\mathbb{R})} = \{\chi_z(t) = e^{zt} \quad (z \in \mathbb{C})\} \simeq \mathbb{C}$$

$$\widehat{G(\mathbb{R})}_u = \{\chi_{i\xi} \quad (\xi \in \mathbb{R})\} \simeq i\mathbb{R}$$

Suggests: $\widehat{G(\mathbb{R})}_u$ = real pts of cplx var $\widehat{G(\mathbb{R})}$. Almost...

$\widehat{G(\mathbb{R})}_h$ = reps with invt form: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}$.

Approximately (Knapp): $\widehat{G(\mathbb{R})}$ = cplx alg var, real pts $\widehat{G(\mathbb{R})}_h$; subset $\widehat{G(\mathbb{R})}_u$ cut out by real algebraic ineqs.

Today: conjecture making inequalities computable.

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Suggests: $\widehat{G(\mathbb{R})}_u =$ real pts of cplx var $\widehat{G(\mathbb{R})}$. Almost...

$\widehat{G(\mathbb{R})}_h =$ reps with invt form: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}$.

Approximately (Knapp): $\widehat{G(\mathbb{R})} =$ cplx alg var, real pts $\widehat{G(\mathbb{R})}_h$; subset $\widehat{G(\mathbb{R})}_u$ cut out by real algebraic ineqs.

Today: conjecture making inequalities computable.

What's a unitary dual look like?

$G(\mathbb{R})$ = real points of complex connected reductive alg G

Problem: find $\widehat{G(\mathbb{R})}_u =$ irr unitary reps of $G(\mathbb{R})$.

Harish-Chandra: $\widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})} =$ "all" irr reps.

Unitary reps $=$ "all" reps with pos def invt form.

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G cplx reductive alg $\supset G(\mathbb{R})$ real form $\supset K(\mathbb{R})$ max cpt.

Rep theory of $G(\mathbb{R})$ modeled on Verma modules...

$H \subset B \subset G$ maximal torus in Borel subgp,

$\mathfrak{h}^* \leftrightarrow$ highest weight reps

$V(\lambda)$ Verma of hwt $\lambda \in \mathfrak{h}^*$, $L(\lambda)$ irr quot

Put cplxification of $K(\mathbb{R}) = K \subset G$, reductive algebraic.

$(\mathfrak{g}, K)$-mod: cplx rep V of $\mathfrak{g}$, compatible alg rep of K.

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X parameter set for irr $(\mathfrak{g}, K)$-mods

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Character formulas

Can decompose Verma module into irreducibles

$$V(\lambda) = \sum_{\mu \leq \lambda} m_{\mu,\lambda} L(\mu) \qquad (m_{\mu,\lambda} \in \mathbb{N})$$

or write a formal character for an irreducible

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Defining Herm dual repn(s)

Suppose V is a $(\mathfrak{g}, K)$-module. Write π for repn map.

Recall Hermitian dual of V

$$V^h = \{\xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v)\}$$

(Also require ξ is K-finite.)

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators.

REQUIRES twist by conjugate linear automorphism of $\mathfrak{g}$.

Assume $\sigma\colon G \to G$ antiholom aut, $\sigma(K) = K$.

Define $(\mathfrak{g}, K)$-module $\pi^{h,\sigma}$ on V^h,

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$V = (\mathfrak{g}, K)$-module, σ antihol aut of G preserving K.

A σ-invt sesq form on V is sesq pairing $\langle , \rangle$ such that

$$\langle X \cdot v, w \rangle = \langle v, -\sigma(X) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, \sigma(k^{-1}) \cdot w \rangle$$

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Proposition

σ-invt sesq form on $V \leftrightarrow (\mathfrak{g}, K)$-map $T\colon V \to V^{h,\sigma}$,

$$\langle v, w \rangle_T = (Tv)(w).$$

Form is Hermitian iff $T^h = T$.
Assume V is irreducible.
$V \simeq V^{h,\sigma} \Leftrightarrow \exists$ invt sesq form $\Leftrightarrow \exists$ invt Herm form.
A σ-invt Herm form on V is unique up to real scalar.

$T \to T^h \rightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\sigma})$.

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$V = (\mathfrak{g}, K)$-module, σ antihol aut of G preserving K.

A σ-invt sesq form on V is sesq pairing $\langle , \rangle$ such that

$$\langle X \cdot v, w \rangle = \langle v, -\sigma(X) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, \sigma(k^{-1}) \cdot w \rangle$$

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Proposition

σ-invt sesq form on V $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-map $T\colon V \to V^{h,\sigma}$:

$$\langle v, w \rangle_T = (Tv)(w).$$

Form is Hermitian iff $T^h = T$.

Assume V is irreducible.

$V \simeq V^{h,\sigma} \Leftrightarrow \exists$ invt sesq form $\Leftrightarrow \exists$ invt Herm form

A σ-invt Herm form on V is unique up to real scalar.

$T \to T^h \leftrightsquigarrow$ real form of cplx line $\mathrm{Hom}_{\mathfrak{g},K}(V, V^{h,\sigma})$.

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$$I(x) = \sum_{y \leq x} m_{y,x} J(y) \qquad (m_{y,x} \in \mathbb{N})$$

for standard $(\mathfrak{g}, K)$-mod $I(x)$.

Want parallel formulas for σ-invt Hermitian forms.

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Form on irr $J(x) \xrightarrow{\text{deformation}}$ Jantzen filt $I_n(x)$ on std, nondeg forms $\langle , \rangle_n$ on I_n / I_{n+1}.

Details (proved by Beilinson-Bernstein):

$$I(x) = I_0 \supset I_1 \supset I_2 \supset \cdots, \qquad I_0/I_1 = J(x)$$

$$I_n/I_{n+1} \text{ completely reducible}$$

$$[J(y) : I_n/I_{n+1}] = \text{coeff of } q^{(\ell(x)-\ell(y)-n)/2} \text{ in KL poly } Q_{y,x}$$

Hence $\langle , \rangle_{I(x)} \stackrel{\text{def}}{=} \sum_n \langle , \rangle_n$, nondeg form on gr $I(x)$.

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Virtual Hermitian forms

$\mathbb{Z}$ = Groth group of vec spaces.

These are mults of irr reps in virtual reps.

$\mathbb{Z}[X]$ = Groth grp of finite length reps.

For invariant forms...

$\mathbb{W} = \mathbb{Z} \oplus \mathbb{Z}$ = Groth grp of fin diml forms.

Ring structure

$$(p, q)(p', q') = (pp' + qq', pq' + q'p).$$

Mult of irr-with-forms in virtual-with-forms is in $\mathbb{W}$:

$\mathbb{W}[X] \approx$ Groth grp of fin lgth reps with invt forms.

Two problems: invt form $\langle , \rangle_J$ may not exist for irr J; and $\langle , \rangle_J$ may not be preferable to $-\langle , \rangle_J$.

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Fix σ-invt Hermitian form $\langle,\rangle_{J(x)}$ on each irr admitting one; recall Jantzen form $\langle,\rangle_n$ on $I(x)_n/I(x)_{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I_n/I_{n-1},\langle,\rangle_n)=\sum_{y\le x} w_{y,x}(n)(J(y),\langle,\rangle_{J(y)}),$$

coeffs $w(n)=(p(n),q(n))\in\mathbb{W}$; summand means

$$p(n)(J(y),\langle,\rangle_{J(y)})\oplus q(n)(J(y),-\langle,\rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^{\sigma}_{y,x}=\sum_n w_{y,x}(n)q^{(l(x)-l(y)-n)/2}\in\mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q=1 \leftrightarrow$ form $\langle,\rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W}\to\mathbb{Z}\leftrightarrow$ KL poly $Q_{y,x}$.

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coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle,\rangle_{J(y)}) \oplus q(n)(J(y), -\langle,\rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q^\sigma_{y,x} = \sum_n w_{y,x}(n) q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle,\rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \to \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$.

Hermitian KL polynomials: multiplicities

Fix σ-invt Hermitian form $\langle , \rangle_{J(x)}$ on each irr admitting one; recall Jantzen form $\langle , \rangle_n$ on $I(x)_n/I(x)_{n+1}$.

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Proposition

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Deforming to $\nu = 0$

Have computable conjectural formula (omitting ℓ_o)

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Polys $P_{x,y}$ are KL polys, computed by `atlas` software.

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$$(J(x), \langle , \rangle^c_{J(x)}) = \sum_{y \le x} (-1)^{l(x)-l(y)} P_{x,y}(s)(\text{gr } I(y), \langle , \rangle^c_{I(y)})$$

for σ^c-invt forms in terms of forms on stds, same inf char.

Polys $P_{x,y}$ are KL polys, computed by `atlas` software.

Std rep $I = I(\nu)$ deps on cont param ν. Put $I(t) = I(t\nu),\ t \ge 0$.

If std rep $I = I(\nu)$ has σ-invt form so does $I(t)$ $(t \ge 0)$.

(signature for $I(t)$) $=$ (signature on $I(t+\epsilon)$), $\epsilon \ge 0$ suff small.

Sig on $I(t)$ differs from $I(t-\epsilon)$ on odd levels of Jantzen filt:

$$\langle , \rangle_{\text{gr } I(t-\epsilon)} = \langle , \rangle_{\text{gr } I(t)} + (s-1) \sum_m \langle , \rangle_{I(t)_{2m+1}/I(t)_{2m+2}}.$$

Each summand after first on right is known comb of stds, all with cont param strictly smaller than $t\nu$. ITERATE...

$$\langle , \rangle^c_J = \sum_{I'(0) \text{ std at } \nu' = 0} v_{J,I'} \langle , \rangle^c_{I'(0)} \qquad (v_{J,I'} \in \mathbb{W}).$$

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From σ_c to σ_0

Cplx conjs σ_c (compact form) and σ_0 (our real form) differ by Cartan involution θ: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(\mathfrak{g}, K)$-mod $J \rightsquigarrow J^\theta$ (same space, rep twisted by θ).

Proposition

J admits σ_0-invt Herm form if and only if $J^\theta \simeq J$. If $T_0 \colon J \xrightarrow{\sim} J^\theta$, and $T_0^2 = \mathrm{Id}$, then

$$\langle v, w \rangle_J^0 = \langle v, T_0 w \rangle_J^c.$$

$T \colon J \xrightarrow{\sim} J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \rightsquigarrow \sigma$-invt Herm form.

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General case

Fix “distinguished involution” δ_0 of G inner to θ

Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.

Can arrange $\theta = \mathrm{Ad}(\tau\delta_0)$, some $\tau \in K$.

Define $K^\Gamma = \mathrm{Cent}_{G^\Gamma}(\tau\delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ $(\mathfrak{g}, K)$-mods V with $D_0 \colon V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \mathrm{Id}$.

Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

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Beilinson-Bernstein localization: $(\mathfrak{g}, K^\Gamma)$-mods $\leftrightsquigarrow$ action of δ_0 on K-eqvt perverse sheaves on G/B.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate σ_c-invt forms to σ_0 invt forms

$$\langle v, w\rangle_V^0 \overset{\text{def}}{=} \langle v, z^{-1/2}\tau\delta_0 \cdot w\rangle_V^c$$

on $(\mathfrak{g}, K^\Gamma)$-mods as in equal rank case.

Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group $G(\mathbb{R})$
- general representation π

and ask whether π is unitary.

Program would say either

- π has no invariant Hermitian form, or
- π has invt Herm form, indef on reps μ_1, μ_2 of K, or
- π is unitary.

Answers to finitely many such questions $\rightsquigarrow$ complete description of unitary dual of $G(\mathbb{R})$.

This would be a good thing.

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Easy Herm KL polys

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