Contragredient representations and characterizing the local Langlands correspondence

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1 Introduction

It is surprising that the following question has not been addressed in the literature: what is the contragredient in terms of Langlands parameters?

Thus suppose G is a connected, reductive algebraic group defined over a local field F, and $G(F)$ is its F-points. According to the local Langlands conjecture, associated to an admissible homomorphism ϕ from the Weil-Deligne group of F into the L-group of $G(F)$ is an L-packet $\Pi(\phi)$, a finite set of (equivalence classes of) irreducible admissible representations of $G(F)$. (If F is archimedean, "equivalence" means "infinitesimal equivalence." If F is non-archimedean, it means "equivalence of smooth vectors.") Conjecturally these L-packets partition the admissible dual.

So suppose π is an irreducible admissible representation, and $\pi \in \Pi(\phi)$. Let π^* be the contragredient, or dual, of π (see (7.1)). The question is: what is the homomorphism ϕ^* such that $\pi^* \in \Pi(\phi^*)$? We also consider the related question of describing the *Hermitian dual* in terms of Langlands parameters. Both of the questions come down to a characterization of the

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local Langlands correspondence. For $\mathbb{R}$ this is the content of Sections 4 and 6, especially Definitions 6.6 and 6.15.

Let ${}^\vee G$ be the complex dual group of G. A Chevalley involution C of ${}^\vee G$ satisfies $C(h) = h^{-1}$, for all h in some Cartan subgroup of ${}^\vee G$. The L-group ${}^L G$ of $G(F)$ is a certain semidirect product ${}^\vee G \rtimes \Gamma$ where Γ is the absolute Galois group of F (or other related groups). We can choose C so that it extends to an involution of ${}^L G$, acting trivially on Γ. We refer to this as the Chevalley involution of ${}^L G$. See Section 2.

We believe the contragredient should correspond to composition with the Chevalley involution of ${}^L G$. To avoid two levels of conjecture, we formulate this as follows.

Conjecture 1.1 *Assume the local Langlands conjecture is known for both π and π^*. Let C be the Chevalley involution of ${}^L G$. Then*

$$\pi \in \Pi(\phi) \Leftrightarrow \pi^* \in \Pi(C \circ \phi).$$

Even the following weaker result is not known:

Conjecture 1.2 *If Π is an L-packet, then so is $\Pi^* = \{\pi^* \mid \pi \in \Pi\}$.*

The local Langlands conjecture is only known, for fixed $G(F)$ and all π, in a limited number cases, notably $GL(n, F)$ over any local field [11], [12], and for any G if $F = \mathbb{R}$ or $\mathbb{C}$. See Langlands's original paper [15], which is summarized in Borel's article [7]. On the other hand it is known for a restricted class of representations for more groups, for example unramified principal series representations of a split p-adic group [7, 10.4].

For discrete series L-packets of real groups, Conjecture 1.1 follows from [18, Lemma 7.4.1], and it is known for some representations of some quasisplit p-adic groups by [13]. It would be reasonable to impose Conjecture 1.1 as a condition on the local Langlands correspondence in cases where it is not known.

We concentrate on the Archimedean case. Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. The contragredient in this case can be realized either via the Chevalley automorphism of ${}^L G$, or via a similar automorphism of $W_{\mathbb{R}}$. Let $\mathbb{C}^*$ be the multiplicative group of nonzero complex numbers. By analogy with the

Chevalley involution C of G, there is a unique $\mathbb{C}^*$-conjugacy class of automorphisms $C_{W_\mathbb{R}}$ of $W_\mathbb{R}$ satisfying $C_{W_\mathbb{R}}(z) = z^{-1}$ for all $z \in \mathbb{C}^*$. See Section 2.

Theorem 1.3 *Let $G(\mathbb{R})$ be the real points of a connected reductive algebraic group defined over $\mathbb{R}$, with L-group ${}^LG$. Suppose $\phi\colon W_\mathbb{R} \to {}^LG$ is an admissible homomorphism, with associated L-packet $\Pi(\phi)$. Then*

$$\Pi(\phi)^* = \Pi(C \circ \phi) = \Pi(\phi \circ C_{W_\mathbb{R}}).$$

In particular $\Pi(\phi)^*$ is an L-packet.

Here is a sketch of the proof.

It is easy to prove in the case of tori. See Section 3.

It is well known that an L-packet Π of (relative) discrete series representations is determined by an infinitesimal and a central character. In fact something stronger is true. Let G_{rad} be the radical of G, i.e., the maximal central torus. Then Π is determined by an infinitesimal character and a character of $G_{\text{rad}}(\mathbb{R})$, which we refer to as a radical character.

In general it is easy to read off the infinitesimal and radical characters of $\Pi(\phi)$, see [7] and Section 6.1. In particular for a relative discrete series parameter Theorem 1.3 reduces to a claim about how C affects the infinitesimal and radical characters. For the radical character this reduces to the case of tori, and Theorem 1.3 follows in this case.

This is the heart of the matter, and the general case follows easily by parabolic induction. In other words, the proof relies on the fact that the parameterization is uniquely characterized by:

1. Infinitesimal character,

2. Radical character,

3. Compatibility with parabolic induction.

In a sense this is the main result of the paper: a self-contained description of the local Langlands classification, and its characterization by (1-3). See Sections 4 and 6. Use of the Tits group (see Section 5) simplifies some technical arguments.

Now consider $GL(n, F)$ for F a local field of characteristic 0. Since $GL(n, F)$ is split, an admissible homomorphism into the L-group is the same thing as a homomorphism into the dual group $GL(n,\mathbb{C})$ (in which the Weil group acts semisimply); that is, ϕ may be identified with an n-dimensional complex representation of the Weil-Deligne group W'_F. In this case L-packets are singletons, so write $\pi(\phi)$ for the representation attached to ϕ.

For the Chevalley involution take $C(g) = {}^tg^{-1}$. Composing any finite-dimensional representation into $GL(n)$ with the inverse transpose gives the contragredient representation; so $C\circ\phi$ is equivalent to the contragredient ϕ^* of ϕ. Over $\mathbb{R}$ Theorem 1.3 says that the Langlands correspondence commutes with the contragredient:

$$\pi(\phi^*) \simeq \pi(\phi)^*. \tag{1.4}$$

This is also true over a p-adic field [11], [12], in which case it is closely related to the functional equations for L and ε factors.

We now consider a variant of (1.4) in the real case. Suppose π is an irreducible representation of $GL(n,\mathbb{R})$. Its Hermitian dual π^h is the unique irreducible representation, admitting a nonzero invariant sesquilinear pairing with π. The representation π admits a nonzero invariant Hermitian form if and only if $\pi \simeq \pi^h$. In this case we say π is Hermitian. See Section 8.

The Hermitian dual arises naturally in the study of unitary representations: the unitary dual is the subset of the fixed points of the involution $\pi \mapsto \pi^h$, consisting of those π for which the invariant form is definite. So it is natural to ask what the Hermitian dual is on the level of Langlands parameters.

There is a natural notion of Hermitian dual of a finite-dimensional representation ϕ of any group: $\phi^h = {}^t\overline{\phi}^{-1}$, and ϕ preserves a nondegenerate Hermitian form if and only if $\phi \simeq \phi^h$.

The local Langlands correspondence for $GL(n,\mathbb{R})$ commutes with the Hermitian dual operation:

Theorem 1.5 *Suppose ϕ is an n-dimensional semisimple representation of $W_{\mathbb{R}}$. Then:*

1. $\pi(\phi^h) = \pi(\phi)^h$,
2. *ϕ is Hermitian if and only if $\pi(\phi)$ is Hermitian,*
3. *ϕ is unitary if and only if $\pi(\phi)$ is tempered.*

See Section 8.

Return now to the setting of general real groups. The space $\mathcal{X}_0$ of conjugacy classes of L-homomorphisms parametrizes L-packets of representations. By introducing some extra data we obtain a space $\mathcal{X}$ which parametrizes irreducible representations [3]. Roughly speaking $\mathcal{X}$ is the set of conjugacy classes of pairs (ϕ, χ) where $\phi \in \mathcal{X}_0$ and χ is a character of the component group of $\mathrm{Cent}_{^\vee G}(\phi(W_\mathbb{R}))$. It is natural to ask for the involution of $\mathcal{X}$ induced by the contragredient.

On the other hand, it is possible to formulate and prove an analogue of Theorem 1.5 for general real groups, in terms of an antiholomorphic involution of ${}^LG$. Also, the analogue of Theorem 1.5 holds in the p-adic case. All of these topics require more machinery. In an effort to keep the presentation as elementary as possible we defer them to later papers.

This paper is a complement to [2], which considers the action of the Chevalley involution of G, rather than ${}^LG$. See Remark 7.6.

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2 The Chevalley Involution

We discuss the Chevalley involution. This is well known, although there isn't a good reference for all of the details we need (Chevalley cites the existence of this automorphism without proof in [9, page 23]). For the convenience of the reader we give complete details. We also discuss a similar involution of $W_\mathbb{R}$.

Throughout this paper G is a connected, reductive algebraic group. We may identify it with its complex points, and write $G(\mathbb{C})$ on occasion to emphasize this point of view. For $x \in G$ write $\mathrm{int}(x)$ for the inner automorphism $\mathrm{int}(x)(g) = xgx^{-1}$.

Proposition 2.1 *Fix a Cartan subgroup H of G. There is an automorphism C of G satisfying $C(h) = h^{-1}$ for all $h \in H$. For any such automorphism $C^2 = 1$, and, for every semisimple element $g \in G$, $C(g)$ is conjugate to g^{-1}.*

Suppose C_1 and C_2 are two such automorphisms defined with respect to Cartan subgroups H_1 and H_2. Then C_1 and C_2 are conjugate by an inner automorphism of G.

The proof uses *based root data* and *pinnings*. For background see [19]. Fix a Borel subgroup B of G, and a Cartan subgroup $H \subset B$. Let $X^*(H), X_*(H)$ be the character and co-character lattices of H, respectively. Let $\Pi, {}^\vee\Pi$ be the sets of simple roots, respectively simple co-roots, defined by B.

The *based root datum defined by* (B, H) is $(X^*(H), \Pi, X_*(H), {}^\vee\Pi)$. There is a natural notion of isomorphism of based root data. A *pinning* is a set $\mathcal{P} = (B, H, \{X_\alpha | \alpha \in \Pi\})$ where, for each $\alpha \in \Pi$, $X_\alpha \neq 0$ is contained in the α-root space $\mathfrak{g}_\alpha$ of $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathrm{Aut}(G)$ be the group of algebraic (equivalently, holomorphic) automorphisms of G, $\mathrm{Int}(G) \subset \mathrm{Aut}(G)$ the inner automorphisms, and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$. Let $\mathrm{Aut}(\mathcal{P})$ be the subgroup of $\mathrm{Aut}(G)$ preserving $\mathcal{P}$. We refer to the elements of $\mathrm{Aut}(\mathcal{P})$ as *$\mathcal{P}$-distinguished automorphisms.*

Theorem 2.2 ([19, Theorem 9.6.2]) *Suppose G, G' are connected, reductive complex groups. Fix pinnings $\mathcal{P} = (B, H, \{X_\alpha\})$ and $\mathcal{P}' = (B', H', \{X'_\alpha\})$. Let D_b, D'_b be the based root data defined by (B, H) and (B', H').*

Suppose $\phi\colon D_b \to D'_b$ is an isomorphism of based root data. Then there is a unique isomorphism $\psi\colon G \to G'$ taking $\mathcal{P}$ to $\mathcal{P}'$ and inducing ϕ on the root data.

The only inner automorphism in $\mathrm{Aut}(\mathcal{P})$ is the identity, and there are isomorphisms

$$\mathrm{Out}(G) \simeq \mathrm{Aut}(D_b) \simeq \mathrm{Aut}(\mathcal{P}) \subset \mathrm{Aut}(G). \tag{2.3}$$

The following consequence of the theorem is quite useful.

Lemma 2.4 *Suppose $\tau \in \mathrm{Aut}(G)$ restricts trivially to a Cartan subgroup H. Then $\tau = \mathrm{int}(h)$ for some $h \in H$.*

Proof. Fix a pinning $(B, H, \{X_\alpha\})$. Then $d\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$ for all α. Therefore we can choose $h \in H$ so that $d\tau(X_\alpha) = \mathrm{Ad}(h)(X_\alpha)$ for all $\alpha \in \Pi$. Then $\tau \circ \mathrm{int}(h^{-1})$ acts trivially on D_b and $\mathcal{P}$. By the theorem $\tau = \mathrm{int}(h)$. □

Proof of the Proposition. Choose a Borel subgroup B containing H and let $D_b = (X^*(H), \Pi, X_*(H), {}^\vee\Pi)$ be the based root datum defined by (B, H). Let B^{op} be the opposite Borel, with corresponding root datum $D_b^{op} = (X^*(H), -\Pi, X_*(H), -{}^\vee\Pi)$.

Choose a pinning $\mathcal{P} = (B, H, \{X_\alpha\})$. Let $\mathcal{P}^{op} = (H, B^{op}, \{X_{-\alpha} | \alpha \in \Pi\})$ where, for each simple root $\alpha \in \Pi$, the root vector $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ is determined by the requirement $\alpha([X_\alpha, X_{-\alpha}]) = 2$.

Let $\phi : D_b \to D_b'$ be the isomorphism of based root data given by -1 on $X^*(H)$. By Theorem 2.2 there is an automorphism $C_{\mathcal{P}}$ of G taking $\mathcal{P}$ to $\mathcal{P}^{op}$ and inducing ϕ. In particular $C_{\mathcal{P}}(h) = h^{-1}$ for $h \in H$. This implies $C_{\mathcal{P}}(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, and since $C_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P}^{op}$ we have $C_{\mathcal{P}}(X_\alpha) = X_{-\alpha}$. Since $C_{\mathcal{P}}^2$ is an automorphism of G taking $\mathcal{P}$ to itself and inducing the trivial automorphism of D_b, the theorem implies that xs$C_{\mathcal{P}}^2 = 1$.

If $g \in G$ is any semisimple element, choose x so that $xgx^{-1} \in H$. Then $C(g) = (C(x^{-1})x)g^{-1}(C(x^{-1})x)^{-1}$.

Suppose $C_1(h) = h^{-1}$ for all $h \in H$. Then $C_1 \circ C_{\mathcal{P}}$ acts trivially on H, so by the lemma $C_1 = \mathrm{int}(h_1) \circ C_{\mathcal{P}}$ for some $h_1 \in H$, which implies $C_1^2 = 1$.

For the final assertion choose $g \in G$ so that $gH_1g^{-1} = H_2$. Then $\mathrm{int}(g) \circ C_1 \circ \mathrm{int}(g^{-1})$ acts by inversion on H_2. By the lemma $\mathrm{int}(g) \circ C_1 \circ \mathrm{int}(g^{-1}) = \mathrm{int}(h_2) \circ C_2$ for some $h_2 \in H_2$. Choose $t \in H_2$ so that $t^2 = h_2$. Then $\mathrm{int}(t^{-1}g) \circ C_1 \circ \mathrm{int}(t^{-1}g)^{-1} = C_2$. □

An involution satisfying the condition of the proposition is known as a *Chevalley involution*. For $\mathcal{P}$ a pinning we refer to the involution $C_{\mathcal{P}}$ of the proof as the *Chevalley involution defined by* $\mathcal{P}$. The proof shows that every Chevalley involution is equal to $C_{\mathcal{P}}$ for some $\mathcal{P}$, and all Chevalley involutions are conjugate. We will abuse terminology slightly and refer to *the* Chevalley involution.

Remark 2.5

1. The Chevalley involution satisfies: $C(g)$ is conjugate to g^{-1} for *all* $g \in G$ [16, Proposition 2.6]. (Lusztig proves the corresponding statement over algebraically closed fields of arbitrary characteristic.)

2. If $G = GL(n)$, $C(g) = {}^tg^{-1}$ is a Chevalley involution. The group of fixed points is $G^C = O(n,\mathbb{C})$, the complexified maximal compact subgroup of $GL(n,\mathbb{R})$. In other words, C is the Cartan involution for $GL(n,\mathbb{R})$. In general the Chevalley involution is the Cartan involution of the split real form of G.

3. Suppose C' is any automorphism such that

$$C'(g) \text{ is } G\text{-conjugate to } g^{-1} \text{ for all semisimple } g \text{ in } G. \qquad (*)$$

   It is not hard to see, using Lemma 2.4, that $C' = \mathrm{int}(x) \circ C$ for some $x \in G$ and some Chevalley involution C. This means that the requirement

(∗) characterizes (via the canonical map $\mathrm{Aut}(G) \to \mathrm{Out}(G)$) the class of the Chevalley involutions in $\mathrm{Out}(G)$.

4. The Chevalley involution is inner if and only if G is semisimple and -1 is in the Weyl group, in which case $C = \mathrm{int}(g_0)$ where $g_0 \in \mathrm{Norm}_G(H)$ represents -1. The proposition implies g_0^2 is central, and independent of all choices. See Lemma 5.4.

Lemma 2.6 *Fix a pinning $\mathcal{P}$. Then $C_{\mathcal{P}}$ commutes with every $\mathcal{P}$-distinguished automorphism.*

This is immediate from the uniqueness statement in Theorem 2.2.

Here is a similar involution of $W_{\mathbb{R}}$. Recall $W_{\mathbb{R}} = \langle \mathbb{C}^*, j\rangle$ with relations $jzj^{-1} = \overline{z}$ and $j^2 = -1$.

Lemma 2.7 *There is an involution $C_{W_{\mathbb{R}}}$ of $W_{\mathbb{R}}$ such that $C_{W_{\mathbb{R}}}(z) = z^{-1}$ for all $z \in \mathbb{C}^*$. Any two such automorphisms are conjugate by* $\mathrm{int}(z)$ *for some $z \in \mathbb{C}^*$.*

Proof. This is elementary. For $z_0 \in \mathbb{C}^*$ define $C_{W,z_0}(z) = z^{-1}$ $(z \in \mathbb{C}^*)$ and $C_{W,z_0}(j) = z_0 j$. From the relations this extends to an automorphism of $W_{\mathbb{R}}$ if and only if $z_0\overline{z}_0 = 1$. Thus $C_{W,1}$ is an automorphism, and $C_{W,z_0} = \mathrm{int}(u) \circ C_{W,1} \circ \mathrm{int}(u^{-1})$, provided $(u/\overline{u})^2 = z_0$. □

3 Tori

Let H be a complex torus, and fix an element $\gamma \in \frac{1}{2}X^*(H)$. Let

(3.1a) $$H_\gamma = \{(h, z) \in H \times \mathbb{C}^* \mid [2\gamma](h) = z^2\}.$$

This is a two-fold cover of H via the map $(h, z) \to h$; write ζ for the nontrivial element in the kernel of this map. We call this the γ-*cover* of H. A character of H_γ is called *genuine* if it takes the value -1 on ζ. Note that

(3.1b) $$\gamma \colon H_\gamma \to \mathbb{C}^*, \quad (h, z) \mapsto z$$

is a genuine character of H_γ, and is a canonical square root of the algebraic character 2γ of H. The genuine algebraic characters of H_γ may be identified with symbols $\gamma + X^*(H)$:

(3.1c) $$\gamma + \kappa_0 \colon H_\gamma \to \mathbb{C}^*, \quad (h, z) \mapsto \kappa_0(h)z \qquad (\kappa_0 \in X^*(H)).$$

Now assume H is defined over $\mathbb{R}$, with Cartan involution θ. The *γ-cover of $H(\mathbb{R})$* is defined to be the inverse image of $H(\mathbb{R})$ in H_γ.

Lemma 3.2 ([6, Proposition 5.8]) *Given $\gamma \in \frac{1}{2}X^*(H)$, the genuine characters of $H(\mathbb{R})_\gamma$ are canonically parametrized by the set of pairs (λ, κ) with*

$$\lambda \in \mathrm{Hom}(\mathfrak{h}, \mathbb{C}), \qquad \kappa \in \gamma + X^*(H)/(1-\theta)X^*(H),$$

subject to the requirement $(1+\theta)\lambda = (1+\theta)\kappa$.

Write $\chi(\lambda, \kappa)$ for the character defined by (λ, κ). This character has differential λ, and its restriction to the maximal compact subgroup is the restriction of the character κ of H_γ. A little more precisely, fix $\kappa_0 \in X^*(H)$ so that $\gamma + \kappa_0$ is a representative of the κ. Then the restriction of $\chi(\lambda, \kappa)$ to the maximal compact subgroup of $H(\mathbb{R})_\gamma$ is the restriction of the algebraic character $\gamma + \kappa_0$ defined in (3.1).

Let ${}^\vee H$ be the dual torus. This satisfies: $X^*({}^\vee H) = X_*(H)$, $X_*({}^\vee H) = X^*(H)$. If H is defined over $\mathbb{R}$, with Cartan involution θ, then θ may be viewed as an involution of $X_*(H)$; its adjoint θ^t is an involution of $X^*(H) = X_*({}^\vee H)$. Let ${}^\vee\theta$ be the automorphism of ${}^\vee H$ induced by $-\theta^t$.

The *L-group of H* is defined as ${}^L H = \langle {}^\vee H, {}^\vee\delta\rangle$ where ${}^\vee\delta^2 = 1$ and ${}^\vee\delta$ acts on ${}^\vee H$ by ${}^\vee\theta$. Part of the data is the distinguished element ${}^\vee\delta$ (more precisely its conjugacy class).

More generally an *E-group for H* is a group ${}^E H = \langle {}^\vee H, {}^\vee\delta\rangle$, where ${}^\vee\delta$ acts on ${}^\vee H$ by ${}^\vee\theta$, and ${}^\vee\delta^2 \in {}^\vee H^{{}^\vee\theta}$. Such a group is determined up to isomorphism by the image of ${}^\vee\delta^2$ in ${}^\vee H^{{}^\vee\theta}/\{h{}^\vee\theta(h) \mid h \in {}^\vee H\}$. Again the data includes the ${}^\vee H$ conjugacy class of ${}^\vee\delta$. See [6, Definition 5.9].

A homomorphism $\phi\colon W_\mathbb{R} \to {}^E H$ is said to be *admissible* if it is continuous and $\phi(j) \in {}^L H \backslash {}^\vee H$. Conjugacy classes of admissible homomorphisms parametrize genuine representations of $H(\mathbb{R})_\gamma$.

Lemma 3.3 ([6, Theorem 5.11]) *In the setting of Lemma 3.2, suppose $(1-\theta)\gamma \in X^*(H)$. View γ as an element of $\frac{1}{2}X_*({}^\vee H)$. Let ${}^E H = \langle {}^\vee H, {}^\vee\delta\rangle$ where ${}^\vee\delta$ acts on ${}^\vee H$ by ${}^\vee\theta$, and ${}^\vee\delta^2 = \exp(2\pi i\gamma) \in {}^\vee H^{{}^\vee\theta}$.*

There is a canonical bijection between the irreducible genuine characters of $H(\mathbb{R})_\gamma$ and ${}^\vee H$-conjugacy classes of admissible homomorphisms $\phi\colon W_\mathbb{R} \to {}^E H$.

If ${}^E H$ is the L-group of H then (by restricting to $H(\mathbb{R}) \subset H(\mathbb{R})_\gamma$) we can replace genuine characters of $H(\mathbb{R})_\gamma$ with characters of $H(\mathbb{R})$.

Sketch of proof. An admissible homomorphism ϕ may be written in the form

$$\begin{aligned} \phi(z) &= z^\lambda \overline{z}^{\,{}^\vee\theta(\lambda)} \\ \phi(j) &= \exp(2\pi i\mu)^\vee\delta \end{aligned} \tag{3.4}$$

for some $\lambda, \mu \in {}^\vee\mathfrak{h}$. Then $\phi(j)^2 = \exp(2\pi i(\mu + {}^\vee\theta\mu) + \gamma)$ and $\phi(-1) = \exp(\pi i(\lambda - {}^\vee\theta\lambda))$, so $\phi(j^2) = \phi(j)^2$ if and only if

$$\kappa := \frac{1}{2}(1 - {}^\vee\theta)\lambda - (1 + {}^\vee\theta)\mu \in \gamma + X_*({}^\vee H) = \gamma + X^*(H). \tag{3.5}$$

In this case $(1 + \theta)\lambda = (1 + \theta)\kappa$; take ϕ to $\chi(\lambda, \kappa)$. □

Write $\chi(\phi)$ for the genuine character of $H(\mathbb{R})_\gamma$ associated to ϕ.

The Chevalley involution C of ${}^\vee H$ (i.e., inversion) extends to an involution of ${}^E H = \langle {}^\vee H, {}^\vee\delta \rangle$, fixing ${}^\vee\delta$ (this uses the fact that $\exp(2\pi i(2\gamma)) = 1$).

Here is the main result in the case of (covers of) tori.

Lemma 3.6 *Suppose $\phi\colon W_\mathbb{R} \to {}^E H$ is an admissible homomorphism, with corresponding genuine character $\chi(\phi)$ of $H(\mathbb{R})_\gamma$. Then*

$$\chi(C \circ \phi) = \chi(\phi)^* \tag{3.7}$$

Proof. Suppose ϕ is given by (3.4), so $\chi(\phi) = \chi(\lambda, \kappa)$ with κ as in (3.5). Then

$$\begin{aligned} (C \circ \phi)(z) &= z^{-\lambda} \overline{z}^{\,-{}^\vee\theta(\lambda)} \\ (C \circ \phi)(j) &= \exp(-2\pi i\mu)^\vee\delta \end{aligned} \tag{3.8}$$

By (3.5) $\chi(C \circ \phi) = \chi(-\lambda, -\kappa) = \chi(\lambda, \kappa)^*$. □

4 L-packets without L-groups

Suppose G is defined over $\mathbb{R}$, with real points $G(\mathbb{R})$. This means that $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ where σ is an antiholomorphic involution. (*Antiholomorphic* means that if f is a locally defined holomorphic function on $G(\mathbb{C})$, then $g \mapsto \overline{f(\sigma(g))}$ is also holomorphic.) Fix a *Cartan involution* θ of G corresponding to $G(\mathbb{R})$, and let $K = G^\theta$. This means that θ and σ commute, and $K(\mathbb{R}) = K \cap G(\mathbb{R}) = K^\sigma = G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$, with complexification K. We have the following picture, where each arrow represents taking the fixed points with respect to the given involution.

$$\begin{array}{ccccc} & & G & & \\ & \overset{\theta}{\swarrow} & & \overset{\sigma}{\searrow} & \\ G^\theta = K & & & & G(\mathbb{R}) = G^\sigma \\ & \underset{\sigma}{\searrow} & & \underset{\theta}{\swarrow} & \\ & & K(\mathbb{R}) & & \end{array}$$

We say that θ *corresponds to* σ, and vice-versa.

We work entirely in the algebraic setting. We consider $(\mathfrak{g}, K)$-modules, and write (π, V) for a $(\mathfrak{g}, K)$-module with underlying complex vector space V. The set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules is a disjoint union of L-packets. In this section we describe L-packets in terms of data for G itself. For the relation with L-parameters see Section 6.

Suppose H is a θ-stable Cartan subgroup of G. After conjugating by K we may assume it is defined over $\mathbb{R}$, which we always do without further comment.

The *imaginary roots* Δ_i, i.e., those fixed by θ, form a root system. Let ρ_i be one-half the sum of a set Δ_i^+ of positive imaginary roots. The two-fold cover H_{ρ_i} of H is defined as in Section 3. It is convenient to eliminate the dependence on Δ_i^+: define $\widetilde{H}$ to be the inverse limit of $\{H_{\rho_i}\}$ over all choices of Δ_i^+. The inverse image of $H(\mathbb{R})$ in H_{ρ_i} is denoted $H(\mathbb{R})_{\rho_i}$, and take the inverse limit to define $\widetilde{H(\mathbb{R})}$. (The existence and uniqueness of the maps defining the inverse limit follows from standard facts about systems of positive roots.)

Definition 4.1 *An* L-datum *is a pair* (H, χ) *where* H *is a* θ*-stable Cartan subgroup of* G*,* χ *is a genuine character of* $\widetilde{H(\mathbb{R})}$*, and* $\langle d\chi, {}^\vee\alpha \rangle \neq 0$ *for all imaginary roots. (The Cartan subgroup is included in this notation for convenience, but it is implicit in the character* χ*; so we may speak of* χ *as an L-datum.*

Associated to each L-datum is an L-packet. We start by defining relative discrete series L-packets.

We say $H(\mathbb{R})$ is *relatively compact* if $H(\mathbb{R}) \cap G_d$ is compact, where G_d is the derived group of G. (It is equivalent to require that all of the roots of H in G be imaginary.) Then $G(\mathbb{R})$ has relative discrete series representations if and only if it has a relatively compact Cartan subgroup.

Suppose $H(\mathbb{R})$ is relatively compact. Choose a set of positive roots Δ^+ and define the *Weyl denominator*

$$D(\Delta^+, h) = e^{\rho}(h) \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha}(h)) \quad (h \in H(\mathbb{R})_\rho). \tag{4.2}$$

This is a genuine function, i.e., it satisfies $D(\Delta^+, \zeta h) = -D(\Delta^+, h)$, and we view it as a function on $\widetilde{H(\mathbb{R})}$.

Let $q = \frac{1}{2}\dim(G_d/K\cap G_d)$. Let $W(K, H) = \mathrm{Norm}_K(H)/H\cap K$; this is isomorphic to the real Weyl group $W(G(\mathbb{R}), H(\mathbb{R})) = \mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$.

Definition 4.3 *Suppose (H, χ) is an L-datum with $H(\mathbb{R})$ relatively compact. Let $\pi = \pi(\chi)$ be the unique (up to equivalence) nonzero relative discrete series representation whose character restricted to the regular elements of $H(\mathbb{R})$ is*

$$\Theta_\pi(h) = (-1)^q D(\Delta^+, \widetilde{h})^{-1} \sum_{w\in W(K,H)} \mathrm{sgn}(w)(w\chi)(\widetilde{h}). \tag{4.4}$$

Here $\widetilde{h} \in \widetilde{H(\mathbb{R})}$ is any inverse image of h, and Δ^+ makes $d\chi$ dominant. Every relative discrete series representation is obtained this way, and $\pi(\chi) \simeq \pi(\chi')$ if and only if χ and χ' are $W(K, H)$-conjugate.

The L-packet of (H, χ) is

$$\Pi_G(\chi) = \{\pi(w\chi) \mid w \in W(G, H)/W(K, H)\}. \tag{4.5}$$

It is a basic result of Harish-Chandra that $\pi(\chi)$ exists and is characterized (among relative discrete series representations) by this character formula. This version of the character formula is a slight variant of the usual one, because of the use of $\widetilde{H(\mathbb{R})}$. See [6] or [1].

By (4.4) the representations in $\Pi_G(\chi)$ all have infinitesimal character $d\chi$. If $\rho = \rho_i$ is one-half the sum of any choice of positive roots (all roots are imaginary), $\chi \otimes e^\rho$ factors to $H(\mathbb{R})$, and the central character of $\Pi_G(\chi)$ is $(\chi \otimes e^\rho)|_{Z(G(\mathbb{R}))}$.

Since $2\rho = 2\rho_i$ is a sum of roots, $e^{2\rho}$ is trivial on the center Z of G, and there is a canonical splitting of the restriction of $\widetilde{H}$ to Z: $z \to (z, 1) \in H_\rho \simeq \widetilde{H}$. Using this splitting the central character of the packet is simply $\chi|_{Z(G(\mathbb{R}))}$.

We are going to show that (as is well known) $\Pi(\chi)$ is precisely the set of relative discrete series representations with the same infinitesimal and central characters as $\pi(\chi)$. In fact something stronger is true.

Let G_{rad} be the radical of G. This maximal central torus is the identity component of the center, and is defined over $\mathbb{R}$. By a *radical character* we mean a character of $G_{\text{rad}}(\mathbb{R})$, and the radical character of an irreducible representation is the restriction of its central character to $G_{\text{rad}}(\mathbb{R})$.

Proposition 4.6 *An L-packet of relative discrete series representations is uniquely determined by an infinitesimal and a radical character.*

What appears in the proof is in fact just the "split radical character," the character of the maximal split torus in the center of G; more precisely, the character on the elements of order two in that split torus.

The proof will be based on the following structural fact.

Lemma 4.7 *Suppose $H(\mathbb{R})$ is a relatively compact Cartan subgroup of $G(\mathbb{R})$. Then*

$$G_{rad}(\mathbb{R}) \subset Z(G(\mathbb{R})) \subset G_{rad}(\mathbb{R})H(\mathbb{R})^0 \subset H(\mathbb{R}). \tag{4.8}$$

Proof. Let A be a maximal split subtorus of H (i.e., $\theta(a) = a^{-1}$ for all $a \in A$). Since $H(\mathbb{R})$ is relatively compact, A is contained in the radical G_{rad} of G. It is well known that $H(\mathbb{R}) = A(\mathbb{R})H(\mathbb{R})^0$. (Such a statement is true for reductive groups [8, Theorem 14.4], with A a maximal split torus; for tori it is elementary.) Therefore $H(\mathbb{R}) = A(\mathbb{R})H(\mathbb{R})^0 = G_{\text{rad}}(\mathbb{R})H(\mathbb{R})^0$. Since $Z(G(\mathbb{R})) \subset H(\mathbb{R})$ this proves the lemma. $\square$

Proof of Proposition 4.6. Suppose (H, χ) is an L-datum as in Definition 4.3, and $\Pi = \Pi_G(\chi)$ the corresponding L-packet of relative discrete series representations. Write $\lambda \in \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ for the differential of χ. The infinitesimal character χ_{inf} of the L-packet is has as representatives exactly all the weights $w\lambda$, with $w \in W(G, H)$.

Write $\chi = (\lambda, \kappa)$ as in Lemma 3.2. The L-packet of γ by definition consists of the representations $\pi(w\lambda, w\kappa)$ (with $w \in W(G, H)$). The set of *all* relative discrete series of infinitesimal character χ_{inf} is equal to

$$\{\pi(w\lambda, w\kappa + \kappa_1)\}.$$

Here the modification term κ_1 is (by Lemma 3.2) subject to the requirement $(1+\theta)\kappa_1 = 0$; that is,

$$\kappa_1 \in X^*(H)^{-\theta}/(1-\theta)X^*(H).$$

The right side here is the group of characters of

$$H^\theta/H_0^\theta = A(\mathbb{R})^\theta/(A(\mathbb{R}) \cap H_0^\theta).$$

We have therefore shown that every relative discrete series representation π_1 of infinitesimal character λ arises by *changing* the radical character of some $\pi(w\chi)$ by κ_1. If the radical character is unchanged—that is, if $\pi(\chi)$ and π_1 have the same radical character—then κ_1 belongs to $(1-\theta)X^*(H)$, and $\pi_1 = \pi(w\chi)$ belongs to the L-packet of π.

□

The following converse to Proposition 4.6 follows immediately from the definitions.

Lemma 4.9 *Assume $G(\mathbb{R})$ has a relatively compact Cartan subgroup, and fix one, denoted $H(\mathbb{R})$. Suppose χ_{inf}, χ_{rad} are infinitesimal and radical characters, respectively. Choose $\lambda \in \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ defining χ_{inf} via the Harish-Chandra homomorphism. Then the L-packet of relative discrete series representations defined by χ_{inf}, χ_{rad} is nonzero if and only if λ is regular, and there is a genuine character of $\widetilde{H(\mathbb{R})}$ satisfying:*

(1) $d\chi = \lambda$

(2) $\chi|_{G_{rad}(\mathbb{R})} = \chi_{rad}$.

The conditions are independent of the choices of $H(\mathbb{R})$ and λ.

In (2) we have used the splitting $Z(G(\mathbb{R})) \to \widetilde{H(\mathbb{R})}$ discussed after (4.5).

We now describe general L-packets. See [1, Section 13] or [5, Section 6].

Definition 4.10 *Suppose (H,χ) is an L-datum. Let A be the identity component of $\{h \in H \mid \theta(h) = h^{-1}\}$ and set $M = \mathrm{Cent}_G(A)$. Let $\mathfrak{a} = \mathrm{Lie}(A)$. Choose a parabolic subgroup $P = MN$ satisfying*

$$\mathrm{Re}\langle d\chi|_{\mathfrak{a}}, {}^\vee\alpha\rangle \geq 0 \textit{ for all roots of } \mathfrak{h} \textit{ in } \mathrm{Lie}(N).$$

Then P is defined over $\mathbb{R}$, and $H(\mathbb{R})$ is a relatively compact Cartan subgroup of $M(\mathbb{R})$. Let $\Pi_M(\chi)$ be the L-packet of relative discrete series representations of $M(\mathbb{R})$ as in Definition 4.3. Define

$$\Pi_G(\gamma) = \bigcup_{\pi\in\Pi_M(\chi)} \{\textit{irreducible quotients of } \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi)\} \tag{4.11}$$

Here we use normalized induction, and pull π back to $P(\mathbb{R})$ via the map $P(\mathbb{R}) \to M(\mathbb{R})$ as usual.

By the discussion following Definition 4.3, and basic properties of induction, the infinitesimal character of $\Pi_G(\chi)$ is $d\chi$, and the central character is $\chi|_{Z(G(\mathbb{R}))}$.

5 The Tits Group

We need a few structural facts provided by the Tits group.

Fix a pinning $\mathcal{P} = (B, H, \{X_\alpha\})$ (see Section 2). For $\alpha \in \Pi$ define $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ by $[X_\alpha, X_{-\alpha}] = {}^\vee\alpha$ as in Section 2. Define $\sigma_\alpha \in W = \exp(\frac{\pi}{2}(X_\alpha - X_{-\alpha})) \in \mathrm{Norm}_G(H)$. The image of σ_α in $W = W(G, H)$ is the simple reflection s_α. Let $H_2 = \{h \in H \mid h^2 = 1\}$.

Definition 5.1 *The* Tits group defined by $\mathcal{P}$ *is the subgroup $\mathcal{T}$ of G generated by H_2 and $\{\sigma_\alpha \mid \alpha \in \Pi\}$.*

Proposition 5.2 ([20]) *The Tits group $\mathcal{T}$ has the given generators, and relations:*

(1) $\sigma_\alpha h \sigma_\alpha^{-1} = s_\alpha(h)$,

(2) the braid relations among the σ_α,

(3) $\sigma_\alpha^2 = {}^\vee\alpha(-1)$.

If $w \in W$ then there is a canonical representative σ_w of w in $\mathcal{T}$ defined as follows. Suppose $w = s_{\alpha_1} \dots s_{\alpha_n}$ is a reduced expression with each $\alpha_i \in \Pi$. Then $\sigma_w = \sigma_{\alpha_1} \dots \sigma_{\alpha_n}$, independent of the choice of reduced expression.

Lemma 5.3 *If w_0 is the long element of the Weyl group, then σ_{w_0} is fixed by any $\mathcal{P}$-distinguished automorphism.*

Proof. Suppose $w_0 = s_{\alpha_1} \dots s_{\alpha_n}$ is a reduced expression. If $\gamma \in \mathrm{Aut}(\mathcal{P})$ it induces an automorphism of the Dynkin diagram, so $s_{\gamma(\alpha_1)} \dots s_{\gamma(\alpha_n)}$ is also a reduced expression for w_0. Therefore $\gamma(\sigma_{w_0}) = \gamma(\sigma_{\alpha_1} \dots \sigma_{\alpha_n}) = \sigma_{\gamma(\alpha_1)} \dots \sigma_{\gamma(\alpha_n)} = \sigma_{w_0}$ by the last assertion of Proposition 5.2. □

Let ${}^\vee\rho$ be one-half the sum of the positive coroots.

Lemma 5.4 *For any $w \in W$ we have*

$$\sigma_w \sigma_{w^{-1}} = \exp(\pi i({}^\vee\rho - w{}^\vee\rho)). \tag{5.5}$$

In particular if w_0 is the long element of the Weyl group,

$$\sigma_{w_0}^2 = \exp(2\pi i{}^\vee\rho) \in Z(G). \tag{5.6}$$

This element of $Z(G)$ is independent of the choice of positive roots, and is fixed by every automorphism of G.

Proof. We proceed by induction on the length of w. If w is a simple reflection s_α then $s_\alpha{}^\vee\rho = {}^\vee\rho - {}^\vee\alpha$, and this reduces to Proposition 5.2(3).

Write $w = s_\alpha u$ with α simple and $\ell(w) = \ell(u) + 1$. Then $\sigma_w = \sigma_\alpha \sigma_u$, $w^{-1} = u^{-1} s_\alpha$, $(\sigma_w)^{-1} = \sigma_{u^{-1}}\sigma_\alpha$, and

(5.7)

$$\begin{aligned}
\sigma_w\sigma_{w^{-1}} &= \sigma_\alpha\sigma_u\sigma_{u^{-1}}\sigma_\alpha \\
&= \sigma_\alpha\sigma_u\sigma_{u^{-1}}\sigma_\alpha^{-1}\exp(\pi i{}^\vee\alpha) \\
&= \sigma_\alpha\exp(\pi i({}^\vee\rho - u{}^\vee\rho))\sigma_\alpha^{-1}\exp(\pi i{}^\vee\alpha) \qquad \text{(by the inductive step)} \\
&= \exp(\pi i(s_\alpha{}^\vee\rho - w{}^\vee\rho))\exp(\pi i{}^\vee\alpha) \\
&= \exp(\pi i({}^\vee\rho - {}^\vee\alpha - w{}^\vee\rho))\exp(\pi i{}^\vee\alpha) = \exp(\pi i({}^\vee\rho - w{}^\vee\rho)).
\end{aligned}$$

The final assertion is easy. □

We thank Marc van Leeuwen for this proof.

We only need what follows for the part of the main theorem involving the Chevalley involution τ of $W_{\mathbb{R}}$. Let $C = C_{\mathcal{P}}$ be the Chevalley involution defined by $\mathcal{P}$.

Lemma 5.8 $C(\sigma_w) = (\sigma_{w^{-1}})^{-1}$

Proof. We proceed by induction on the length of w.

Since $C(X_\alpha) = X_{-\alpha}$ $(\alpha \in \Pi)$, we conclude $C(\sigma_\alpha) = \sigma_\alpha^{-1}$.

Suppose $w = s_\alpha u$ with $\text{length}(w) = \text{length}(u) + 1$. Then $\sigma_w = \sigma_\alpha\sigma_u$, and $C(\sigma_w) = C(\sigma_\alpha)C(\sigma_u) = \sigma_\alpha^{-1}(\sigma_{u^{-1}})^{-1}$. On the other hand $w^{-1} = u^{-1}s_\alpha$, so $\sigma_{w^{-1}} = \sigma_{u^{-1}}\sigma_\alpha$, and taking the inverse gives the result. $\square$

Fix a $\mathcal{P}$-distinguished involution τ of G. Consider the semidirect product $G \rtimes \langle\delta\rangle$ where $\delta^2 = 1$ and δ acts on G by τ. By Lemma 2.6, $C = C_{\mathcal{P}}$ extends to the semidirect product, fixing δ. Since τ normalizes H and B, it defines an automorphism of W, satisfying $\tau(\sigma_w) = \sigma_{\tau(w)}$.

Lemma 5.9 *Suppose $w \in W$ satisfies $w\tau(w) = 1$. Then*
(a) $C(\sigma_w\delta) = (\sigma_w\delta)^{-1}$
(b) Suppose $g \in \text{Norm}_G(H)$ is a representative of w. Then $C(g\delta)$ is H-conjugate to $(g\delta)^{-1}$.

Proof. By the previous lemma, and using $\tau(w) = w^{-1}$, we compute

$$C(\sigma_w\delta)\sigma_w\delta = (\sigma_{w^{-1}})^{-1}\sigma_{\tau(w)} = (\sigma_{\tau(w)})^{-1}\sigma_{\tau(w)} = 1.$$

For (b) write $g = h\sigma_w\delta$ with $h \in H$. Then $C(g) = C(h\sigma_w\delta) = h^{-1}(\sigma_w\delta)^{-1} = h^{-1}(h\sigma_w\delta)^{-1}h$. $\square$

6 L-parameters

Fix a Cartan involution θ of G. Let ${}^\vee G$ be the connected, complex dual group of G. The L-group ${}^LG$ of G is $\langle {}^\vee G, {}^\vee\delta\rangle$ where ${}^\vee\delta^2 = 1$, and ${}^\vee\delta$ acts on ${}^\vee G$ by a homomorphism ${}^\vee\theta_0$, which we now describe. See [7], [6], or [4, Section 2].

Fix Borel and Cartan subgroups B_0, H_0 and let

$$D_b = (X^*(H_0), \Pi, X_*(H_0), {}^\vee\Pi)$$

be the corresponding based root datum. Similarly choose ${}^\vee B_0, {}^\vee H_0$ for ${}^\vee G$ to define ${}^\vee D_b$. We identify $X^*(H_0) = X_*({}^\vee H_0)$ and $X_*(H_0) = X^*({}^\vee H_0)$. Also fix a pinning ${}^\vee\mathcal{P} = ({}^\vee B_0, {}^\vee H_0, \{X_{{}^\vee\alpha}\})$ for ${}^\vee G$. See Section 2.

An automorphism μ of D_b consists of a pair

$$(\tau, \tau^t) \in \text{Aut}(X^*(H_0)) \times \text{Aut}(X_*(H_0)),$$

where the transpose is defined with respect to the perfect pairing

$$X^*(H_0) \times X_*(H_0) \to \mathbb{Z}.$$

By definition τ and τ^t preserve $\Pi, {}^\vee\Pi$ respectively. Interchanging (τ, τ^t) defines a transpose isomorphism $\mathrm{Aut}(D_b) \simeq \mathrm{Aut}({}^\vee D_b)$, denoted $\mu \to \mu^t$. Compose with the embedding $\mathrm{Aut}({}^\vee D_b) \hookrightarrow \mathrm{Aut}({}^\vee G)$ defined by ${}^\vee\mathcal{P}$ (Section 2) to define a map:

$$\mu \to \mu^t\colon \mathrm{Aut}(D_b) \hookrightarrow \mathrm{Aut}({}^\vee G). \tag{6.1}$$

Suppose σ is a real form corresponding to θ (see the beginning of Section 4). Choose $g \in G$ conjugating $\sigma(B_0)$ to B_0 and $\sigma(H_0)$ to H_0. Then $\tau = \mathrm{int}(g) \circ \sigma \in \mathrm{Aut}(D_b)$. Let ${}^\vee\theta_0 = \tau^t \in \mathrm{Aut}({}^\vee G)$. See [7]. For example, if $G(\mathbb{R})$ is split, taking B_0, H_0 defined over $\mathbb{R}$ shows that ${}^L G = {}^\vee G \times \Gamma$ (direct product).

Alternatively, using θ itself gives a version naturally related to the most compact Cartan subgroup. Let γ be the image of θ in $\mathrm{Out}(G) \simeq \mathrm{Aut}(D_b)$. The automorphism $-w_0$ of $X^*(H)$, taking γ to $-(w_0(\gamma))$, induces an automorphism of D_b, also denoted $-w_0$. Thus $-w_0\gamma \in \mathrm{Aut}(D_b)$, and we may define:

$${}^\vee\theta_0 = (-w_0\gamma)^t \in \mathrm{Aut}({}^\vee G). \tag{6.2}$$

This is the approach of [6] and [4]. It is not hard to see the elements $\tau, \gamma \in \mathrm{Aut}(D_b)$ satisfy $\tau = -w_0\gamma$, so the two definitions of ${}^L G$ agree.

Lemma 6.3 *The following conditions are equivalent:*

(1) $G(\mathbb{R})$ has a compact Cartan subgroup,

(2) ${}^\vee\theta_0$ is inner to the Chevalley involution;

(3) There is an element $y \in {}^L G \backslash {}^\vee G$ such that $yhy^{-1} = h^{-1}$ for all $y \in {}^\vee H_0$.

Proof. The equivalence of (2) and (3) is immediate. Let C be the Chevalley involution of ${}^\vee G$ with respect to ${}^\vee\mathcal{P}$. It is easy to see (6.2) is equivalent to: the image of ${}^\vee\theta_0 \circ C$ in $\mathrm{Out}({}^\vee G) \simeq \mathrm{Aut}({}^\vee D_b)$ is equal to γ^t. So the assertion is that $G(\mathbb{R})$ has a compact Cartan subgroup if and only if $\gamma = 1$, i.e., θ is an inner automorphism. The direction $\Rightarrow$ holds by Lemma 2.4. For the other direction, if $\theta = \mathrm{int}(x)$ for $x \in G(\mathbb{C})$, let H be any Cartan subgroup of G containing x. Then θ_x acts trivially on H, i.e., H is the complexification of a compact Cartan subgroup of $G(\mathbb{R})$. $\square$

A homomorphism $\phi\colon W_{\mathbb{R}} \to {}^L G$ is said to be *quasiadmissible* if it is continuous, $\phi(\mathbb{C}^*)$ consists of semisimple elements, and $\phi(j) \in {}^L G \backslash {}^\vee G$ [7, 8.2]. We will see in Sections 6.2 and 6.3 that every quasiadmissible homomorphism is associated to an L-packet $\Pi_G(\phi)$, which depends only on the ${}^\vee G$-conjugacy class of ϕ. We say ϕ is *admissible* if it satisfies the additional relevancy condition [7, 8.2(ii)]. The admissible condition (unlike quasiadmissibility) is sensitive to the real forms of G, and guarantees that $\Pi_G(\phi)$ is nonempty. If $G(\mathbb{R})$ is quasisplit every quasiadmissible homomorphism is admissible.

After conjugating by ${}^\vee G$ we may assume $\phi(\mathbb{C}^*) \subset {}^\vee H_0$. Let ${}^\vee S$ be the centralizer of $\phi(\mathbb{C}^*)$ in ${}^\vee G$. Since $\phi(\mathbb{C}^*)$ is connected, abelian and consists of semisimple elements, ${}^\vee S$ is a connected reductive complex group, and ${}^\vee H_0$ is a Cartan subgroup of ${}^\vee S$. Conjugation by $\phi(j)$ is an involution of ${}^\vee S$, so $\phi(j)$ normalizes a Cartan subgroup of ${}^\vee S$. Equivalently some ${}^\vee S$-conjugate of $\phi(j)$ normalizes ${}^\vee H_0$; after this change we may assume $\phi(W_{\mathbb{R}}) \subset \mathrm{Norm}_{{}^\vee G}({}^\vee H_0)$.

Therefore

(6.4)(a) $\phi(z) = z^{\lambda}\overline{z}^{\lambda'}$ (for some $\lambda, \lambda' \in X_*({}^\vee H_0) \otimes \mathbb{C},\ \lambda - \lambda' \in X_*({}^\vee H_0)$)

(6.4)(b) $\phi(j) = h\sigma_w {}^\vee\delta$ (for some $w \in W, h \in {}^\vee H_0$).

Here (a) is shorthand for $\phi(e^s) = \exp(s\lambda + \overline{s}\lambda') \in {}^\vee H_0$ ($s \in \mathbb{C}$), and the condition on $\lambda - \lambda'$ guarantees this is well defined. In (b) we're using the element σ_w of the Tits group representing w (Proposition 5.2).

Conversely, given λ, λ', w and h, (a) and (b) give a well-defined homomorphism $\phi\colon W_{\mathbb{R}} \to {}^L G$ if and only if

(6.4)(c) ${}^\vee\theta := \mathrm{int}(h\sigma_w {}^\vee\delta)$ is an involution of ${}^\vee H_0$,

(6.4)(d) $\lambda' = {}^\vee\theta(\lambda)$,

(6.4)(e) $h {}^\vee\theta(h)(\sigma_w {}^\vee\theta_0(\sigma_w)) = \exp(\pi i(\lambda - {}^\vee\theta(\lambda))$.

Furthermore (c) is equivalent to

(6.4)(c′) $w {}^\vee\delta(w) = 1$.

6.1 Infinitesimal and radical characters

Suppose ϕ is as in (6.4). We attach two invariants to an admissible homomorphism ϕ.

Infinitesimal Character of ϕ

View λ as an element of $X^*(H_0) \otimes \mathbb{C}$ via the identification $X_*({}^\vee H_0) = X^*(H_0)$. The $W(G, H_0)$-orbit of λ is independent of all choices, so it defines an infinitesimal character for G, denoted $\chi_{\mathrm{inf}}(\phi)$, via the Harish-Chandra homomorphism.

Radical character of ϕ

Recall (Section 4) G_{rad} is the radical of G, and the radical character of a representation is its restriction to $G_{\mathrm{rad}}(\mathbb{R})$.

Dual to the inclusion $\iota\colon G_{\mathrm{rad}} \hookrightarrow G$ is a surjection ${}^\vee\iota\colon {}^\vee G \twoheadrightarrow {}^\vee[G_{\mathrm{rad}}]$. For an L-group for G_{rad} we can take ${}^L G_{\mathrm{rad}} = \langle {}^\vee[G_{\mathrm{rad}}], {}^\vee\delta\rangle$. Thus ${}^\vee\iota$ extends to a natural surjection ${}^\vee\iota\colon {}^L G \to {}^L G_{\mathrm{rad}}$ (taking ${}^\vee\delta$ to itself). Then ${}^\vee\iota \circ \phi\colon W_{\mathbb{R}} \to {}^L G_{\mathrm{rad}}$, and this defines a character of $G_{\mathrm{rad}}(\mathbb{R})$ by the construction of Section 3. We denote this character $\chi_{rad}(\phi)$. See [7, 10.1] and [15, page 20].

6.2 Relative Discrete Series L-packets

By a *Levi subgroup* of ${}^L G$ we mean the centralizer ${}^d M$ of a torus ${}^\vee T \subset {}^\vee G$, which meets both components of ${}^L G$ [7, Lemma 3.5]. We will soon see that an L-packet $\Pi_G(\phi)$ consists of relative discrete series representations if and only if $\phi(W_{\mathbb{R}})$ is not contained in a proper Levi subgroup.

Lemma 6.5 *Suppose ϕ is as in* (6.4)*. If $\phi(W_{\mathbb{R}})$ is not contained in a proper Levi subgroup then λ is regular and $G(\mathbb{R})$ has a relatively compact Cartan subgroup.*

See [7, Lemma 11.1] and [15, Lemmas 3.1 and 3.3].

Proof. Assume $\phi(W_{\mathbb{R}})$ is not contained in a proper Levi subgroup. Let ${}^\vee S = \mathrm{Cent}_{{}^\vee G}(\phi(\mathbb{C}^*))$ as in the discussion preceding (6.4). Then ${}^\vee\theta = \mathrm{int}(\phi(j))$ is an involution of ${}^\vee S$, and of its derived group ${}^\vee S_d$. There cannot be a torus in ${}^\vee S_d$, fixed (pointwise) by ${}^\vee\theta$; its centralizer would contradict the assumption. Since any involution of a semisimple group fixes a torus, this implies ${}^\vee S_d = 1$, i.e., ${}^\vee S = {}^\vee H_0$, which implies λ is regular.

Similarly, there can be no torus in ${}^\vee H_0 \cap {}^\vee G_d$ fixed by ${}^\vee\theta$. This implies ${}^\vee\theta(h) = h^{-1}$ for all $h \in {}^\vee H_0 \cap {}^\vee G_d$. By Lemma 6.3 applied to the derived group, $G(\mathbb{R})$ has a relatively compact Cartan subgroup. $\square$

Definition 6.6 *In the setting of Lemma 6.5, $\Pi_G(\phi)$ is the L-packet of relative discrete series representations determined by infinitesimal character $\chi_{inf}(\phi)$ and radical character $\chi_{rad}(\phi)$ (see Proposition 4.6).*

Lemma 6.7 $\Pi_G(\phi)$ *is nonempty.*

Proof. After conjugating θ if necessary we may assume H_0 is θ-stable. Write ϕ as in (6.4)(a) and (b). By the discussion of infinitesimal character above $\chi_{\text{inf}}(\phi)$ is defined by λ, viewed as an element of $X^*(H_0) \otimes \mathbb{C}$.

Choose positive roots making λ dominant. By Lemma 4.9 it is enough to construct a genuine character of $H(\mathbb{R})_\rho$ satisfying $d\Lambda = \lambda$ and $\Lambda|_{G_{\text{rad}}(\mathbb{R})} = \chi_{\text{rad}}$. For this we apply Lemma 3.3, using the fact that $\phi(W_\mathbb{R}) \subset \langle {}^\vee H_0, \sigma_w {}^\vee\delta\rangle$. We being by identifying $\langle {}^\vee H_0, \sigma_w {}^\vee\delta\rangle$ as an E-group.

First we claim $w = w_0$. By (6.4)(b) and (c) ${}^\vee\theta|_{{}^\vee H_0} = w {}^\vee\theta_0|_{{}^\vee H_0}$. By (6.2), for $h \in {}^\vee H_0 \cap {}^\vee G_d$ we have:

$$
{}^\vee\theta(h) = w {}^\vee\theta_0(h) = w(-w_0\gamma^t)(h) = ww_0\gamma^t(h)^{-1}. \tag{6.8}
$$

Since $G_d(\mathbb{R})$ has a compact Cartan subgroup, γ is trivial on $H_0 \cap G_d$ and $\gamma^t(h) = h$. On the other hand, as in the proof of Lemma 6.5, ${}^\vee\theta(h) = h^{-1}$. Therefore $ww_0 = 1$, i.e., $w = w_0$.

Next we compute

$$
\begin{aligned}
(\sigma_{w_0}{}^\vee\delta)^2 &= w_0\theta_0(w_0) \\
&= w_0^2 \quad \text{(by (5.3), since } {}^\vee\theta_0 \text{ is distinguished)} \\
&= \exp(2\pi i\rho) \quad \text{(by (5.6))}.
\end{aligned} \tag{6.9}
$$

Thus, in the terminology of Section 3, $\langle {}^\vee H_0, \sigma_{w_0}{}^\vee\delta\rangle$ is identified with the E-group for H defined by ${}^\vee\rho$. Consequently $\phi\colon W_\mathbb{R} \to \langle {}^\vee H_0, \sigma_{w_0}{}^\vee\delta\rangle$ defines a genuine character χ of $H(\mathbb{R})_\rho$.

By construction $d\chi = \lambda$. The fact that $\chi|_{G_{\text{rad}}(\mathbb{R})} = \chi_{\text{rad}}$ is a straightforward check. Here are the details.

Write h of (6.4)(b) as $h = \exp(2\pi i\mu)$. We use the notation of Section 3, especially (3.4). Using the fact that $\sigma_{w_0}{}^\vee\delta$ is the distinguished element of the E-group of H_0 we have $\chi = \chi(\lambda, \kappa)$ where

$$
\kappa = \frac{1}{2}(1 - {}^\vee\theta)\lambda - (1 + {}^\vee\theta)\mu \in \rho + X^*(H_0). \tag{6.10}
$$

Write $p\colon X^*(H_0) \to X^*(G_{\mathrm{rad}})$ for the map dual to inclusion $G_{\mathrm{rad}} \to H_0$. Recall (Section 4) there is a canonical splitting of the cover of $Z(G(\mathbb{R}))$; using this splitting $\chi(\rho,\rho)|_{Z(G(\mathbb{R}))} = 1$, and

$$\chi|_{G_{\mathrm{rad}}(\mathbb{R})} = \chi(p(\lambda), p(\kappa-\rho)).$$

On the other hand, by the discussion of the character of $G_{\mathrm{rad}}(\mathbb{R})$ above, the E-group of G_{rad} is $\langle {}^\vee G_{\mathrm{rad}}, {}^\vee\delta\rangle$. The map $p\colon X^*(H_0) \to X^*(G_{\mathrm{rad}})$ is identified with a map $p\colon X_*({}^\vee H_0) \to X_*({}^\vee G_{\mathrm{rad}})$. Then ${}^\vee\iota(\phi(j)) = {}^\vee\iota(h\sigma_w{}^\vee\delta) = {}^\vee\iota(h){}^\vee\delta = \exp(2\pi i p(\mu)){}^\vee\delta$. Let

$$\kappa' = \frac{1}{2}(1-{}^\vee\theta)p(\lambda) - (1+{}^\vee\theta)p(\mu) \in X^*(G_{\mathrm{rad}})$$

Thus $\kappa' = p(\kappa-\rho)$. By the construction of Section 3 applied to $\phi\colon W_{\mathbb{R}} \to {}^\vee G_{\mathrm{rad}}$, $\chi_{\mathrm{rad}} = \chi(p(\lambda),\kappa') = \chi(p(\lambda), p(\kappa-\rho)) = \chi|_{G_{\mathrm{rad}}(\mathbb{R})}$. □

Remark 6.11 The fact that $(\sigma_{w_0}{}^\vee\delta)^2 = \exp(2\pi i\rho)$ is the analogue of [15, Lemma 3.2].

We can read off the central character of the L-packet from the construction. We defer this until we consider general L-packets (Lemma 6.17).

6.3 General L-packets

See [7, Section 11.3] and [15, pages 40–58].

Recall (see the beginning of the previous section) a Levi subgroup ${}^dM$ of ${}^LG$ is the centralizer of a torus ${}^\vee T$, which meets both components of ${}^LG$. An admissible homomorphism ϕ may factor through various Levi subgroups ${}^dM$. We first choose ${}^dM$ so that $\phi\colon W_{\mathbb{R}} \to {}^dM$ defines a relative discrete series L-packet of M.

Choose a maximal torus ${}^\vee T \subset \mathrm{Cent}_{{}^\vee G}(\phi(W_{\mathbb{R}}))$ and define

$$(6.12)\qquad {}^\vee M = \mathrm{Cent}_{{}^\vee G}({}^\vee T),\ {}^dM = \mathrm{Cent}_{{}^LG}({}^\vee T).$$

Then ${}^dM = \langle {}^\vee M, \phi(j)\rangle$, so ${}^dM$ is a Levi subgroup, and $\phi(W_{\mathbb{R}}) \subset {}^dM$.

Suppose $\phi(W_{\mathbb{R}}) \subset \mathrm{Cent}_{{}^dM}({}^\vee U)$ where ${}^\vee U \subset {}^\vee M$ is a torus. Then ${}^\vee U$ centralizes $\phi(W_{\mathbb{R}})$ and ${}^\vee T$, so ${}^\vee U{}^\vee T$ is a torus in $\mathrm{Cent}_{{}^\vee G}(\phi(W_{\mathbb{R}}))$. By maximality

${}^\vee U \subset {}^\vee T$ and $\mathrm{Cent}_{{}^d M}({}^\vee U) = {}^d M$. Therefore $\phi(W_{\mathbb{R}})$ is not contained in any proper Levi subgroup of ${}^d M$.

Lemma 6.13 *The group* ${}^d M$ *is independent of the choice of* ${}^\vee T$*, up to conjugation by* $\mathrm{Cent}_{{}^\vee G}(\phi(W_{\mathbb{R}}))$.

Proof. Conjugation by $\phi(j)$ defines an involution of the connected reductive group ${}^\vee S = \mathrm{Cent}_{{}^\vee G}(\phi(\mathbb{C}^*))$ (see the discussion after Lemma 6.3), and $\mathrm{Cent}_{{}^\vee G}(\phi(W_{\mathbb{R}}))$ is the fixed points of this involution. Thus ${}^\vee T$ is a maximal torus in (the identity component of) this reductive group, and any two such tori are conjugate by $\mathrm{Cent}_{{}^\vee G}(\phi(W_{\mathbb{R}}))$. □

The idea is to identify ${}^d M$ with the L-group of a Levi subgroup $M'(\mathbb{R})$ of $G(\mathbb{R})$. Then, since $\phi(W_{\mathbb{R}}) \subset {}^d M$ is not contained in any proper Levi subgroup, it defines a relative discrete series L-packet for $M'(\mathbb{R})$. We obtain Π_G by induction. Here are the details.

We need to identify ${}^d M = \langle {}^\vee M, \phi(j) \rangle$ with an L-group. A crucial technical point is that after conjugating we may assume ${}^d M = \langle {}^\vee M, {}^\vee \delta \rangle$, making this identification clear.

Lemma 6.14 ([7], Section 3.1) *Suppose* S *is a* ${}^\vee\theta_0$*-stable subset of* ${}^\vee\Pi$*. Let* ${}^\vee M_S$ *be the corresponding Levi subgroup of* ${}^\vee G$*:* ${}^\vee H_0 \subset {}^\vee M_S$*, and* S *is a set of simple roots of* ${}^\vee H_0$ *in* ${}^\vee M_S$*. Let* ${}^d M_S = {}^\vee M_S \rtimes \langle {}^\vee \delta \rangle$*, a Levi subgroup of* ${}^L G$*.*

Let $M_S \supset H_0$ *be the Levi subgroup of* G *with simple roots* $\{\alpha \mid {}^\vee\alpha \in S\} \subset \Pi$*. Suppose some conjugate* M' *of* M_S *is defined over* $\mathbb{R}$*. Write* ${}^L M' = {}^\vee M' \rtimes \langle {}^\vee \delta_{M'} \rangle$*. Then conjugation induces an isomorphism* ${}^L M' \simeq {}^d M_S$*, taking* ${}^\vee \delta_{M'}$ *to* ${}^\vee \delta$*.*

Any Levi subgroup of ${}^L G$ *is* ${}^\vee G$*-conjugate to* ${}^d M_S$ *for some* ${}^\vee\theta_0$*-stable set* S*.*

We refer to the Levi subgroups ${}^d M_S$ of the lemma (where S is ${}^\vee\theta_0$-stable) as *standard Levi subgroups.*

Definition 6.15 *Suppose* $\phi\colon W_{\mathbb{R}} \to {}^L G$ *is an admissible homomorphism. Choose a maximal torus* ${}^\vee T$ *in* $\mathrm{Cent}_{{}^\vee G}(\phi(W_{\mathbb{R}}))$*, and define* ${}^\vee M, {}^d M$ *by* (6.12)*.*

After conjugating by ${}^\vee G$*, we we may assume* ${}^d M$ *is a standard Levi subgroup. Let* $M(\mathbb{C})$ *be the corresponding standard Levi subgroup of* $G(\mathbb{C})$*.*

Assume there is a subgroup M' conjugate to M, which is defined over $\mathbb{R}$; otherwise define $\Pi(\phi)$ to be empty. Let $\Pi_{M'}(\phi)$ be the L-packet for $M'(\mathbb{R})$ defined by $\phi\colon W_\mathbb{R} \to {}^dM \simeq {}^LM'$. (cf. Lemma 6.14). Define the L-packet for G attached to ϕ $\Pi_G(\phi)$ *by induction from $\Pi_{M'}(\phi)$ as in Definition 4.10.*

Lemma 6.16 *The L-packet $\Pi_G(\phi)$ is independent of all choices.*

Proof. By Lemma 6.13 the choice of ${}^\vee T$ is irrelevant: another choice leads to an automorphism of ${}^dM$ fixing $\phi(W_\mathbb{R})$ pointwise.

It is straighforward to see that the other choices, including another Levi subgroup M'', would give an element $g \in G(\mathbb{C})$ such that $\mathrm{int}(g)\colon M' \to M''$ is defined over $\mathbb{R}$, and this isomorphism takes $\Pi_{M'}(\phi)$ to $\Pi_{M''}(\phi)$.

First of all we claim $M'(\mathbb{R})$ and $M''(\mathbb{R})$ are $G(\mathbb{R})$-conjugate. To see this, let $H'(\mathbb{R})$ be a relatively compact Cartan subgroup of $M'(\mathbb{R})$. Then $H''(\mathbb{R}) = gH'(\mathbb{R})g^{-1}$ is a relatively compact Cartan subgroup of $M''(\mathbb{R})$. In fact $H'(\mathbb{R})$ and $H''(\mathbb{R})$ are $G(\mathbb{R})$-conjugate: two Cartan subgroups of $G(\mathbb{R})$ are $G(\mathbb{C})$-conjugate if and only if they are $G(\mathbb{R})$-conjugate. Therefore $M'(\mathbb{R}), M''(\mathbb{R})$, being the centralizers of the split components of $H'(\mathbb{R})$ and $H''(\mathbb{R})$, are also $G(\mathbb{R})$-conjugate.

Therefore, since the inductive step is not affected by conjugating by $G(\mathbb{R})$, we may assume $M' = M''$. Then $g \in \mathrm{Norm}_{G(\mathbb{C})}(M')$, and furthermore $g \in \mathrm{Norm}_{G(\mathbb{C})}(M'(\mathbb{R}))$.

Now $gH'(\mathbb{R})g^{-1}$ is another relatively compact Cartan subgroup of $M'(\mathbb{R})$, so after replacing g with gm for some $m \in M'(\mathbb{R})$ we may assme $g \in \mathrm{Norm}_{G(\mathbb{C})}(H'(\mathbb{R}))$. It is well known that

$$\mathrm{Norm}_{G(\mathbb{C})}(H'(\mathbb{R})) = \mathrm{Norm}_{M'(\mathbb{C})}(H'(\mathbb{R}))\,\mathrm{Norm}_{G(\mathbb{R})}(H'(\mathbb{R})).$$

For example see [22, Proposition 3.12] (where the group in question is denoted $W(R)^\theta$), or [17, Theorem 2.1]. Since conjugation by $M'(\mathbb{C})$ does not change infinitesimal or central characters, by Proposition 4.6 it preserves $\Pi_{M'}(\phi)$. (See Lemma 6.18). As above $G(\mathbb{R})$ has no effect after the inductive step. This completes the proof. □

We now give the formula for the central character of $\Pi_G(\phi)$. This follows immediately from the preceding discussion, and (6.10) applied to M.

Lemma 6.17 *Write ϕ as in* (6.4)*(a) and (b), and suppose $h = \exp(2\pi i\mu)$, with $\mu \in X_*({}^\vee H_0) \otimes \mathbb{C} \simeq X^*(H_0) \otimes \mathbb{C}$. Let ρ_i be one-half the sum of any set* *of positive roots of* $\{\alpha \,|\, {}^\vee\theta\alpha = -\alpha\}$*, Set*

$$\tau = \frac{1}{2}(1 - {}^\vee\theta)\lambda - (1 + {}^\vee\theta)\mu + \rho_i \in X^*(H_0).$$

Then the central character of $\Pi_G(\phi)$ is $\tau|_{Z(G(\mathbb{R}))}$.

For the local Langlands classification to be well-defined it should be natural with respect to automorphisms of G. This is the content of the next lemma.

Suppose $\tau \in \text{Aut}(G)$ is an involution which commutes with θ. The automorphism τ acts on the pair $(\mathfrak{g}, K)$, and defines an involution on the set of irreducible $(\mathfrak{g}, K)$-modules, which preserves L-packets.

On the other hand, consider the image of τ under the sequence of maps $\text{Aut}(G) \to \text{Out}(G) \simeq \text{Aut}(D_b) \to \text{Aut}({}^\vee G)$; the final arrow is the transpose (6.1). This extends to an automorphism of ${}^L G$ which we denote τ^t. For example $\tau^t = 1$ if and only if τ is an inner automorphism.

Lemma 6.18 *Suppose* ϕ *is an admissible homomorphism. Then* $\Pi_G(\phi)^\tau = \Pi_G(\tau^t \circ \phi)$*.*

Remark 6.19 Suppose τ is an inner automorphism of $G = G(\mathbb{C})$. It may not be inner for K, and therefore it may act nontrivially on irreducible representations. So it isn't entirely obvious that τ preserves L-packets (which it must by the lemma).

For example $\text{int}(\text{diag}(i, -i))$ normalizes $SL(2, \mathbb{R})$, and $K(\mathbb{R}) = SO(2)$, and interchanges the two discrete series representations in an L-packet.

Proof. This is straightforward from our characterization of the correspondence. Suppose τ is inner. Then it preserves infinitesimal and radical characters, and commutes with parabolic induction. Therefore it preserves L-packets. On the other hand $\tau^t = 1$.

In general, this shows (after modifying τ by an inner automorphism) we may assume τ is distinguished. Then it is easy to check the assertion on the level of infinitesimal and radical characters, and it commutes with parabolic induction. The result follows. We leave the details to the reader. □

7 Contragredient

Suppose (π, V) is a $(\mathfrak{g}, K)$-module. Define a$(\mathfrak{g}, K)$-module structure on $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ by

(7.1)(a) $$\pi^*(X)(f)(v) = -f(\pi(X)v) \quad (v \in V, X \in \mathfrak{g})$$

and

(7.1)(b) $$\pi^h(k)(f)(v) = f(\pi(k^{-1})v) \quad (v \in V, k \in K).$$

Then the *dual* of (π, V) is defined to be the $(\mathfrak{g}, K)$-module (π^*, V^*), where $V^* \subset \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the subspace of K-finite functionals. See [21, Definition 8.5.1].

Following [10] we say that a $(\mathfrak{g}, K)$-module (π', V') is contragredient to (π, V) if there is a nondegenerate bilinear form $B\colon V \times V' \to \mathbb{C}$, respecting the actions of $\mathfrak{g}$ and K. If (π, V) is irreducible then (π', V') is contragredient to (π, V) if and only if (π', V') is isomorphic to the dual (π^*, V^*). Consequently we follow the common practice of using the terms "dual" and "contragredient" interchangeably.

The proof of Theorem 1.3 is now straightforward. We restate the theorem here.

Theorem 7.2 *Let $G(\mathbb{R})$ be the real points of a connected reductive algebraic group defined over $\mathbb{R}$, with L-group ${}^LG$. Let C be the Chevalley involution of ${}^LG$ (Section 2) and let τ be the Chevalley involution of $W_{\mathbb{R}}$ (Lemma 2.7). Suppose $\phi\colon W_{\mathbb{R}} \to {}^LG$ is an admissible homomorphism, with associated L-packet $\Pi(\phi)$. Let $\Pi(\phi)^* = \{\pi^* \mid \pi \in \Pi(\phi)\}$. Then:*

(a) $\Pi(\phi)^* = \Pi(C \circ \phi)$

(b) $\Pi(\phi)^* = \Pi(\phi \circ \tau)$

Proof. Let $\mathcal{P} = ({}^\vee B_0, {}^\vee H_0, \{X_{{}^\vee\alpha}\})$ be the pinning used to define ${}^LG$. After conjugating by ${}^\vee G$ we may assume $C = C_{\mathcal{P}}$ (Proposition 2.1). As in the discussion before (6.4), we are free to conjugate ϕ so that $\phi(\mathbb{C}^*) \in {}^\vee H_0$, and $\phi(W_{\mathbb{R}}) \subset \mathrm{Norm}_{{}^\vee G}({}^\vee H_0)$.

First assume $\Pi(\phi)$ is an L-packet of relative discrete series representations. Then $\Pi(\phi)$ is determined by its infinitesimal character $\chi_{\mathrm{inf}}(\phi)$ and its radical character $\chi_{\mathrm{rad}}(\phi)$ (Definition 6.6). It is easy to see that the infinitesimal character of $\Pi(\phi)^*$ is $-\chi_{\text{inf}}(\phi)$, and the radical character is $\chi_{\text{rad}}(\phi)^*$. So it is enough to show $\chi_{\text{inf}}(C \circ \phi) = -\chi_{\text{inf}}(\phi)$ and $\chi_{\text{rad}}(C \circ \phi) = \chi_{\text{rad}}(\phi)^*$. The first is obvious from (6.4)(a), the definition of $\chi_{\text{inf}}(\phi)$, and the fact that C acts by -1 on the Lie algebra of ${}^\vee H_0$. The second follows from the fact that C factors to the Chevalley involution of ${}^L G_{\text{rad}}$, and the torus case (Lemma 3.6).

Now suppose ϕ is any admissible homomorphism such that $\Pi_G(\phi)$ is nonempty. As in Definition 4.10 we may assume $\phi(W_{\mathbb{R}}) \subset {}^d M$ where ${}^d M$ is a standard Levi subgroup of ${}^L G$. Choose M' as in Definition 6.15 and write $\Pi_{M'} = \Pi_{M'}(\phi)$ as in that Definition.

Write socle (resp. co-socle) for the set of irreducible submodules (resp. quotients) of an admissible representation.

Choose $P = M'N$ as in Definition 4.10 to define

$$\text{(7.3)(a)} \qquad \Pi_G(\phi) = \text{cosocle}(\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\Pi_{M'})) = \bigcup_{\pi \in \Pi_{M'}} \text{cosocle}(\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi)).$$

It is immediate from the definitions that $C_{\mathcal{P}}$ restricts to the Chevalley involution of ${}^\vee M$. Therefore by the preceding case $\Pi_{M'}(C \circ \phi) = \Pi_{M'}(\phi)^*$. Compute $\Pi_G(C \circ \phi)$ using Definition 4.10; this time the positivity condition in Definition 4.10 forces us to to use the opposite parabolic $\overline{P} = M'\overline{N}$:

$$\text{(7.3)(b)} \qquad \Pi_G(C \circ \phi) = \text{cosocle}(\text{Ind}_{\overline{P}(\mathbb{R})}^{G(\mathbb{R})}(\Pi_{M'}(\phi)^*)).$$

Here is the proof of part (a), we justify the steps below.

$$\text{(7.3)(c)} \qquad \begin{aligned} \Pi_G(C \circ \phi)^* &= [\text{cosocle}(\text{Ind}_{\overline{P}(\mathbb{R})}^{G(\mathbb{R})}(\Pi_{M'}(\phi)^*))]^* \\ &= [\text{cosocle}(\text{Ind}_{\overline{P}(\mathbb{R})}^{G(\mathbb{R})}(\Pi_{M'}(\phi))^*)]^* \\ &= [\text{socle}(\text{Ind}_{\overline{P}(\mathbb{R})}^{G(\mathbb{R})}(\Pi_{M'}(\phi)))^*]^* \\ &= \text{cosocle}(\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\Pi_{M'}(\phi))) = \Pi_G(\phi). \end{aligned}$$

The first step is just the contragredient of (7.3)(b). For the second, integration over $G(\mathbb{R})/P(\mathbb{R})$ is a pairing between $\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi^*)$ and $\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi)^*$, and gives

$$\text{(7.4)} \qquad \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi^*) \simeq \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi)^*.$$

For the next step use $\mathrm{cosocle}(X^*) = \mathrm{socle}(X)^*$. Then the double dual cancels for irreducible representations, and it is well known that the theory of intertwining operators gives:

$$\mathrm{socle}(\mathrm{Ind}_{\overline{P}(\mathbb{R})}^{G(\mathbb{R})}(\pi)) = \mathrm{cosocle}(\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi)). \tag{7.5}$$

Finally plugging in (7.3)(a) gives part (a) of the theorem.

For (b) we show that $C \circ \phi$ is ${}^\vee G$-conjugate to $\phi \circ \tau$.

Recall τ is any automorphism of $W_\mathbb{R}$ acting by inverse on $\mathbb{C}^*$, and any two such τ are conjugate by $\mathrm{int}(z)$ for $z \in \mathbb{C}^*$. Therefore the statement is independent of the choice of τ. It is convenient to choose $\tau(j) = j^{-1}$, i.e., $\tau = \tau_{-1}$ in the notation of the proof of Lemma 2.7.

By (6.4)(a) $(C \circ \phi)(z) = C(\phi(z)) = z^{-\lambda}\overline{z}^{-\lambda'}$. Recall (Lemma 2.7) that $\tau(z) = z^{-1}$ for all $z \in \mathbb{C}^* \subset W_\mathbb{R}$, so $(\phi \circ \tau)(z) = \phi(z^{-1}) = z^{-\lambda}\overline{z}^{-\lambda'}$. Therefore it is enough to show $C(\phi(j))$ is ${}^\vee H_0$-conjugate to $\phi(\tau(j))$, which equals $\phi(j)^{-1}$ by our choice of τ.

Since $\phi(j)$ normalizes ${}^\vee H_0$, $\phi(j) = g^\vee\delta$ with $g \in \mathrm{Norm}_{{}^\vee G}({}^\vee H_0)$. Then $g^\vee\delta(g)\phi(j)^2 = \phi(-1) \in {}^\vee H_0$. Therefore the image w of g in W satisfies $w\theta_0(w) = 1$. Apply Lemma 5.9(b). □

Remark 7.6 The main result of [2], together with Lemma 6.18, gives an alternative proof of Theorem 1.3. By [2, Theorem 1.2], there is a "real" Chevalley involution $C_\mathbb{R}$ of G, which is defined over $\mathbb{R}$. This satisfies: $\pi^{C_\mathbb{R}} \simeq \pi^*$ for any irreducible representation π. The transpose automorphism $C_\mathbb{R}^t$ of ${}^L G$ of Lemma 6.18 is the Chevalley automorphism of ${}^L G$. Then Lemma 6.18 applied to $C_\mathbb{R}$ implies Theorem 1.3.

8 Hermitian Dual

Suppose π is an admissible representation of $G(\mathbb{R})$. We briefly recall what it means for π to have an invariant Hermitian form, and the notion of the Hermitian dual of π. See [14] for details, and for the connection with unitary representations.

Let σ be the antiholomorphic involution of $\mathbb{G}$ with fixed points $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G(\mathbb{R}))$. We can and do assume that σ commutes with the Cartan involution θ, and write σ for the corresponding automorphism of $K(\mathbb{C})$, so $K(\mathbb{C})^\sigma$ is a maximal compact subgroup of $K(\mathbb{R})$.

We say a $(\mathfrak{g}, K)$-module (π, V), or simply π, is Hermitian if there is a nondegenerate Hermitian form $(\, , \,)$ on V, satisfying

$$(\pi(X)v, w) + (v, \pi(\sigma(X))w) = 0 \quad (v, w \in V, X \in \mathfrak{g}), \tag{8.1)(a}$$

and

$$(\pi(k)v, \pi(\sigma(k))w) = (v, w) \quad (v, w \in V, k \in K). \tag{8.1)(b}$$

Define the Hermitian dual (π^h, V^h) as follows (compare (7.1)). Define a representation of $\mathfrak{g}$ on the space of conjugate-linear functions $V \to \mathbb{C}$ by

$$\pi^h(X)(f)(v) = -f(\pi(\sigma(X))v) \quad (v \in V, X \in \mathfrak{g}) \tag{8.2)(a}$$

and

$$\pi^h(k)(f)(v) = f(\pi(\sigma(k^{-1}))v) \quad (v \in V, k \in K). \tag{8.2)(b}$$

Let V^h be the K-finite functions; then (π^h, V^h) is a $(\mathfrak{g}, K)$-module. If π is irreducible then π is Hermitian if and only if $\pi \simeq \pi^h$.

Fix a Cartan subgroup H of G. Identify an infinitesimal character χ_{inf} with (the Weyl group orbit of) an element $\lambda \in \text{Hom}(\mathfrak{h}, \mathbb{C})$, by the Harish-Chandra homomorphism. Define $\lambda \to \overline{\lambda}$ with respect to the real form $X^*(H) \otimes \mathbb{R}$ of $\text{Hom}(\mathfrak{h}, \mathbb{C})$, and write $\overline{\chi_{\text{inf}}}$ for the corresponding action on infinitesimal characters. This is well-defined, independent of all choices.

For simplicity we restrict to $GL(n, \mathbb{R})$ from now on.

Lemma 8.3 *Suppose π is an admissible representation of $GL(n, \mathbb{R})$, admitting an infinitesimal character $\chi_{inf}(\pi)$, and a central character $\chi(\pi)$. Then:*

(a) $\chi_{inf}(\pi^h) = -\overline{\chi_{inf}(\pi)}$,

(b) $\chi(\pi^h) = \chi(\pi)^h$,

(c) Suppose $P(\mathbb{R}) = M(\mathbb{R})N(\mathbb{R})$ is a parabolic subgroup of $GL(n, \mathbb{R})$, and π_M is an admissible representation of $M(\mathbb{R})$. Then $\text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_M^h) \simeq \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi_M)^h$.

In (b) $\chi(\pi)^h$ refers to the Hermitian dual of the one-dimensional representation of $Z(G(\mathbb{R})) = \mathbb{R}^*$.

Proof. The first assertion is easy if π is a minimal principal series representation. Since any irreducible representation embeds in a minimal principal series, (a) follows. Statement (b) is elementary, and (c) is an easy variant of (7.4). We leave the details to the reader. □

Now suppose ϕ is a finite-dimensional representation of $W_{\mathbb{R}}$ on a complex vector space V. Define a representation ϕ^h on the space V^h of conjugate linear functions $V \to \mathbb{C}$ by: $\phi(w)(f)(v) = f(\phi(w^{-1})v)$ $(w \in W_{\mathbb{R}}, f \in V^h, v \in V)$. Choosing dual bases of V, V^h, identify $GL(V), GL(V^h)$ with $GL(n, \mathbb{C})$, to write $\phi^h = {}^t\overline{\phi}^{-1}$.

It is elementary that ϕ has a nondegenerate invariant Hermitian form if and only if $\phi \simeq \phi^h$.

Recall from the introduction that irreducible admissible representations of $GL(n, \mathbb{R})$ are parametrized by equivalence classes of n-dimensional semisimple representations of $W_{\mathbb{R}}$. Write $\phi \to \pi(\phi)$ for this correspondence.

Lemma 8.4 *Suppose ϕ is an n-dimensional semisimple representation of $W_{\mathbb{R}}$. Then*

1. $\chi_{inf}(\phi^h) = -\overline{\chi_{inf}(\phi)}$,
2. $\chi_{rad}(\phi^h) = \chi_{rad}(\phi)^h$.

Proof. Let ${}^\vee H$ be the diagonal torus in $GL(n, \mathbb{C})$. As in (6.4), write $\phi(z) = z^\lambda \overline{z}^{\lambda'}$, so $\chi_{\mathrm{inf}}(\phi) = \lambda$. On the other hand $\phi^h(z) = \overline{z^\lambda \overline{z}^{\lambda'}}^{-1}$, and it is easy to see this equals $z^{-\overline{\lambda'}}\overline{z}^{-\overline{\lambda}}$, where $\overline{\lambda}$ is complex conjugatation with respect to $X_*({}^\vee H) \otimes \mathbb{R}$. Therefore $\chi_{\mathrm{inf}}(\phi^h) = -\overline{\lambda'}$. Then (1) follows from the fact that, by (6.4)(d), λ is $GL(n, \mathbb{C})$-conjugate to λ'.

The second claim comes down to the case of tori, which we leave to the reader.

□

Proof of Theorem 1.5. The equivalence of (1) and (2) follow from the preceding discussion. For (3), it is well known (and a straightforward exercise) that $\pi(\phi)$ is tempered if and only if $\phi(W_{\mathbb{R}})$ is bounded [7, 10.3(4)], which is equivalent to ϕ being unitary.

The proof of (1) is parallel to that of Theorem 1.3, using the previous two lemmas, and our characterization of the Langlands classification in terms of infinitesimal character, radical character, and compatibility with parabolic induction. We leave the few remaining details to the reader. □

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