4. LIE ALGEBRA COHOMOLOGY

This is a quick-and-dirty introduction to Lie algebra cohomology. A reasonable setting in which to work is this: k is a base field (any characteristic is OK, and algebraic closure doesn't matter). We fix a Lie algebra $\mathfrak{g}$ over k, a subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, and an ideal $\mathfrak{u}$ of $\mathfrak{q}$:

$$\mathfrak{g} \supset \mathfrak{q} \supset \mathfrak{u}, \qquad [\mathfrak{q}, \mathfrak{u}] \subset \mathfrak{u}. \tag{4.1}$$

The main example we'll be interested in is $\mathfrak{q}$ a parabolic subalgebra of a reductive $\mathfrak{g}$, with nil radical $\mathfrak{u}$; but not even finite-dimensionality of the algebras matters for the definitions.

The definition of Lie algebra cohomology lives in the world of modules over rings. We write

$$\mathcal{M}(\mathfrak{g}) = \text{category of modules (over } k\text{) for the Lie algebra } \mathfrak{g}. \tag{4.2}$$

This is the same thing as the category of modules over the associative ring $U(\mathfrak{g})$, so we can use freely ideas from that subject. Our point of view is largely that of [CE]: we will manage with nothing more abstract than categories of modules, and derived categories will remain only a distant rumor. Complete details and much more can be found also in [Knapp].

A *covariant functor* $\mathcal{F}$ from a category $\mathcal{A}$ to another category $\mathcal{B}$ associates to each object M of $\mathcal{A}$ an object $\mathcal{F}(M)$ of $\mathcal{B}$; and to each map f from objects M to N of $\mathcal{A}$, associates a map $\mathcal{F}(f)$ from $\mathcal{F}(M)$ to $\mathcal{F}(N)$. These associations are required to send the identity map to the identity map, and to respect composition. A *contravariant functor* $\mathcal{G}$ is formally similar except that it reverses all arrows, associating to each map g from M to N a map $\mathcal{G}(g)$ from $\mathcal{F}(N)$ to $\mathcal{F}(M)$.

We can now say that $\mathfrak{u}$-cohomology is a collection of functors $H^i(\mathfrak{u}, \cdot)$, for $i = 0, 1, 2, \dots$, each going from $\mathcal{M}(\mathfrak{g})$ to $\mathcal{M}(\mathfrak{q}/\mathfrak{u})$. Before defining these functors, I will state three of their fundamental properties. (One natural perspective is that these properties *are* the definition; the general arguments of homological algebra show how to construct uniquely functors with these properties. But I will write a more mundane definition.)

$$H^0(\mathfrak{u}, M) = \{m \in M \mid x \cdot m = 0, \text{all } x \in \mathfrak{u}\} =_{\text{def}} M^{\mathfrak{u}}. \tag{4.3)(a}$$

Whenever

$$0 \to A \to B \to C \to 0 \tag{4.3)(b}$$

is a short exact sequence of $\mathfrak{g}$-modules, there is long exact sequence of $\mathfrak{q}/\mathfrak{u}$-modules

$$\begin{array}{ccccccl} 0 & \to & H^0(\mathfrak{u}, A) & \to & H^0(\mathfrak{u}, B) & \to & H^0(\mathfrak{u}, C) \\ & \to & H^1(\mathfrak{u}, A) & \to & H^1(\mathfrak{u}, B) & \to & H^1(\mathfrak{u}, C) \\ & \to & H^2(\mathfrak{u}, A) & \to & H^2(\mathfrak{u}, B) & \to & H^2(\mathfrak{u}, C) \quad \cdots \end{array} \tag{4.3)(c}$$

Here the maps within each row (like $H^i(\mathfrak{u}, B) \to H^i(\mathfrak{u}, C)$) arise by applying the functor H^i to the maps in (4.3)(b). The maps from one row to the next (like $H^i(\mathfrak{u}, C) \to H^{i+1}(\mathfrak{u}, A)$) are called *connecting homomorphisms*, and are something new.

To state the third fundamental property of cohomology, we need a notion from module theory. A (left) module I for a ring R is called *injective* if the contravariant functor $\mathrm{Hom}_R(\cdot, I)$ (from R-modules to abelian groups) is exact. Making this explicit in our case, the requirement on I is that if M is any $\mathfrak{g}$-module and S is any submodule, then any $\mathfrak{g}$-module homomorphism from S to I can be extended to a $\mathfrak{g}$-module homomorphism from M to I. Pictorially, the assertion is that if we are given maps corresponding to the first two arrows in the following diagram, then there exists a map corresponding to the last (vertical) arrow so that the diagram commutes.

$$\begin{array}{ccc} S & \hookrightarrow & M \\ & \searrow & \downarrow \\ & & I \end{array} \tag{I}$$

The notion of injective module is formally very similar to the (possibly more familiar) notion of projective module. An R-module P is called *projective* if the covariant functor $\mathrm{Hom}_R(P, \cdot)$ from R-modules to abelian groups is exact. This means that if N is any R-module and Q any quotient of N, then any R-module homomorphism from P to Q lifts to a homomorphism from P to N. Here is the corresponding picture; the assertion is that if we are given the two arrows on the right, then we can find a vertical map on the left making the diagram commute.

$$\begin{array}{ccc} N & \twoheadrightarrow & Q \\ \uparrow & \nearrow & \\ P & & \end{array} \tag{P}$$

The third fundamental property is that whenever I is an injective $\mathfrak{g}$-module, then

$$H^i(\mathfrak{u}, I) = 0, \qquad \text{all } i > 0. \tag{4.3)(d}$$

I'll turn next to a direct and mundane definition of Lie algebra cohomology. One almost trivial bit of linear algebra is helpful.

Lemma 4.4. *Suppose V and W are vector spaces over a field k. Write $\wedge^i V$ for the ith exterior power of V, the quotient of the ith tensor power by the relation making multiplication antisymmetric. Then there is a natural identification*

$$\mathrm{Hom}_k(\wedge^i V, W) \simeq \{\omega\colon \underbrace{V \times \cdots \times V}_{i \text{ factors}} \to W \mid \omega \textit{ is linear in each variable and}$$
$$\textit{vanishes if two arguments are equal}\}.$$

Definition 4.5. In the setting (4.1), suppose M is a $\mathfrak{g}$-module. Regard

$$\mathrm{Hom}_k(\wedge^i \mathfrak{u}, M) \tag{4.5)(a}$$

as a $\mathfrak{q}$-module, using Hom of the (ith exterior power of the) adjoint action of $\mathfrak{q}$ on $\mathfrak{u}$ into the (restriction of the action of $\mathfrak{g}$ to the) action of $\mathfrak{q}$ on M. Define

$$d\colon \mathrm{Hom}_k(\wedge^i \mathfrak{u}, M) \to \mathrm{Hom}_k(\wedge^{i+1} \mathfrak{u}, M), \tag{4.5)(b}$$

$$d\omega(X_0 \dots X_i) = \sum_{j=0}^{i} (-1)^j X_j \cdot \omega(X_0 \dots \widehat{X}_j \dots X_i) + \sum_{0 \le j < k \le i} (-1)^{j+k} \omega([X_j, X_k], X_0 \dots \widehat{X_j} \dots \widehat{X_k} \dots X_i). \tag{4.5)(c}$$

Here a hat over an argument means that it is omitted. We use the identification in Lemma 4.4 throughout. An elementary calculation shows that d intertwines the action of $\mathfrak{q}$, and that $d^2 = 0$. The maps d therefore make the spaces (4.5)(a) into a chain complex of $\mathfrak{q}$-modules. We define

$$\begin{aligned} H^i(\mathfrak{u}, M) &= i\text{th cohomology of the complex} \\ &= \ker d \text{ on } \mathrm{Hom}_k(\wedge^i \mathfrak{u}, M)/\mathrm{im}\, d \text{ on } \mathrm{Hom}_k(\wedge^{i-1}\mathfrak{u}, M) \end{aligned} \tag{4.5)(d}$$

As I said in class, the formula for d mimics a standard one in differential geometry. There ω is an i-form on a manifold, regarded as a (special kind of) i-linear map from vector fields to smooth functions. At this point one should pause to verify that the functors H^i of Definition 4.5 are really functors, and that they satisfy the requirements (4.3). If you accept the assertion in the definition that d^2 is easily seen to be zero, and that d respects the action of $\mathfrak{q}$, what follows is only that $H^i(\mathfrak{u}, M)$ is a $\mathfrak{q}$-module. We need to see that $\mathfrak{u}$ acts by zero on the cohomology. For that, suppose $X \in \mathfrak{u}$. *Interior multiplication by* X is a map

$$\iota(X)\colon \mathrm{Hom}_k(\wedge^i \mathfrak{u}, M) \to \mathrm{Hom}_k(\wedge^{i-1}\mathfrak{u}, M), \tag{4.6)(a}$$

$$(\iota(X)\omega)(X_2 \dots X_i) = \omega(X, X_2 \dots X_i). \tag{4.6)(b}$$

Write $\theta(X)$ for the action of X on the complex of Definition 4.5. Then another straightforward calculation shows that

$$d\iota(X) + \iota(X)d = \theta(X), \tag{4.6)(c}$$

both sides being linear maps from $\mathrm{Hom}_k(\wedge^i \mathfrak{u}, M)$ to itself. (This is again an analogue of a formula from differential geometry.) If now ω is an i-cocycle (an element of the kernel of d), then it follows that

$$X \cdot \omega = \theta(X)\omega = d(\iota(X)\omega), \tag{4.6)(d}$$

so that $X \cdot \omega$ is a coboundary. (In differential geometry, the conclusion is that the action of vector fields on differential forms by Lie derivative is zero on de Rham cohomology.) This shows that $H^i(\mathfrak{u}, M)$ is $\mathfrak{q}/\mathfrak{u}$-module.

The complex defining the cohomology is functorial: any $\mathfrak{g}$-module map from M to M' defines a $\mathfrak{q}$-module map of complexes

$$\mathrm{Hom}_k(\wedge^* \mathfrak{u}, M) \to \mathrm{Hom}_k(\wedge^* \mathfrak{u}, M'),$$

which induces $\mathfrak{q}$-module maps on cohomology. In the setting of (4.3)(b), the sequence of complexes

$$0 \to \mathrm{Hom}_k(\wedge^* \mathfrak{u}, A) \to \mathrm{Hom}_k(\wedge^* \mathfrak{u}, B) \to \mathrm{Hom}_k(\wedge^* \mathfrak{u}, C) \to 0$$

is exact, and the existence of connecting homomorphisms as required in (4.3)(c) follows by the usual diagram chase.

The property (4.3)(d) (that higher cohomology vanishes for injective $\mathfrak{g}$-modules) requires a little more work; and you may wonder why anyone one should bother. For pedagogical reasons, I'll outline the work first; but you may wish to skip ahead to the Casselman-Osborne theorem, to see a reason to bother.

In order to verify (4.3)(d), we need to understand something about what injective $\mathfrak{g}$-modules look like. The theory for projective modules is formally parallel but perhaps a little more familiar, so I will discuss that at the same time. The general results about rings and modules here are from [CE], II.6.

Definition 4.7. Suppose R and S are associative rings with unit, and $\phi\colon S \to R$ is a ring homomorphism. If M is any (left) S-module, then

$$ {}_{\phi}M =_{\text{def}} R \otimes_S M \tag{4.7)(a} $$

is a (left) R-module. This defines a covariant functor from S-modules to R-modules, called *extension of scalars*. This functor is a left adjoint to the "forgetful" functor $\mathcal{F}$ making any R-module into an S-module. This statement means that if N is any left R-module, then there is a natural isomorphism (the "adjunction formula")

$$ \mathrm{Hom}_R({}_{\phi}M, N) \simeq \mathrm{Hom}_S(M, \mathcal{F}N). \tag{4.7)(b} $$

The isomorphism identifies a map t on the right with T on the left if

$$ T(r \otimes m) = r \cdot t(m), \qquad t(m) = T(1 \otimes m). \tag{4.7)(c} $$

The functor $\mathcal{F}$ is obviously exact, because it doesn't change N as a set. The exactness of $\mathcal{F}$ and the adjunction formula together imply that extension of scalars is right exact. If R is flat as a right S-module—for example, if it is free—then the tensor product formula shows that extension of scalars is exact.

Similarly, we can define a left R-module

$$ {}^{\phi}M =_{\text{def}} \mathrm{Hom}_S(R, M) \tag{4.7)(d} $$

Here the action of R on the right is given by

$$ (r \cdot \mu)(x) = \mu(xr) \qquad (\mu \in {}^{\phi}M, r \in R, x \in R). \tag{4.7)(e} $$

This is also a covariant functor from S-modules to R-modules, which we will call *coextension of scalars*. (There doesn't seem to be a good name for it. Cartan and Eilenberg say "contravariant extension of scalars," which seems terrible because it's a covariant functor.) This functor is a right adjoint to the forgetful functor: if N is any left R-module, then there is a natural isomorphism (the "adjunction formula")

$$ \mathrm{Hom}_R(N, {}^{\phi}M) \simeq \mathrm{Hom}_S(\mathcal{F}N, M). \tag{4.7)(f} $$

This time the isomorphism identifies a map T on the left with t on the right if

$$ T(n)(r) = r \cdot t(n), \qquad t(n) = T(n)(1). \tag{4.7)(g} $$

The exactness of $\mathcal{F}$ and the adjunction formula together imply that coextension of scalars is left exact. If R is projective as a left S-module—for example, if it is free—then the definition of ${}^{\phi}M$ as a Hom shows that coextension of scalars is exact.

Proposition 4.8. *([CE], Proposition II.6.1) Suppose we are in the setting of Definition 4.7.*

(1) *If P is a projective S-module, then the extension of scalars*

$$_\phi P = R \otimes_S P$$

is a projective R-module.

(2) *Suppose that M is any R-module, that P is an S-module, and that $q\colon P \to \mathcal{F}M$ is a surjective map of S-modules. Then the R-module homomorphism*

$$Q\colon {_\phi P} \to M$$

corresponding to q under the adjunction isomorphism (4.7)(b) is a surjective map of R-modules.

(3) *If I is an injective S-module, then the coextension of scalars*

$$^\phi I = \mathrm{Hom}_S(R, I)$$

is an injective R-module.

(4) *Suppose that M is any R-module, that I is an S-module, and that $j\colon \mathcal{F}M \to I$ is an injective map of S-modules. Then the R-module homomorphism*

$$J\colon M \to {^\phi I}$$

corresponding to j under the adjunction isomorphism (4.7)(f) is an injective map of R-modules.

Proof. For (1), according to the adjunction formula (4.7)(b),

$$\mathrm{Hom}_R({_\phi P}, N) \simeq \mathrm{Hom}_S(P, \mathcal{F}N).$$

The right side is exact in N, because the forgetful functor is exact and P is projective. The proof for (3) is identical. For (2), the formula (4.7)(c) shows that Q is surjective (already on the subset of elements $1 \otimes p$) if q is. The proof for (4) is identical. □

One of the basic facts of homological algebra is that any R-module is a quotient of a projective R-module, and a submodule of an injective R-module. Proposition 4.8 reduces the proof of this fact to the case $R = \mathbb{Z}$; a proof in that case can be found in [CE], VII.5. The rings we are interested in all contain fields; and in that case Proposition 4.8 reduces the fact to the case of fields. Any module over a field (that is to say, any vector space) is both projective and injective (assuming the axiom of choice); for injectivity, this is just the statement that any linear map defined on a subspace can be extended linearly to the whole vector space. Here is a statement in the case of enveloping algebras.

Corollary 4.9. *Suppose that $\mathfrak{g}$ is a Lie algebra over a field k.*

(1) *Suppose V is a vector space over k. Then $\mathrm{Hom}_k(U(\mathfrak{g}), V)$ is an injective $\mathfrak{g}$-module.*
(2) *Any $\mathfrak{g}$-module is a submodule of some $\mathrm{Hom}_k(U(\mathfrak{g}), V)$.*
(3) *Any injective $\mathfrak{g}$-module is a direct summand of some $\mathrm{Hom}_k(U(\mathfrak{g}), V)$.*

Proof. Since V is an injective k-module, (1) and (2) follow from Proposition 4.8. For (3), given an injective $\mathfrak{g}$-module I, construct an inclusion $I \hookrightarrow \mathrm{Hom}_k(U(\mathfrak{g}), V)$, and write I' for the image of I. According to the injectivity of I, the $\mathfrak{g}$-module isomorphism $I' \to I$ (the inverse of the inclusion) extends to a $\mathfrak{g}$-module map

$$\xi\colon \mathrm{Hom}_k(U(\mathfrak{g}), V) \to I.$$

It's easy to see that $\mathrm{Hom}_k(U(\mathfrak{g}), V)$ is the direct sum of I' and the kernel of ξ. $\square$

Here is the proof of property (4.3)(d).

Theorem 4.10. *In the setting (4.1), suppose I is an injective $U(\mathfrak{g})$-module. Then $H^i(\mathfrak{u}, I) = 0$ for $i > 0$.*

Sketch of proof. By Corollary 4.9, (and since the cohomology of a direct sum is obviously the direct sum of the cohomologies) we may assume that $I = \mathrm{Hom}_k(U(\mathfrak{g}), V)$. By the Poincaré-Birkhoff-Witt Theorem, $U(\mathfrak{g})$ is free as a right $U(\mathfrak{u})$-module. We can therefore find a vector subspace W of $U(\mathfrak{g})$ so that $U(\mathfrak{g}) \simeq W \otimes_k U(\mathfrak{u})$. It follows that

$$I \simeq \mathrm{Hom}_k(U(\mathfrak{u}), \mathrm{Hom}_k(W, V)) \simeq \mathrm{Hom}_k(U(\mathfrak{u}), H)$$

as a $U(\mathfrak{u})$-module; here H is the vector space $\mathrm{Hom}_k(W, V)$.

Now consider the complex computing Lie algebra cohomology of I. The terms are

$$\mathrm{Hom}_k(\wedge^i \mathfrak{u}, \mathrm{Hom}_k(U(\mathfrak{u}), H)) \simeq \mathrm{Hom}_k(U(\mathfrak{u}) \otimes \wedge^i \mathfrak{u}, H).$$

The differentials arise as $\mathrm{Hom}(\cdot, H)$ of boundary operators

$$\partial\colon U(\mathfrak{u}) \otimes \wedge^{i+1}\mathfrak{u} \to U(\mathfrak{u}) \otimes \wedge^i \mathfrak{u}, \tag{4.11)(a}$$

$$\begin{aligned}\partial(u \otimes X_0 \wedge \cdots \wedge X_i) = &\sum_j (-1)^j u X_j \otimes X_0 \wedge \cdots \widehat{X_j} \cdots \wedge X_i \\ &+ \sum_{j<k} (-1)^{j+k} u \otimes [X_j, X_k] \wedge \cdots \widehat{X_j} \cdots \widehat{X_k} \cdots \wedge X_i.\end{aligned} \tag{4.11)(b}$$

It is therefore sufficient to prove that the homology of this last complex vanishes except in degree 0.

For that, recall the standard increasing filtration of $U(\mathfrak{u})$: the subspace $U_n(\mathfrak{u})$ is the span of products of at most n terms in $\mathfrak{u}$. If we define

$$C^i_n = U_{n-i}(\mathfrak{u}) \otimes \wedge^i \mathfrak{u}, \tag{4.11)(c}$$

then it is clear from (4.11)(b) that

$$C^i_0 \subset C^i_1 \subset \cdots, \qquad \bigcup_n C^i_n = U(\mathfrak{u}) \otimes \wedge^i \mathfrak{u}, \qquad \partial\colon C^{i+1}_n \to C^i_n. \tag{4.11)(d}$$

The Poincaré-Birkhoff-Witt Theorem allows us to identify

$$C^i_n / C^i_{n-1} \simeq S^{n-i}(\mathfrak{u}) \otimes \wedge^i \mathfrak{u}. \tag{4.11)(e}$$

According to (4.11)(d), the boundary operator defines by passage to the quotient

$$\partial_n \colon S^{n-(i+1)}(\mathfrak{u}) \otimes \wedge^{i+1}\mathfrak{u} \to S^{n-i}(\mathfrak{u}) \otimes \wedge^{i}\mathfrak{u}. \tag{4.11)(f}$$

This is a Koszul complex. It is a standard and elementary fact that this complex is exact except at $i = n = 0$ (where the homology is one-dimensional). I have not found a really convenient reference for this statement; it is the content of IV.6 in [Knapp].

Another elementary argument (also in IV.6 of [Knapp]) passes from the exactness of the associated graded complex back to exactness of (4.11)(a), and proves the theorem. □

If I get ambitious I'll add an account of the Casselman-Osborne theorem.

References

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, New Jersey, 1956.

[Knapp] A. Knapp, *Lie Groups, Lie Algebras, and Cohomology*, Mathematical Notes 34, Princeton University Press, Princeton, New Jersey, 1988.