CLASSICAL GROUPS

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1. Orthogonal groups

These notes are about "classical groups." That term is used in various ways by various people; I'll try to say a little about that as I go along. Basically these are groups of matrices with entries in fields or division algebras.

To warm up, I'll recall a definition of the orthogonal group.

Definition 1.1. Suppose $n \geq 1$ is an integer. The *real orthogonal group* $O(n)$ is

$$O(n) = \left\{\text{all } n \times n \text{ real matrices } g \text{ such that } {}^t gg = I\right\}.$$

Here if A is an $m \times n$ matrix, ${}^tA$ is the $n \times n$ matrix whose (j,i) entry is equal to the (i,j) entry of A.

Here is another way to formulate the definition. Write

$$\mathbb{R}^n = \{\text{column vectors of size } n\} = \{n \times 1 \text{ matrices}\} \tag{1.2a}$$

and identify

$$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \simeq \{m \times n \text{ real matrices } A\} =_{\mathrm{def}} M_{m\times n}(\mathbb{R}) \tag{1.2b}$$

by matrix multiplication on the left. For the (most important) case of square matrices, we will write

$$M_n(\mathbb{R}) =_{\mathrm{def}} M_{n\times n}(\mathbb{R}); \tag{1.2c}$$

this is a real algebra under matrix multiplication, and a real Lie algebra under commutator of matrices:

$$[A, B] = AB - BA \quad (A, B \in M_n(\mathbb{R}). \tag{1.2d}$$

Write

$$(,) = (,)_n, \quad (v, w) = {}^tv \cdot w, \qquad (v, w \in \mathbb{R}^n) \tag{1.2e}$$

for the standard inner product on $\mathbb{R}^n$. Then

$$(Av, w)_m = \left(v, {}^tAw\right)_n \qquad (v \in \mathbb{R}^n, w \in \mathbb{R}^m, A \in M_{m\times n}(\mathbb{R}). \tag{1.2f}$$

With this notation, it is very easy to check that

$$O(n) = \{\text{linear transformations of } \mathbb{R}^n \text{ preserving } (,)_n\}. \tag{1.3}$$

The orthogonal group is the first classical group. The zeroth classical group is

$$GL(n,\mathbb{R}) = \{\text{all invertible } n\times n \text{ matrices}\}$$
$$= \{\text{invertible linear transformations of } \mathbb{R}^n\}\,. \tag{1.4}$$

This is a Lie group or a real algebraic group; from either of these perspectives, we compute

$$\mathrm{Lie}(GL(n,\mathbb{R})) =_{\mathrm{def}} \mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R}),$$
$$\dim_{\mathbb{R}}(GL(n,\mathbb{R})) = n^2; \tag{1.5}$$

the Lie algebra structure is the commutator of matrices defined above.

The descriptions above make it clear that $O(n)$ is a (closed) Lie subgroup of $GL(n,\mathbb{R})$, and a real algebraic subgroup. We can compute

$$\mathrm{Lie}(O(n)) =_{\mathrm{def}} \mathfrak{o}(n) = \{X \in M_n(\mathbb{R}) \mid {}^tX = -X\},$$
$$\dim_{\mathbb{R}}(O(n)) = n(n-1)/2. \tag{1.6}$$

The goal of these notes is to write down descriptions like Definition 1.1, (1.3), and (1.6) of other classical groups.

Already in this case, one can see where ambiguity might arise about what constitutes a classical group. There is a short exact sequence (recall that $n \ge 1$)

$$1 \to SO(n) \longrightarrow O(n) \xrightarrow{\det} \{\pm 1\} \to 1 \tag{1.7}$$

which is the definition of the *special orthogonal group* $SO(n)$. It is the identity component of $O(n)$, and therefore has the same dimension and the same Lie algebra.

Because there are lots of nice theorems about connected compact Lie groups, some people prefer $SO(n)$ to $O(n)$, and like to call $SO(n)$ a classical group. Others are interested in simply connected groups, or only in the Lie algebra, and so like to call the double cover $Spin(n)$ of $SO(n)$ a classical group. But there are some subtle theorems about $O(n)$ that actually fail for $SO(n)$. An example is the fact that $O(n)$ acts irreducibly on the space of complex harmonic polynomials of degree m whenever this space is not zero. There are similar difficulties with $Spin(n)$. I will therefore take the second point of view and regard $O(n)$ as the *only* compact classical orthogonal group.

2. Unitary groups

To get another family of groups "like" the orthogonal groups, we can replace the field $\mathbb{R}$ by $\mathbb{C}$. Write

$$\mathbb{C}^n = \{\text{complex column vectors of size } n\}$$
$$= \{n\times 1 \text{ complex matrices})\} \tag{2.1a}$$

and identify

$$\text{(2.1b)} \qquad \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \simeq \{m \times n \text{ complex matrices } A\} =_{\text{def}} M_{m\times n}(\mathbb{C})$$

by matrix multiplication on the left. For square matrices, we write

$$\text{(2.1c)} \qquad M_n(\mathbb{C}) =_{\text{def}} M_{n\times n}(\mathbb{C});$$

this is a complex algebra under matrix multiplication, and a complex Lie algebra under commutator of matrices:

$$\text{(2.1d)} \qquad [A, B] = AB - BA \quad (A, B \in M_n(\mathbb{C}).$$

In this setting we have a complex Lie group, or complex algebraic group

$$\text{(2.1e)} \qquad \begin{aligned} GL(n, \mathbb{C}) &= \{\text{all invertible } n \times n \text{ complex matrices}\} \\ &= \{\text{invertible linear transformations of } \mathbb{C}^n\}. \end{aligned}$$

We compute

$$\text{(2.1f)} \qquad \begin{aligned} \mathrm{Lie}(GL(n, \mathbb{C})) &=_{\text{def}} \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C}), \\ \dim_{\mathbb{C}}(GL(n, \mathbb{C})) &= n^2; \end{aligned}$$

the Lie algebra structure is the commutator of matrices. The ordinary real Lie algebra is the same vector space with the complex structure forgotten; so

$$\text{(2.1g)} \qquad \dim_{\mathbb{R}}(GL(n, \mathbb{C})) = 2n^2.$$

Write

$$\text{(2.1h)} \qquad \langle, \rangle = \langle, \rangle_n, \quad \langle v, w\rangle = {}^t\overline{v} \cdot \overline{w}, \qquad (v, w \in \mathbb{C}^n)$$

for the standard Hermitian form on $\mathbb{C}^n$. Then

$$\text{(2.1i)} \qquad \langle Av, w\rangle_m = \langle v, {}^t\overline{A}w\rangle_n \qquad (v \in \mathbb{C}^n, w \in \mathbb{C}^m, A \in M_{m\times n}(\mathbb{C})).$$

It is convenient to define

$$\text{(2.1j)} \qquad A^h =_{\text{def}} {}^t\overline{A} \qquad (A \in M_{m\times n}(\mathbb{C})),$$

the *Hermitian transpose* of A.

Definition 2.2. Suppose $n \geq 1$ is an integer. The *unitary group* $U(n)$ is

$$U(n) = \left\{\text{all } n \times n \text{ complex matrices } g \text{ such that } g^h g = I\right\}.$$

Equivalently,

$$U(n) = \{\text{linear transformations of } \mathbb{C}^n \text{ preserving } \langle, \rangle_n\}.$$

It is clear from the definition that $U(n)$ is a closed real Lie subgroup, or a real algebraic subgroup, of $GL(n, \mathbb{C})$. Because of the complex conjugates appearing in the definition, $U(n)$ is *not* a complex Lie subgroup or a complex algebraic group. We calculate

$$\text{(2.3a)} \qquad \mathrm{Lie}(U(n)) =_{\text{def}} \mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^h = -X\},$$

the set of *skew-Hermitian* complex matrices. The condition on X is that each diagonal entry is purely imaginary, and the entries above the diagonal are the complex conjugates of the corresponding entries below the diagonal. We calculate

$$\dim_{\mathbb{R}}(U(n)) = n + 2(n(n-1)/2) = n^2. \tag{2.3b}$$

The group $U(n)$ is a classical group. Also interesting (but not for us a classical group) is the *special unitary group*, defined by the short exact sequence

$$1 \to SU(n) \longrightarrow U(n) \xrightarrow{\det} U(1) \to 1; \tag{2.3c}$$

here $U(1)$ is the circle group of complex numbers of absolute value 1. The (connected) subgroup $SU(n)$ is the commutator subgroup of $U(n)$. We can calculate

$$\mathrm{Lie}(SU(n)) =_{\mathrm{def}} \mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X^h = -X, \mathrm{tr}(X) = 0\}, \tag{2.3d}$$

the Lie algebra of skew Hermitian matrices of trace zero. Consequently

$$\dim_{\mathbb{R}}(SU(n)) = n^2 - 1. \tag{2.3e}$$

3. Quaternion groups

For the last family of compact classical groups, we replace the field by the division algebra $\mathbb{H}$. Write

$$\begin{aligned} \mathbb{H}^n &= \{\text{quaternion column vectors of size } n\} \\ &= \{n \times 1 \text{ quaternion matrices})\}. \end{aligned} \tag{3.1a}$$

We regard $\mathbb{H}^n$ as a *right* vector space over $\mathbb{H}$; scalar multiplication is on the right, one coordinate at a time. Then and identify

$$\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^m) \simeq \{m \times n \text{ quaternion matrices } A\} =_{\mathrm{def}} M_{m\times n}(\mathbb{H}) \tag{3.1b}$$

by matrix multiplication on the left. For square matrices, we write

$$M_n(\mathbb{C}) =_{\mathrm{def}} M_{n\times n}(\mathbb{C}); \tag{3.1c}$$

this is a real algebra under matrix multiplication, and a real Lie algebra under commutator of matrices:

$$[A, B] = AB - BA \quad (A, B \in M_n(\mathbb{H}). \tag{3.1d}$$

(There is no obvious notion of a Lie algebra over a division ring; square matrices over a division ring form an algebra or a Lie algebra only over the *center* of the division ring, which in this case is $\mathbb{R}$.) In this setting we have a real Lie group, or real algebraic group

$$\begin{aligned} GL(n, \mathbb{H}) &= \{\text{all invertible } n \times n \text{ quaternion matrices}\} \\ &= \{\text{invertible linear transformations of } \mathbb{H}^n\}. \end{aligned} \tag{3.1e}$$

We compute

$$\begin{aligned} \mathrm{Lie}(GL(n, \mathbb{H})) &=_{\mathrm{def}} \mathfrak{gl}(n, \mathbb{H}) = M_n(\mathbb{H}), \\ \dim_{\mathbb{R}}(GL(n, \mathbb{H})) &= 4n^2; \end{aligned} \tag{3.1f}$$

the Lie algebra structure is the commutator of matrices.

Recall the definition of conjugation on $\mathbb{H}$:

$$\overline{a+bi+cj+dk} = a - bi - cj - dk, \qquad \overline{h_1 h_2} = \overline{h_2}\,\overline{h_1}. \tag{3.1g}$$

That is, conjugation is an *antiautomorphism* of the real algebra $\mathbb{H}$.

Write

$$\langle , \rangle = \langle , \rangle_n, \quad \langle v, w \rangle = {}^t\overline{v} \cdot \overline{w}, \qquad (v, w \in \mathbb{H}^n) \tag{3.1h}$$

for the standard Hermitian form on $\mathbb{H}^n$. Then

$$\langle Av, w \rangle_m = \langle v, {}^t\overline{A}w \rangle_n \qquad (v \in \mathbb{H}^n, w \in \mathbb{C}^m, A \in M_{m\times n}(\mathbb{C})). \tag{3.1i}$$

It is convenient to define

$$A^h =_{\text{def}} {}^t\overline{A} \qquad (A \in M_{m\times n}(\mathbb{H})), \tag{3.1j}$$

the *Hermitian transpose* of A.

Definition 3.2. Suppose $n \geq 1$ is an integer. The *quaternionic unitary group* $Sp(n)$ is

$$Sp(n) = \left\{ \text{all } n \times n \text{ quaternion matrices } g \text{ such that } g^h g = I \right\}.$$

Equivalently,

$$Sp(n) = \{\text{linear transformations of } \mathbb{H}^n \text{ preserving } \langle , \rangle_n\}.$$

Another natural and commonly used notation for this group is $U(n, \mathbb{H})$.

It is clear from the definition that $Sp(n)$ is a closed real Lie subgroup, or a real algebraic subgroup, of $GL(n, \mathbb{H})$.

$$\text{Lie}(Sp(n)) =_{\text{def}} \mathfrak{sp}(n) = \{X \in M_n(\mathbb{H}) \mid X^h = -X\}, \tag{3.3a}$$

the set of *skew-Hermitian* quaternion matrices. The condition on X is that each diagonal entry is purely imaginary (that is, has real part zero), and the entries above the diagonal are the conjugates of the corresponding entries below the diagonal. We calculate

$$\dim_{\mathbb{R}}(Sp(n)) = 3n + 4(n(n-1)/2) = 2n^2 + n. \tag{3.3b}$$

The group $Sp(n)$ is a classical group. There is no $\mathbb{H}$-valued "determinant" for quaternionic matrices; all that can be defined is the real determinant of the matrix regarded as a real-linear transformation of a $4n$-dimensional space. This determinant is automatically one on $Sp(n)$, so there is no smaller "quaternionic special unitary group."

4. Natural inclusions

The inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ define group inclusions

$$O(n) \hookrightarrow U(n) \hookrightarrow Sp(n), \tag{4.1}$$

just by regarding a matrix over a small field as a matrix over a larger field (or division algebra). Similarly, the identifications $\mathbb{H} \simeq \mathbb{C}^2 \simeq \mathbb{R}^4$ and $\mathbb{C} \simeq \mathbb{R}^2$ define inclusions

$$Sp(n) \hookrightarrow U(2n) \hookrightarrow O(4n), \qquad U(n) \hookrightarrow O(2n). \tag{4.2}$$

These inclusions actually take values in the determinant one subgroups:

$$Sp(n) \hookrightarrow SU(2n) \hookrightarrow SO(4n), \qquad U(n) \hookrightarrow SO(2n). \tag{4.3}$$

In case $n = 1$, we find

$$\dim Sp(1) = 2 \cdot 1^2 + 1 = 3 = 2^2 - 1 = \dim SU(2); \tag{4.4a}$$

It follows that the first inclusion above is an isomorphism

$$Sp(1) \simeq SU(2), \quad a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & -c - di \\ c - di & a - bi \end{pmatrix}. \tag{4.4b}$$

Another way to write this isomorphism explicitly is
(4.4c)
$$Sp(1) = \text{unit quaternions} = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\} \simeq S^3,$$
the unit sphere in $\mathbb{R}^4$. Similarly

$$SU(2) = \left\{ \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}. \tag{4.4d}$$

The isomorphism of $SU(2)$ with $Sp(1)$ sends the complex numbers (z, w) to the quaternion $z + wj$.