Arthur representations and the unitary dual

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Mathematical Legacy of James Arthur

August 11–15, 2025

Outline

Introduction and outline

Real parabolic induction

Cohomological parabolic induction

Unipotent representations

Slides at `http://www-math.mit.edu/~dav/paper.html`

Most interesting problem in mathematics...

...is understanding the unitary dual of G.

Meaning: all ways G can be automorphisms of a Hilbert space.

Interesting since math is full of interesting Hilbert spaces.

Gelfand (1930s): should solve this for G locally compact.

Mackey (1950s): enough to consider G simple.

(many parents) (1890–1970): best is G reductive.

My work is mostly about unitary dual of real reductive G.

I've stolen lots of excellent ideas about this problem, without ever completely solving it.

Subject for today: what I've stolen from Arthur.

Since it was advertised, I'll mention. . .

Conjecture. Suppose G is a real reductive algebraic group, and π is a unitary representation of G having integral infinitesimal character. Then π is an Arthur representation.

I know no counterexs for G classical, and there are none for G_2 or E_6.

For split F_4, conj fails for at most two reps; for split E_7, at most six; and for split E_8, at most 27.

Why at most? Adams, Barbasch, and I gave a def of Arthur's reps, but the definition is hard to calculate.

In cases above we have a complete list of unitary reps and a nearly complete list of Arthur reps.

Quick calculations by Jeff Adams seem to prove that the F_4 reps above are not Arthur, so the conjecture seems to fail for many real exceptional G. (True for complex F_4.)

What I will really do today

Describe three big ideas about unitary reps:

1. real parabolic induction (Gelfand);
2. cohomological parabolic induction (Zuckerman); and
3. unipotent reps (Arthur).

Show how they almost compute Arthur's reps.

If time allows, I will discuss the fact that these three big ideas originate in three different countries:

1. one of which no longer exists;
2. one of which may not exist much longer; and
3. Canada!

How rep theory works: motivation

Representation theory is useful because it concern all ways G can act on a Hilbert space.

Goal is to find very small collection of such actions, which magically exhausts the possibilities.

Parallel problem: find all ways G can act (transitively).

These are exactly conjugacy classes of subgroups $H \subset G$.

Too many H to make interesting statements.

Theorem If $H \subset G$ are algebraic, then $\exists$ deformation (limit of conjugates) H_0 of H that is a normal subgroup of a parabolic subgroup of G.

This must be well known, but I do not know a reference.

Impossible problem of describing all subgps $\rightsquigarrow$ tractable problem of describing normal subgroups of parabolics.

How rep theory works: big picture

Suppose $P(\mathbb{C})$ is a parabolic subgroup of reductive $G(\mathbb{C})$.

The quotient variety $G(\mathbb{C})/P(\mathbb{C})$ is projective algebraic.

$P(\mathbb{C})$ has a unipotent radical $U(\mathbb{C})$, a unipotent normal subgroup.

The quotient group $L(\mathbb{C}) = P(\mathbb{C})/U(\mathbb{C})$ is again reductive.

Rep (π_L, V_{π_L}) of $L(\mathbb{C}) \rightsquigarrow G(\mathbb{C})$-eqvt vector bundle

$$\mathcal{V}_{\pi_L} = G(\mathbb{C}) \times_{P(\mathbb{C})} V_{\pi_L} \to G(\mathbb{C})/P(\mathbb{C}).$$

Space (π_G, V_G) = sections of $\mathcal{V}_{\pi_L}$ is rep of $G(\mathbb{C})$.

Theorem (Borel-Weil, Harish-Chandra, Langlands). If π_L is (nearly) irreducible rep of $L(\mathbb{C})$, then π_G is (nearly) irreducible rep of $G(\mathbb{C})$. Every irreducible rep of $G(\mathbb{C})$ arises in this way.

How rep theory works: the easy fine print

Preceding slide required G and P complex.

Suppose now that $G =$ real points of $G(\mathbb{C})$, and that $P(\mathbb{C})$ is defined over $\mathbb{R}$. Then G/P is a compact manifold.

Unitary rep $(\pi_L, V_{\pi_L}) \rightsquigarrow$ Hermitian vector bundle $\mathcal{V}_{\pi_L}$.

Space of $V_{\pi_G} = L^2$ sections of $\mathcal{V}_{\pi_L}$ is a Hilbert space, and (π_G, V_{π_G}) is unitary rep of G.

Theorem (Gelfand, Harish-Chandra). If π_L is nearly irr unitary of L, then π_G is nearly irr unitary of G.

But it is very far from true that every irr unitary of G arises in this way.

MORAL of thm: unitary of Levi $L(\mathbb{R}) \rightsquigarrow$ unitary of $G(\mathbb{R})$.

But not all real Levis come from real parabolics!

How rep theory works: more difficult fine print

Levi subgp of real parabolic is cent in $G(\mathbb{R})$ of split torus.

Suppose $P(\mathbb{C}) \subset G(\mathbb{C})$ is opposite to its cplx conjugate $\overline{P(\mathbb{C})}$. Then $L(\mathbb{C}) = P(\mathbb{C}) \cap \overline{P(\mathbb{C})}$ is defined $/\mathbb{R}$, so $L(\mathbb{R}) = G(\mathbb{R}) \cap P$ is real Levi (not of a real parabolic).

$P(\mathbb{C})$ defines $G(\mathbb{R})$-invt cplx structure on $G(\mathbb{R})/L(\mathbb{R})$.

Such a Levi is cent in $G(\mathbb{R})$ of compact torus.

Unitary rep $(\pi_L, V_{\pi_L}) \rightsquigarrow$ holom Herm vector bdle $\mathcal{V}_{\pi_L}$.

Natural unitary rep of $G(\mathbb{R})$: L^2 holomo secs of $\mathcal{V}_{\pi_L}$.

But this is almost always ZERO.

Harish-Chandra (1954) found all nonzero exs: holom disc series.

In case $L(\mathbb{R}) = T$ compact torus, HC found desired the unitary reps $\pi_{G(\mathbb{R})}$...indirectly.

$\sim$1965, Kostant, Langlands suggested looking for $\pi_{G(\mathbb{R})}$ on higher Dolbeault cohom with coeffs in $\mathcal{V}_{\pi_L}$.

Complex geometry for such parabolics

In the setting $P(\mathbb{C}) \subset G(\mathbb{C})$ opposite $\overline{P(\mathbb{C})}$ we call $P(\mathbb{C})$ θ-stable, with real Levi $L(\mathbb{R}) = P(\mathbb{C}) \cap G(\mathbb{R})$.

Suppose $G(\mathbb{R}) = GL(2n, \mathbb{R}) \supset L(\mathbb{R}) = GL(n, \mathbb{C})$;

$$X = G(\mathbb{R})/L(\mathbb{R}) = \{\text{complex structures on } \mathbb{R}^{2n}\},$$

cplx mfld of dim n^2. Have a compact cplx submfld

$$Z = O(2n)/U(n) = \{\text{orthogonal complex structures on } \mathbb{R}^{2n}\}$$

of dimension $s = (n^2 - n)/2$.

Theorem (Schmid). X is $s+1$ complete, meaning as close to Stein as Z allows. Therefore

1. Dolbeault cohom on X vanishes above degree s.
2. Dolbeault cohom in degree s is a right exact functor

$$\text{holom vec bdles on } X \longrightarrow \text{vector spaces}$$

Precisely analogous results hold for any θ-stable $P(\mathbb{C})$.

Representations on Dolbeault cohomology

Suppose $P(\mathbb{C}) \subset G(\mathbb{C})$ is θ-stable, with Levi $L(\mathbb{R})$

$X = G(\mathbb{R})/L(\mathbb{R})$ cplx manifold.

Have an exact functor

rep (π_L, V_{π_L}) of $L(\mathbb{R}) \rightsquigarrow$ holom vector bdle $\mathcal{V}_{\pi_L}$ on X.

Schmid's $s+1$ complete theorem says that

$$\text{rep } (\pi_L, V_{\pi_L}) \text{ of } L(\mathbb{R}) \rightsquigarrow H^{0,s}(X, \mathcal{V}_{\pi_L})$$

is right exact functor from reps of $L(\mathbb{R})$ to reps of $L(\mathbb{R})$.

Technical difficulties abound...

1. $\mathrm{im}(\overline{\partial})$ not obviously closed, so $H^{0,s}$ not obviously Hausdorff.
2. no unitary structure can exist on $H^{0,s}$.
3. lower degree Dolbeault coh makes $H^{0,s}$ not left exact.

Zuckerman fixed everything...

Continue with $P(\mathbb{C}) \subset G(\mathbb{C})$ θ-stable, Levi $L(\mathbb{R})$.

The difficulties with cohomological induction are analytic.

Analysis is really difficult. Proof: it's what Jim does.

Harish-Chandra (another analyst) (1953) showed many rep theory problems are equiv to algebraic analogues.

Zuckerman (1977) found alg analogue of Dolbeault.

1. (sheaf of germs of holomorphic sections of $\mathcal{V}_{\pi_L}$) $\rightsquigarrow$ (formal power series sections at eL on G/L)
2. (global sections functor) $\rightsquigarrow$ (K-finite vectors).

In this way he made cohomological induction functors

$$R^i\colon ((\mathfrak{l}, L\cap K)\text{-mod}) \to ((\mathfrak{g}, K)\text{-mod}) \qquad (0 \le i \le s)$$

Henceforth twist $\mathcal{V}_{\pi_L}$ by square root of canonical bundle.

There should be similar functors for rational Levis in p-adic G. Lots of the best p-adic rep theory in past fifty years is more or less in this direction, but I don't think there is a general result.

. . . and here's what he found.

$P(\mathbb{C}) = L(\mathbb{C})U(\mathbb{C}) \subset G(\mathbb{C})$ θ-stable, Levi $L(\mathbb{R}) \supset H(\mathbb{R})$ Cartan.

$$\mathfrak{h} = (\text{center of } \mathfrak{l}) \oplus (\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]) = \mathfrak{h}^z \oplus \mathfrak{h}^d$$

Suppose π_L is a finite length $(\mathfrak{l}, L \cap K)$-module, infinitesimal character $\gamma = \gamma^z + \gamma^d \in \mathfrak{h}^*$.

γ_L^z is just character on center of Lie alg rep π_L.

Theorem (Zuckerman 1977 *et al.*)

1. Each $R^i(\pi_L)$ is fin lgth $(\mathfrak{g}, K)$-mod of infl char
$$\gamma = \gamma^z + \gamma^d \in \mathfrak{h}^*$$
2. If γ wkly nonpos on coroots of $U(\mathbb{C})$, then $R^i(\pi_L) = 0$ for $i < s$. If also π_L is unitary, then so is $R^s(\pi_L)$.
3. If γ^z wkly nonpos on coroots of $U(\mathbb{C})$ and π_L small enough, then $R^i(\pi_L) = 0$ for $i < s$. If also π_L is unitary, so is $R^s(\pi_L)$.

I still need to tell you

(a) what "small enough" means, and
(b) how this helps construct Arthur reps.

What do negativity conditions mean?

$P = LU$, $H \subset L$ Cartan, rep π_L of L, infl char $\gamma = \gamma^z + \gamma^d$.

Had two conditions:

1. γ antidominant on coroots of U, and
2. γ^z antidominant on coroots of U

Take $G = GL(2n)$, $L = GL(n) \times GL(n)$, $H = \mathbb{C}^\times)^{2n}$.

Take $\pi_L = (\mathrm{triv} \otimes |\det|^a) \otimes \big(\mathrm{triv} \otimes |\det|^b\big)$.

In standard basis $\{e_i \mid 1 \le i \le n\}$, $\{f_j \mid 1 \le j \le n\}$ of $\mathbb{C}^{2n}$, the n^2 coroots of H in U are $e_i - f_j \quad (1 \le i, j \le n)$.

Infl char $\gamma = \gamma^z + \gamma^d$ of π_L is

$\gamma^z = [(a, \ldots, a)(b, \ldots, b)]$, $\quad \gamma^d = [(n-1, n-3, \ldots, -n+1)(n-1, \ldots -n+1)]/2$.

Condition (1) means $(a - b) \ge n - 1$.

Condition (2) is the weaker condition $(a - b) \ge 0$.

For rational P, conv of intertwiners $\leftsquigarrow$ (strict) cond (1)...

...or from weaker cond (2) plus temperedness of π^d.

What does "small enough" mean?

Definition (Vogan 1984). Rep π is weakly unipotent if it has minimal infl char in its coherent family (family defined by Jantzen, Zuckerman, Schmid).

Examples: triv rep; any rep of infl char zero; unitarily induced from wk unip on Levi of real parabolic.

Examples all of you care about: suppose ψ is a real Arthur parameter "trivial" on $\mathbb{C}^\times \subset W_\mathbb{R}$. Then Arthur packet $A(\psi)$ consists entirely of weakly unipotent representations.

$P(\mathbb{C}) = L(\mathbb{C})U(\mathbb{C}) \subset G(\mathbb{C})$ θ-stable, Levi $L(\mathbb{R}) \supset H(\mathbb{R})$ Cartan.

Theorem (Vogan 1984) If γ^z wkly nonneg on coroots of $U(\mathbb{C})$ and π_L weakly unipotent and unitary, then $R^i(\pi_L) = 0$ for $i < s$, and $R^s(\pi_L)$ is unitary.

Executive summary: (summary accessible to administrators) Weakly unipotent unitary reps on θ-stable Levis can always be cohom induced to cohom induced to unitary.

Just need to choose $P(\mathbb{C})$ making central character weakly antidominant.

An advertisement for youth

Arthur's conjectures led to wonderful families of interesting weakly unipotent unitary reps. . . probably all the reps relevant to automorphic forms on linear groups.

But there are more: to begin with, reps relevant to automorphic forms on nonlinear groups. (I hope we will hear from Wee Teck Gan about the case of metaplectic groups.)

Ivan Losev and a host of others (including Lucas Mason-Brown, Dmytro Matvieievskyi, and Dougal Davis) have developed more general constructions of weakly unipotent representations, and in many cases proven that these representations are unitary.

I am convinced that these representations—together, of course, with others obtained from them by real and cohomological parabolic induction—have a large role to play in our understanding of the unitary dual.

This is to remind me to sketch the construction of general Arthur packets from unipotent ones.