Affine Weyl group alcoves

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Outline

Introduction

Integer parts. . .

. . . and also ordering

Partitioning $\mathbb{R}^n$ into facets

Facets and unitary representations

Slides eventually at
`http://www-math.mit.edu/~dav/paper.html`

What's this about?

I'll talk about decomposing $\mathbb{R}^n$ using symmetries.

Question is how can you use symmetries to put any vector into the simplest possible form?

Simple version: symms are chging signs of some coords, and adding an integer to a coord.

Next: add the symms exchanging any two coords.

Having tried to explain the simplification process in those two examples, I will talk about a general mathematical setting where the same ideas apply.

The math secret code word is affine Weyl group.

In the unlikely event that I finish those topics before four o'clock, I will finish with my mathematical reasons for looking at such simplification problems.

Reducing modulo $\mathbb{Z}$

How can you simplify $v \in \mathbb{R}^n$ by adding ints to any coord and chging sgns of any coord?

First process allows moving any v to

$$W\overline{A} =_{\text{def}} [-1/2, 1/2]^n.$$

$$(8/3, -4/5, 2) \quad \xrightarrow{+(-3,1,-2)} \quad (-1/3, 1/5, 0).$$

Then the second process allows moving v to

$$\overline{A} =_{\text{def}} [0, 1/2]^n.$$

$$(-1/3, 1/5, 0) \quad \xrightarrow{(-,+,\pm)} \quad (1/3, 1/5, 0).$$

Let's write that as a Theorem

Define $T = \mathbb{Z}^n$, translations of $\mathbb{R}^n$ by integers.

Define $W = (\pm 1)^n$, coord sign changes in $\mathbb{R}^n$.

Recall $\overline{A} = [0, 1/2]^n$, $W\overline{A} = [-1/2, 1/2]^n$.

Theorem

1. For all $v \in \mathbb{R}^n$ $\quad \exists t \in \mathbb{Z}^n$ so $v^1 =_{\text{def}} t + v \in W\overline{A}$.
2. t is unique except in coords with $v_i \in \mathbb{Z} + 1/2$.
3. For all $v^1 \in W\overline{A}$ $\quad \exists w \in \pm 1^n$ so $w \cdot v^1 =_{\text{def}} v^0 \in \overline{A}$.
4. w is unique except in coords where $v_i^1 = 0$.
5. v_0 is unique.

Symm grp we want is $W \rtimes T$, a semidirect product.

This is an affine Weyl group of type $(\widetilde{A}_1)^n$.

But the main point is statements in Theorem.

Let's draw that as a picture

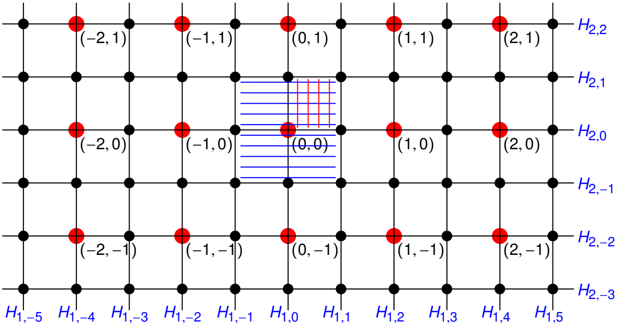
I'm interested in the hyperplanes in $\mathbb{R}^n$

$$H_{i,m} = \{v \in \mathbb{R}^n \mid 2v_i = m\} \qquad (1 \le i \le n, \quad m \in \mathbb{Z}).$$

For each hyperplane, I'm interested in the reflection

$$\begin{aligned} s_{i,m}(v) &= v - (2v_i - m)e_i \\ &= (v_1, \cdots, v_{i-1}, -v_i + m, v_{i+1}, \cdots, v_n). \end{aligned}$$

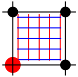
$s_{i,m}$ chgs sign of ith coord and translates by m.



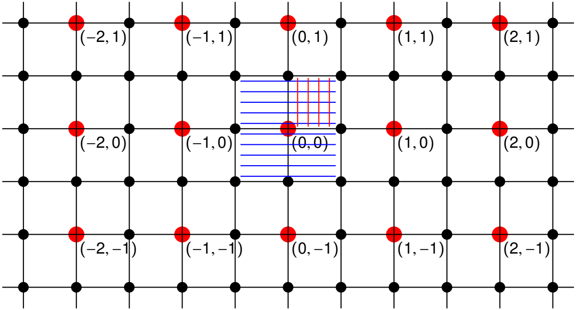
How does the picture prove the theorem?

Start with any $v \in \mathbb{R}^n$.

Want to use hyperplane reflections to move v to

$\overline{A} =$ 

Whenever v is on the wrong side of a hyperplane $H_{i,m}$ from $\overline{A}$, reflect v in that hyperplane, moving it closer to A.



That wasn't complicated enough to be *math*

Add more symmetries: interchanging coords of v.

How can you simplify $v \in \mathbb{R}^n$ by adding integers, chging coord sgns, and permuting coords?

First process (still) allows moving any v to

$$W\overline{A} =_{\text{def}} [-1/2, 1/2]^n.$$

$$(7/4, -3/5, 3/2) \xrightarrow{+(-2,1,-2)} (-1/4, 2/5, -1/2).$$

Last two processes move v to (much smaller)

$$\overline{A} =_{\text{def}} \left\{ v \in \mathbb{R}^n \mid 1/2 \ge v_1 \ge v_2 \ge \cdots \ge v_n \ge 0 \right\}.$$

$$(-1/4, 2/5, -1/2) \longrightarrow (1/2, 2/5, 1/4).$$

$\overline{A}$ is an n-simplex, volume $= 1/(2^n \cdot n!)$

This too is a Theorem

Define $T = \mathbb{Z}^n$, translations of $\mathbb{R}^n$ by integers.

Define $W = S_n \ltimes (\pm 1)^n$, coord perms and sign changes.

Our new $\overline{A} = \{1/2 \geq v_1 \geq \cdots \geq v_n \geq 0\}$, $W\overline{A} = [-1/2, 1/2]^n$.

Unit cube $W\overline{A}$ is union of $2^n \cdot n!$ translates of simplex $\overline{A}$.

Theorem

1. For all $v \in \mathbb{R}^n$ $\exists t \in \mathbb{Z}^n$ so $v^1 =_{\text{def}} t + v \in W\overline{A}$.
2. t is unique except in coords with $v_i \in \mathbb{Z} + 1/2$.
3. For all $v^1 \in W\overline{A}$ $\exists w \in W$ so $w \cdot v^1 =_{\text{def}} v^0 \in \overline{A}$.
4. w is unique unless $v_i^1 = 0$ or $\pm v_i^1 \pm v_j^1 = 0$.
5. v_0 is unique.

Symmetry grp we want is $W \ltimes T$, a semidirect product.

This is an affine Weyl group of type $\widetilde{B}_n$.

But the main point is statements in Theorem.

Draw the new Theorem as a picture

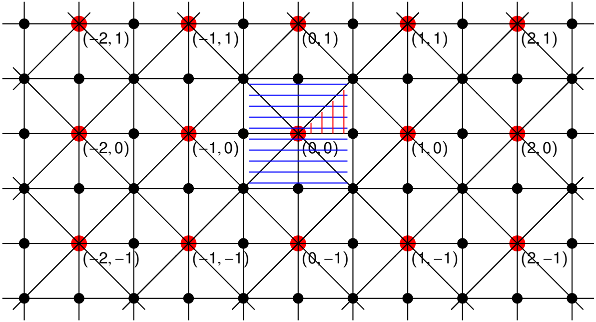
New hyperplanes are

$$H_{i,\pm j,m} = \{v \in \mathbb{R}^n \mid v_i \pm v_j = m\} \qquad (1 \le i, j \le n, \quad m \in \mathbb{Z}).$$

For each hyperplane, we want the reflection

$$s_{i,\pm j,m}(\cdots, v_i, \cdots, v_j, \cdots) = (\cdots, \pm v_j + m, \cdots, \pm v_i \pm m, \cdots).$$

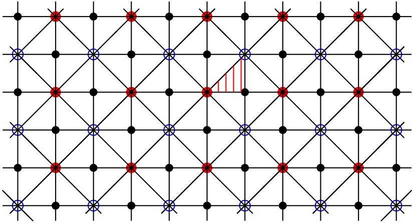
$s_{i,\pm j,m}$ interchanges ith and jth coords, multiplying both by $\pm$, and translates by $m(e_i \pm e_j)$.



What's a facet?

The reflection hyperplanes (like $\{v_i + v_j = m\}$ each divide $\mathbb{R}^n$ into three pieces: the hyperplane itself, and two open pieces.

These hyperplanes divide $\mathbb{R}^n$ into facets. Here's $\mathbb{R}^2$.



Each open triangle is a facet, called an alcove. An alcove has three kinds of 1-diml facets as edges, and three kinds of 0-diml facets as vertices.

There are three kinds of 0-diml facets:

1. integral (p, q) (p and q in $\mathbb{Z}$);
2. half-integral $(p + 1/2, q + 1/2)$; and
3. mixed $(p + 1/2, q)$ or $(p, q + 1/2)$.

There are three kinds of 1-diml facets (black open intervals):

1. horiz or vert, with one red and one black endpoint;
2. horiz or vert, with one blue and one black endpoint; and
3. diagonal (always with one red and one blue endpoint).

Everything you always wanted to know about facets

$T = \mathbb{Z}^n$, transl of $\mathbb{R}^n$; $W = S_n \ltimes (\pm 1)^n$ = type B_n Weyl group.

$\widetilde{W} = W \ltimes T$ = affine Weyl group.

Everything below works for W = any Weyl group, T = root lattice.

An alcove is a conn component of $\mathbb{R}^n$ – (all refl hyperplanes).

Theorem $\widetilde{W}$ acts simply transitively on alcoves.

1. The fundamental alcove A is the n-simplex
$$A = \{1/2 \geq v_1 \geq \cdots \geq v_n \geq 0\}.$$
2. The $n+1$ vertices of A are $f_m = (\underbrace{1/2, \cdots, 1/2}_{m \text{ terms}}, \underbrace{0, \cdots, 0}_{n-m \text{ terms}})$.
3. Each alcove is an n-simplex, so has $\binom{n+1}{d+1}$ d-faces.
4. Every d-diml facet is a d-face of some alcove.

This theorem provides a computer-effective way to list all facets.

I'll return to that after explaining why one might want a list of facets.

And now for something completely different

My favorite problem in the whole world is the unitary dual problem.

Start with a group G; look for all ways that G can act by isometries of Hilbert spaces.

Quantum mech systems live on Hilbert space, so unitary rep $\leftrightsquigarrow$ symmetry of quantum systems.

How can you look for unitary reps?

I'll explain how looking for unitary reps of simple Lie groups leads to geometry of facets.

Two important subgroups for $GL(n,\mathbb{R})$

$$K(\mathbb{R}) = O(n) = \text{orthogonal group},$$
$$A = \text{positive diagonal matrices},$$
$$A^+ = \text{positive diag mats with decreasing entries.}$$

Any invertible $n \times n$ real g has a polar decomposition

$$g = k_1 a k_2, \qquad (a \in A^+, \quad k_i \in O(n)).$$

Matrix a is unique. Diagonal entries of a are the singular values of g. Largest singular value is

$$a_1 = \max_{v \in \mathbb{R}^n \setminus 0} \frac{||gv||}{||v||},$$

the largest amount that g can stretch a vector.

Similarly, a_n is the least that g can shrink a vector.

Since $K(\mathbb{R})$ is compact, polar decomp says that A—better, A^+—enumerates all ways to go to infinity in $GL(n,\mathbb{R})$.

So what can you do with KAK?

$$K = O(n) = \text{orthogonal group},$$
$$A = \text{positive diagonal matrices},$$
$$A^+ = \text{positive diag mats with decreasing entries}.$$

Study harmonic functions on the unit disc by boundary values: limiting behavior in radial directions.

Same applies to functions on $GL(n, \mathbb{R}) = KAK$: helps to study limiting behavior in the A variable, particularly along A^+.

(approximate) Theorem (Harish-Chandra). Nice fn ϕ on $GL(n, \mathbb{R})$ is exponential at infinity: have an asymptotic expansion

$$\phi(k_1 a k_2) \sim c(k_1, k_2) a^\nu + \text{ lower terms}, \quad (a \in A^* \to \infty)$$

with $\nu \in \mathbb{C}^n$. Here $a^\nu = a_1^{\nu_1} \cdots a_n^{\nu_n}$.

HC/Langlands idea: reps of $GL(n, R)$ are indexed by $\nu \in \mathbb{C}^n$ describing their asymptotic behavior at infinity.

which reps are unitary $\leftrightsquigarrow$ which facet ν is in!

How do you make reps of $GL(n,\mathbb{R})$?

Reps of G on fns on homogeneous spaces G/H.

Better: sections of vector bundles $\mathcal{E} \to G/H$.

Best space to use for $GL(n,\mathbb{R})$:

$$X = \text{complete flags } 0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{R}^n.$$

X has n real line bundles $\mathcal{E}_i$, fiber V_i/V_{i-1}.

$\nu \in \mathbb{R}^n \rightsquigarrow$ real line bundle $\mathcal{E}_\nu = |\mathcal{E}_1|^{\nu_1} \otimes \cdots \otimes |\mathcal{E}_n|^{\nu_n}$

$\pi_\nu =$ sections of $\mathcal{E}_\nu \otimes D^{1/2}$, nice rep of $GL(n,\mathbb{R})$.

Here $D^{1/2}$ is half-density bundle on X, useful normalization.
If $\rho = ((n-1)/2, (n-3)/2, \cdots, -(n-1)/2)$, then $D^{1/2} = \mathcal{E}_\rho$.

Theorem (HC, Helgason, Helgason-Johnson). Say $\nu_1 \geq \cdots \geq \nu_n$.

1. π_ν has asymptotic behavior $a^{\nu-\rho}$ at infinity on A^+.
2. π_ν bdd $\iff \nu \in \rho -$ (nonneg combs of pos roots $e_i - e_j$).
3. π_ν herm $\iff \nu = (v_1, \cdots, v_m, \{0\}, -v_m, \cdots, -v_1)$.
4. In (3), whether π_ν is unitary $\leftrightsquigarrow$ facet of v.

How to classify unitary reps of $GL(2m, \mathbb{R})$

Unitary reps of $GL(2m, \mathbb{R})$ indexed by some facets in

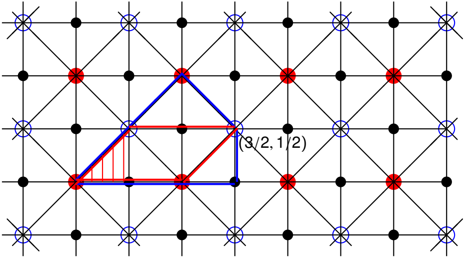
$$v = (v_1, \ldots, v_m), \qquad v_1 \geq \cdots \geq v_n \geq 0$$

$$v_1 + v_2 \cdots + v_p \leq (m - 1/2) + (m - 3/2) + \cdots + (m - (2p - 1)/2)$$

So to describe unitary representations, need to

1. enumerate finite # facets satisfying inequalities; and
2. for each facet, test whether one v in facet is unitary.

Test (2) is possible using `atlas` software.



Blue quadrilateral is the candidates allowed by Helgason-Johnson: 7 alcoves, 29 facets. Red parallelogram FPP is a better bound found by Dan Barbasch: 4 alcoves, 19 facets.

Same ideas $\rightsquigarrow$ unitary duals for all real reductive G.

So what's the program?

General G: pos roots $R^+ \subset \mathbb{R}$ vec space $\mathfrak{h}^*_{\mathbb{R}}$ replaces $\mathbb{R}^n$.

Weyl group W replaces $S_n \ltimes \{\pm 1\}^n$.

Hyperplanes are $H_{\alpha^\vee, m} = \{\gamma \in \mathfrak{h}^* \mid \langle \gamma, \alpha^\vee \rangle = m\}$.

FPP is $\{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \gamma, \alpha^\vee \in [0, 1],\ \text{all } \alpha \text{ simple}\}$.

Need to

1. compute partition of FPP into facets
2. for one v in each facet, test unitarity of finitely many reps of infl char v.

For E_7, number of facets in FPP is about 38 million; compute them in few hours.

For E_8, number of facets in FPP is about 30 billion; compute in a month or so.

test is harder...