Affine subspaces

I'm assuming that (right) vector spaces, subspaces, and dimension are already defined. Here are the basic facts about affine subspaces. (The motivating example is that a one-dimensional vector subspace of the plane is a line through the origin. A one-dimensional affine subspace is any line.)

Definition 1. Suppose V is a right vector space over a field k. A subset $S \subset V$ is called an *affine subspace* if we can find a vector subspace $W \subset V$ and an element $s_1 \in V$ so that

$$S = W + s_1 = \{w + s_1 \mid w \in W\}. \tag{A}$$

One bad feature of this definition is that it asks you to go look for W and s_1, without explaining how to find them. The first theorem fixes that. It says in particular that the vector space W is uniquely determined by S.

Theorem 2. *Suppose S is a nonempty subset of a vector space V. Then the following three conditions on S are equivalent.*

(1) *The set S is an affine subspace of V.*
(2) *For every $s_0 \in S$, the set*

$$W(s_0) = \{s - s_0 \mid s \in S\}$$

is a vector subspace of V.

(3) *For some $s_0 \in S$, $W(s_0)$ is a vector subspace of V.*

When these conditions are satisfied, the various subspaces $\{W(s_0) \mid s_0 \in S\}$ are all the same; write $W(S)$ for this common space. Then S can be written as in (A) above if and only if $W = W(S)$, and $s_1 \in S$.

Proof. Suppose that (1) holds, so that $S = W + s_1$. Now fix $s_0 \in S$. Then $s_0 = w_1 + s_1$ for some $w_1 \in W$, so

$$\begin{aligned} W(s_0) &= \{s - s_0 \mid s \in S\} = \{(w + s_1) - (w_1 + s_1) \mid w \in W\} \\ &= \{w - w_1 \mid w \in W\} = W. \end{aligned}$$

This gives (2). That (2) implies (3) is immediate. Assume (3). Taking $W = W(s_0)$ and $s_1 = s_0$, we find immediately that S can be written as in (A) above, as required for (1). This completes the proof of equivalence. The remaining assertions have essentially been proved along the way. $\square$

Definition 3. The *dimension* of an affine subspace $S \subset V$ is equal to the dimension of the corresponding vector subspace $W(S)$. Two affine subspaces S and S' of V are said to be *parallel* if the corresponding affine subspaces are the same.

Corollary 4. *Suppose S is an affine subspace of V, and $v \in V$. Then there is a unique affine subspace of V parallel to S and containing v, namely $W(S) + v$.*

This is really almost obvious from Theorem 2, so I won't write a proof.

Here's a useful abstract characterization of affine subspaces. Notice first of all that zero-dimensional affine subspaces of V are just points of V. This fact and a "functorial" turn of mind suggest the following theorem.

Theorem 5. *Suppose V and V' are vector spaces, and $T: V \to V'$ is a linear transformation. Then the inverse image $S = T^{-1}(v')$ of any point in the image of T is an affine subspace of V. The corresponding vector space $W(S)$ is equal to the kernel of T.*

Conversely, suppose S is any affine subspace of V, with corresponding vector subspace $W(S)$. Let

$$T: V \to V/W(S)$$

be the canonical quotient map. Then T carries S to a single point v' of $V/W(S)$, and $S = T^{-1}(v')$.

Again this is very easy, so I'll skip the proof. Another useful fact is

Theorem 6. *Suppose S and S' are affine subspaces of V, with associated vector subspaces W and W'. If $S \cap S'$ is non-empty, then it is an affine subspace with associated vector space $W \cap W'$.*

This follows directly from Definition 1, or from Theorem 5.

As a final bit of direction for the homework problems: an (affine) line in a vector space is by definition a one-dimensional affine subspace.

For lines in a plane, there is a wonderful dichotomy: two lines must either be parallel or meet in a point. Here is a theorem/definition that provides a generalization to higher dimensions.

Theorem 7. *Suppose W and W' are vector subspaces of a vector space V. The following conditions on W and W' are equivalent.*

(1) *The vector space V is the sum of the subspaces W and W'; that is, every vector $v \in V$ can be written $v = w + w'$, with $w \in W$ and $w' \in W'$.*
(2) *For any affine subspaces S and S' associated to W and W', the intersection $S \cap S'$ is non-empty.*

When these conditions are satisfied, we say that W and W' (or corresponding affine subspaces S and S') are transverse.

Suppose now that W and W' have finite codimension in V; that is, that the quotient vector spaces V/W and V/W' are finite-dimensional. Write r and r' for their dimensions. Then transversality is also equivalent to any of the following three conditions.

(3) *The dimension of $V/(W \cap W')$ is equal to $r + r'$.*
(4) *The dimension of $W/(W \cap W')$ is equal to r'.*
(5) *The dimension of $W'/(W \cap W')$ is equal to r.*

In each case the indicated dimension is less than or equal to the corresponding number, independent of transversality.

Proof. Suppose first that $V = W + W'$; let $S = W + s$ and $S' = W' + s'$ be corresponding affine subspaces. Write $v = s - s'$ as a sum of vectors in W and W': $s - s' = w + w'$. Then $-w + s = w' + s'$ is obviously in $S \cap S'$; so the intersection is non-empty, as we wished to show. Next, suppose that (2) holds. Fix $v \in V$, and consider the affine subspaces $S = W - v$ and $S' = W'$. By (2) the intersection of these subspaces is non-empty; so we can find $w \in W$ and $w' \in W'$ so that $w - v = w'$. This means that $v = w - w'$, so $v \in W + W'$. Since v was arbitrary, $W + W' = V$, as we wished to show.

Now suppose that W and W' have finite codimension in V. There is a natural linear transformation

$$T\colon V \to V/W \oplus V/W', \qquad T(v) = (v + W, v + W').$$

The kernel of T is clearly $W \cap W'$, so there is an inclusion of vector spaces

$$\overline{T}\colon V/(W \cap W') \to V/W \oplus V/W'.$$

The direct sum on the right has dimension $r + r'$; it follows that $V/(W \cap W')$ has dimension at most $r + r'$. This proves the inequality in (3). Obviously equality holds if and only if T is surjective. I'll let you find the proof that T is surjective if and only if $W + W' = V$; this gives the equivalence of (3) with (1). For the equivalence of (3) and (4), consider the natural quotient map

$$Q\colon V/(W \cap W') \to V/W.$$

Obviously Q is surjective, and the kernel of Q is $W/(W \cap W')$. Therefore

$$\dim V/(W \cap W') = \dim W/(W \cap W') + \dim V/W,$$

or

$$\dim W/(W \cap W') = \dim V/(W \cap W') - r.$$

Since the first term on the right is at most $r + r'$, the left is at most r'. This proves the inequality in (4), and that equality holds in (4) exactly when it does in (3). The equivalence of (3) and (5) is identical. $\square$

Two lines in a plane are clearly either parallel (if the corresponding lines through the origin coincide) or transversal (otherwise). The Theorem says that transversal lines in a plane must always meet in an affine subspace of codimension 2; that is, in a single point.