Lie group actions and vector fields

Suppose that a Lie group G acts smoothly on a manifold M. Recall that this means that the *action map*

$$a: G \times M \to M, \qquad a(g,m) = g \cdot m$$

is a smooth map. I believe that I stated quickly in class that whenever there is such a smooth action, it can be differentiated to give a Lie algebra homomorphism

$$\xi: \mathfrak{g} \to \mathrm{Vec}(M), \quad \xi_X(m) = (da_m)_e(X) \qquad (X \in \mathfrak{g},\ m \in M).$$

(There is an unfortunate possibility that I have lost a sign in this formula, but I hope not.)

A little more explicitly, the notation means that a_m is the restriction of a to the G variable: $a_m(g) = g \cdot m$. Therefore the differential of a_m at the identity in G (what's written $(da_m)_e$) is a linear map from $T_e(G)$ (which is $\mathfrak{g}$) to $T_m(M)$. For each $X \in \mathfrak{g}$, $(da_m)_e(X)$ is therefore a tangent vector at m to M. The equation therefore defines a vector field ξ_X on M.

What I explained in class was that this map $X \mapsto \xi_X$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M.

I will try to make a longer version of this note later on including references to the text, and perhaps some proofs and examples.