Nonunitarity certificates

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Outline

What's the plan?

G real reductive Lie $\supset K$ maximal compact.

Assume $G =$ real pts of conn reductive cplx algebraic group.

Want to describe $\widehat{G}_u =$ unitary dual: equiv classes of irreducible unitary representations. This is hard.

Harish-Chandra: larger set $\widehat{G}_a =$ adm dual easier.

Start with Langlands' parametrization of $\widehat{G}_a$.

Unitary dual $\rightsquigarrow$ understand $\widehat{G}_u \subset \widehat{G}_a$.

Do this in two steps:

1. Understand tempered dual $\widehat{G}_t \subset \widehat{G}_a$. This is easy.
2. Understand $\widehat{G}_u$ as small deformation of $\widehat{G}_t$.

Plan is that discussion of (1) should re-do some of Nigel's lectures; and that the details of that discussion will arm us with tools for approaching (2).

Elevator pitch for the talk

$\widehat{G}_a =$ union of cplx vec spaces indexed by $\widehat{K}$.

$\widehat{G}_t =$ union of real forms of these vec spaces.

$\widehat{G}_u = \widehat{G}_t \;\cup$ small imaginary deformations.

Example: $G = SL(2,\mathbb{C})$, $K = SU(2)$, $\widehat{K} \simeq \mathbb{N}$.

$\widehat{G}_a = \{(n,\nu) \in \mathbb{N} \times \mathbb{C}\}$

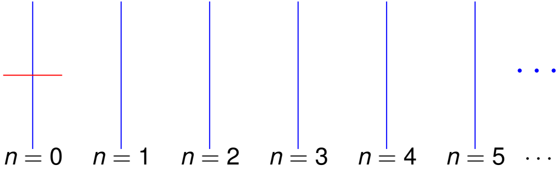
One complex line for each $n \in \widehat{K} \simeq \mathbb{N}$.

$\widehat{G}_t = \{(n,\nu) \in \mathbb{N} \times i\mathbb{R}\}$ unitary princ series

One real line for each $n \in \widehat{K} \simeq \mathbb{N}$.

$\widehat{G}_u = \{(n,\nu) \in \mathbb{N} \times i\mathbb{R}\} \cup \{(0,\nu) \mid \nu \in [-1,1]\}$

Deform (only) first line by adding compact interval.



What's the admissible dual look like?

admissible rep $\pi \rightsquigarrow$ Cartan subgroup $H(\pi) = T(\pi)A(\pi)$

$\rightsquigarrow$ character $\nu(\pi)\colon A(\pi) \to \mathbb{C}^\times$

$\rightsquigarrow$ character $\lambda(\pi)\colon T(\pi) \to \mathbb{C}^\times$

$\rightsquigarrow \Pi_{\text{im}}(\pi)$ of simple singular imag roots

Character $\nu(\pi)$ controls growth of mat coeffs of π at infinity.

π tempered $\iff$ real part of $\nu(\pi)$ is zero.

π bounded $\leftrightsquigarrow$ real part of $\nu(\pi)$ is in "$W \cdot \rho$."

Pair $(\lambda(\pi), \Pi_{\text{im}}(\pi)) \leftrightsquigarrow$ lowest K-types of π

differential of $\lambda(\pi) \approx$ highest wt of LKTs

HC, Langlands, Knapp, Zuckerman: invts determine π;

Also show which $(H(\pi), \nu(\pi), (\lambda(\pi), \Pi_{\text{im}}(\pi)))$ occur.

Which $(H, \nu, (\lambda, \Pi_{\text{im}}))$?

A parameter is a tuple as above satisfying

requirements on $p = (H, \nu, (\lambda, \Pi_{\text{im}}))$:

1. $H = TA$ any θ-stable Cartan subgroup
2. $\nu \in \mathfrak{a}^*_{\mathbb{C}} \simeq \widehat{A}$
3. $\lambda \in [X^*(H) + \rho] \, / \, [(1-\theta)X^*(H)] \simeq \widehat{T} + \rho$
4. Π_{im} = simple system for imaginary roots zero on λ.
5. Π_{im} consists of noncompact roots. (NONZERO)

Last, we will often impose ONE of the following conditions.

6. $\nu = 0$ on real $\alpha^\vee \implies \langle \lambda - \rho, \alpha^\vee \rangle$ even. (FINAL)
7. $\lambda \neq 0$ on every imaginary $\beta^\vee$. (M-REGULAR)

(1)–(3) $\rightsquigarrow$ $(\lambda, \nu) =$ any character of H (up to ρ shift).

Set $MA = \text{Cent}_G(A)$, cuspidal Levi subgroup of G.

HC theory of discrete series $\rightsquigarrow$ limit of discrete series rep

$$\delta = \delta(p) = \delta(T, (\lambda, \Pi_{\text{im}})) \in \widehat{M}$$

$$\rightsquigarrow I(p) = I(H, \nu, (\lambda, \Pi_{\text{im}})) = \text{Ind}_P^G(\delta \otimes \nu \otimes 1) = \text{standard rep}$$

Condition (5) $\iff \delta \neq 0$.

Condition (6) $\iff \delta$ is a discrete series representation.

Standard reps are stars of the Langlands classification.

How does Langlands classification work?

Start with parameter $p = (H, \nu, (\lambda, \Pi_{\mathrm{im}}))$

Purpose of parameter is $\rightsquigarrow$ standard representation

$$I(p) = I(H, \nu, (\lambda, \Pi_{\mathrm{im}})) = \mathrm{Ind}_P^G(\delta \otimes \nu \otimes 1)$$

Standard rep has finite set of lowest K-types, all with mult one, depending only on δ. The Langlands factor of $I(p)$ is

$J(p) =$ sum of comp factors containing a lowest K-type

Irreducibles in $J(p)$ are part of a Langlands L-packet.

Theorem (Langlands) Each irr rep of G is a summand of $J(p_{\mathrm{reg}})$ for exactly one M-regular parameter p_{reg}.

Theorem (Knapp-Zuckerman) If p_{fin} is a final parameter, then $J(p_{\mathrm{fin}}$ is irreducible. Each irr rep of G appears in this way for exactly one nonzero final parameter p_{fin}.

So there is a finite-to-one correspondence

$\widehat{G}_a \longrightarrow$ nonzero M-reg params mod K conjugacy

summands of $J(p) \longrightarrow$ parameter p_{reg}

Similarly, there is a bijection

$\widehat{G}_a \longleftrightarrow$ nonzero final params mod K conjugacy

$J(p_{\mathrm{fin}}) \longleftrightarrow$ parameter p_{fin}

Real parabolics

Parameter $p = (H = TA, \nu, (\lambda, \Pi_{\text{im}})) \rightsquigarrow MA = \text{Cent}_G(A)$ Levi subgroup of real parabolic.

$MA = \langle H, \text{roots zero on } A\rangle = \langle H, \text{imaginary roots}\rangle$.

$p \rightsquigarrow$ limit of discrete series $\delta \in \widehat{M}_t$ unitary.

Unitarity of $p \leftrightsquigarrow$ unitarity of induction from MA to G.

Easy case: $J(p)$ tempered $\iff \nu \in i\mathfrak{a}^*$.

Extend this: how does ν fail to be pure imaginary?

Define $(MA)_{\text{re}} = \langle H, \text{roots real on } \nu\rangle \supset MA$.

Theorem (see Knapp "Overview"). $J_G(p)$ is unitary $\iff$

1. $J_{M_{\text{re}}}(p)$ is unitary, and
2. $\nu_{\text{re}} = \nu|_{A_{\text{re}}}$ is unitary.

Theorem is reduction of unitary dual to real infl char.

By definition of M_{re}, $J_{M_{\text{re}}}(p)$ has real infl char.

Unitary reps of nonreal infl char only from real parab ind.

θ-stable parabolics

Cplxification of real parab subalg $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ satisfies

$$\theta(\mathfrak{m} + \mathfrak{a}) = \mathfrak{m} + \mathfrak{a}, \qquad \theta(\mathfrak{n}) = \mathfrak{n}^{\text{op}}.$$

Cplx parab subalg $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is θ-stable if

$$\theta(\mathfrak{l}) = \mathfrak{l}, \qquad \theta(\mathfrak{u}) = \mathfrak{u}.$$

Seek to relate unitarity of $J(p)$ to θ-stable parabolics.

Param $p = (H = TA, \nu, (\lambda, \Pi_{\text{im}})) \rightsquigarrow$ θ-stab parab $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$,

$$L = \langle H, \text{roots zero on } T \rangle = \langle H, \text{real roots} \rangle$$

$$\Delta^\vee(\mathfrak{u}, H) \supset \text{coroots positive on } \lambda$$

Condition does not specify $\mathfrak{u}$ uniquely, but that will not matter; like indeterminacy of N in parabolic MAN attached to p.

p for G (inf char γ) $\rightsquigarrow$ p_L for L (inf char $\gamma - \rho(\mathfrak{u})$).

L is split, $I(p_L)$ = minimal principal series for L.

$I(p_L) \overset{\text{coh ind}}{\rightsquigarrow} I(p)$. How does cohom ind affect unitarity?

What does FINAL mean?

Example: $G = SL(2, R)$, $K = SO(2)$, $\widehat{K} \simeq \mathbb{Z}$.

$H_s = T_s A_s = \{\pm I\} \times \mathbb{R}^{>0}$, $X^*(H) = \mathbb{Z}$, $\rho = 1$, $\mathfrak{a} \simeq \mathbb{R}$.

Parameter p_s on H_s is $(\lambda_s = \epsilon_s, \nu_s)$, $\epsilon_s \in \mathbb{Z}/2\mathbb{Z}$, $\nu_s \in \mathbb{C}$.

All p_s are M-regular since no imaginary roots.

p_s is final UNLESS ϵ_s is even and $\nu = 0$.

Standard rep $I(p_s)$ is principal series $I((\epsilon_s - 1) \otimes \nu_s)$.

(1) If ϵ_s odd, K-types of $I(p_s)$ are $\{\mu_{2m}\}$. Only LKT is triv = μ_0; $J(p_s)$ is spherical comp factor.

(2a) If ϵ_s even, K-types of $I(p_s)$ are $\{\mu_{2m+1}\}$. LKTs are = $\{\mu_{\pm 1}\}$.

(2b) If $\nu \neq 0$ (p_s FINAL) then $\{\mu_{\pm 1}$ both appear in one composition factor $J(p_s)$.

(2c) If $\nu = 0$ (p_s NOT FINAL) then $J(p_s) = J(p_s)^+ \oplus J(p_s)^-$, each summand with one LKT.

What does M-REGULAR mean?

Continued example: $G = SL(2,R)$, $K = SO(2)$, $\widehat{K} \simeq \mathbb{Z}$.

$H_c = SO(2) = T$, $X^*(H) = \mathbb{Z}$, $\rho = 1$, $\mathfrak{a} = 0$.

Param p_c on H_c is $\lambda_c = n_c \in \mathbb{Z}$ AND choice of $\epsilon_c = \pm 1$ if $n_c = 0$.

All p_c are final since no real roots; p_c M-regular iff $n_c \neq 0$.

In this case $I(p_c) =$ discrete series with HC parameter n_c.

K-types of $I(p_c)$ are $\mu_{n_c + \mathrm{sgn}(n_c)(2m+1)}$ ($m \in \mathbb{N}$.

Always irr, with unique lowest K-type $\mu_{n_c + \mathrm{sgn}(n_c)}$.

Just two parameters are not M-regular: $(0,+)$ and $(0,-)$

Standard rep $I(0,+)$ is hol limit of disc ser, K-types $\{1,3,5,7\cdots\}$, LKT = +1

Standard rep $I(0,-)$ is antihol limit of disc ser, K-types $\{-1,-3,-5,-7\cdots\}$, LKT = -1

Always irr, with unique lowest K-type $\mu_{n_c + \mathrm{sgn}_c}$.

Cayley transforms of parameters I

$p = (H, \nu, (\lambda, \Pi_{\text{im}}))$ nonzero NONFINAL parameter.

Means there is a real coroot $\alpha^\vee$ with

$$\langle \nu, \alpha^\vee \rangle = 0, \qquad \langle \lambda, \alpha^\vee \rangle \text{ even}.$$

$p \approx$ param for reducible temp princ ser for $SL(2, \mathbb{R})_{\alpha^\vee}$.

Same $SL(2, \mathbb{R})_{\alpha^\vee}$ provides more compact Cartan

$$H_c = \text{Cayley}(H, \alpha) = T_c A_c, \quad A_c = \ker(\alpha|_A).$$

NONFINAL condition guarantees that we can Cayley transform p to two parameters, at least one nonzero

$$p_c^\pm = (H_c, \nu_c, (\lambda_c, \Pi^\pm_{\text{im,c}})).$$

$\lambda_c \leftrightarrow \lambda + m\alpha$, m chosen so $\langle \lambda_c, \alpha_c^\vee \rangle = 0$; $p_c^\pm$ NOT M-reg.

Hecht-Schmid identity $I(p) = I(p_c^+) + I(p_c^-)$.

Technicality: because of disconnectedness (*e.g.* $GL(2, \mathbb{R})$), might be one parameter p_c. Char ident is then $I(p) = I(p_c)$.

Gives one-to-several map

non-final nonzero params $\longrightarrow$ non-M-regular nonzero params.

Cayley transforms of parameters II

$p = (H, \nu, (\lambda, \Pi_{\text{im}}))$ nonzero NON-M-REGULAR parameter.

Means there is an imaginary coroot $\beta^\vee \in \Pi_{\text{im}}$ with

$$\langle \lambda, \beta^\vee \rangle = 0.$$

Nonzero assumption guarantees $\beta^\vee$ is noncompact.
$p \approx$ param for limit of discrete series for $SL(2, \mathbb{R})_{\beta^\vee}$.

Same $SL(2, \mathbb{R})_{\beta^\vee}$ provides more split Cartan

$$H_s = \text{Cayley}(H, \beta) = T_s A_s, \quad T_s \supset \ker(\beta|_T).$$

We can Cayley transform p to one nonzero parameter

$$p_s = (H_s, \nu_s, (\lambda_s, \Pi_{\text{im,s}})).$$

Here ν_s extends ν by 0 on the span of the real root α_s, and $\lambda_s \leftrightarrow \lambda$, which is zero and therefore even on $\alpha^\vee$; so $\alpha_s^\vee$ exhibits p_s as non-final.

Gives one-to-several (because of choice of $\beta^\vee$) map

non-M-regular nonzero params $\longrightarrow$ non-final nonzero params.

Knapp-Stein R-groups

Suppose $p = (H, \nu, (\lambda, \Pi_{\text{im}}))$ is M-regular $\rightsquigarrow$ discrete series rep of M.

Packets of parameters

Described how NON-FINALITY of a nonzero parameter allowed moving it to one or more parameters on a more compact Cartan.

In the same way, NON-*M*-REGULARITY of a nonzero parameter allowed moving it to one or more parameters on a more split Cartan.

Doing both things provides equivalence relation on nonzero parameters. Equivalence classes are *R*-packets.

Theorem Suppose G is real reductive.

1. Each *R*-packet contains a unique *M*-regular parameter p_{fin}, which may be characterized as living on the most split Cartan for the packet.
2. Each *R*-packet has exactly 2^r final params p_{fin}, which may be characterized as living on the most compact Cartan for the packet.