Generalizing endoscopic transfer

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Outline

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Examples of endoscopic groups

Slides at
http://www-math.mit.edu/~dav/paper.html/
Joint work with Jeffrey Adams and Lucas Mason-Brown generalizing endoscopic transfer for reductive groups.

Our results concern real reductive groups.

Subject is a morass of technical difficulties, many of which are much worse for $\mathbb{R}$ than for $p$-adic fields.

Example: need to change def of Langlands parameter/$\mathbb{R}$.

I’ll avoid some difficulties by discussing mostly non-archimedean local field $k$, and connected reductive algebraic $G/k$.

Avoid remaining difficulties by ignoring them.
What’s the plan?

Study rep theory of reductive algebraic $G$.

Typically $G$ defined over a local field $k$, but details later.

Endoscopic group: smaller reductive $H$, often $H \not\subset G$.

Examples:

$$G = \text{Sp}(2(p + q), \mathbb{R}), \quad H = \text{SO}(p, p) \times \text{Sp}(2q, \mathbb{R})$$
$$G = \text{Sp}(2(p + r), \mathbb{R}), \quad H = \text{GL}(p, \mathbb{R}) \times \text{Sp}(2r, \mathbb{R}).$$

Endoscopic transfer: (virtual $H$-reps) $\rightarrow$ (virt $G$-reps).

Will define slightly larger class of such $H \not\subset G$.

New examples:

$$G = \text{Sp}(2(p + q + r), \mathbb{R}) \quad H = \text{U}(p, q) \times \text{Sp}(2r, \mathbb{R})$$
$$G = \text{GL}(2p + q, \mathbb{R}) \quad H = \text{GL}(p, \mathbb{C}) \times \text{GL}(q, \mathbb{R})$$
What’s this have to do with Harish-Chandra?

Harish-Chandra’s work on discrete series was rooted in what Hermann Weyl did for compact groups: (Weyl integration) + (Schur orthog) \(\leadsto\) (Weyl char formula).

Harish-Chandra’s work was the same, except that every step required radically new ideas.

One such idea was his method of descent. If \(s \in G\) semisimple, then \(H = G^s\) is again reductive.

Harish-Chandra descent describes any character \(\Theta_G\) near \(s\) in terms of a new character \(\Theta_H\) on \(H\).

In formal language, he defined a linear map descent

\[
(K(G) = \text{virtual reps of } G) \longrightarrow (K(H) = \text{virtual reps of } H).
\]

**Endoscopic transfer** is Harish-Chandra descent applied in the Langlands L-group.
How do you name a group? (case of $\overline{k}$)

To ask about a group $G$, you need first to give it a name.

Lie, Chevalley and Grothendieck solved this problem:

(reductive algebraic group $G$) / algebraically closed $\overline{k}$ ↔ based root datum $\mathcal{R}(G) = (X^*, \Pi, X_*, \Pi^\vee)$.

$X^*$ and $X_*$ are dual lattices: chars/ cochars of max torus in $G$.

finite sets $\Pi \subset X^*$ and $\Pi^\vee \subset X_*$: simple roots/simple coroots.

Any lattice is isomorphic to $\mathbb{Z}^n$, so the name $\mathcal{R}(G)$ of $G$ is two finite collections of $n$-tuples of integers.

Two names are the same iff first collections differ by invertible integer matrix $M$, and second collections differ by $tM^{-1}$.

Example: $GL(2)$ is given by $\Pi = \{(1, -1)\}$, $\Pi^\vee = \{(1, -1)\}$.

Example: the exceptional group $G_2$ is given by

$\Pi = \{(1, 0), (0, 1)\}$, $\Pi^\vee = \{(2, -1), (-3, 2)\}.$
How do you name a group? (case of $k$)

A reductive $G/\overline{k}$ named by the (combinatorial) based root datum $\mathcal{R}(G)$: two finite sets of $n$-tuples of integers.

Defining $G/k$ gives action of $\Gamma = \text{Gal}(\overline{k}/k)$ on $\mathcal{R}(G)$.

Concretely: repn of $\Gamma$ by $n \times n$ integer matrices $\mu(\sigma)$ so

$$\mu(\sigma) \cdot \Pi = \Pi, \quad ^t\mu(\sigma)^{-1} \cdot \Pi^\vee = \Pi^\vee,$$

respecting axioms for a based root datum.

Shorthand: action of $\Gamma$ on the Dynkin diagram of $G$.

$k$-forms of $G$ are inner if $\sim$ same action of $\Gamma$ on $\mathcal{R}(G)$.

Example A rank two unitary group/$k$ starts with a separable quadratic extension of $k$; that is, subgroup $\Gamma_0 \subset \Gamma$ of index two.

Representation of $\Gamma$ on $\mathbb{Z}^2$ is

$$M(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\sigma \in \Gamma_0), \quad M(\sigma) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (\sigma \notin \Gamma_0)$$

All unitary grps w fixed quad ext form a single inner class.
L-group

Defining $G / \text{any } k \xrightarrow{\sim} \text{action of } \Gamma = \text{Gal}(\bar{k}/k) \text{ on } \mathcal{R}(G)$.

Axioms for based root data are symmetric in $(X^*, \Pi) \leftrightarrow (X^*, \Pi^\vee)$.

Dual based root datum is $\mathcal{R}^\vee = (X^*, \Pi^\vee, X^*, \Pi)$.

Gives reductive algebraic dual group $^\vee G$ and

\[ L\text{-group } ^L G = ^\vee G \rtimes \Gamma, \quad \text{(defined over } \mathbb{Z}). \]

Langlands’ insight (local Langlands conjecture):

\[ (\text{analytic rep theory/K of } G(k)) \leftrightarrow (\text{alg geom of } ^L G(K)). \]

Typically $K = \mathbb{C}$ and $k$ is local.

Complex reps of $G(k) \leftrightarrow \text{complex alg geom of } ^L G(\mathbb{C})$

Endoscopic (and generalized endoscopic) groups $H$ correspond to subgroups $^E H \subset ^L G$.

By local Langlands, relating $\hat{H}(k)$ to $\hat{G}(k)$ means relating alg geom of $^L G(\mathbb{C})$ to alg geom of subgroup $^E H(\mathbb{C})$.

Easy! But what the hell is $\leftrightarrow$?
Weil-Deligne group

$k$ $p$-adic field, $\Gamma = \text{Gal}(\overline{k}/k)$, $G$ conn reductive alg/k.

$L$-$G$ complex $L$-group: $1 \to \mathcal{V} G \to L G \to \Gamma \to 1$.

Local Langlands explains irr reps $\widehat{G(k)}$ using $L G$.

Recall: finite residue field $\mathbb{F}_q$ of $k \leadsto$ natural surjection

$$1 \to I_k \to \Gamma \to \widehat{\mathbb{Z}} = \langle \text{Frob} \rangle \to 1.$$ 

Inertia subgroup $I_k$ is profinite compact.

Weil group $W_k = (\text{dense})$ preimage in $\Gamma$ of $\langle \text{Frob} \rangle$:

$$1 \to I_k \to W_k \to \mathbb{Z} = \langle \text{Frob} \rangle \to 1.$$

Weil-Deligne group $W'_k = W_k \ltimes \mathbb{C}$: here $I_k$ acts trivially on $\mathbb{C}$, and Frob acts by multiplication by $q$. 
Langlands parameters

Recall $\Gamma = \text{Gal}(\overline{k}/k)$, and $W_k \subset \Gamma$ is a dense subgroup.

Have two short exact sequences

\[
\begin{array}{cccccc}
1 & \to & \breve{\mathbf{G}} & \to & L\mathbf{G} & \to & \Gamma & \to & 1 \\
& & \uparrow & & \uparrow \phi' & & \uparrow \\
1 & \to & \mathbb{C} & \to & W'_k & \to & W_k & \to & 1
\end{array}
\]

Langlands parameter is a group homomorphism $\phi' : W'_k \to L\mathbf{G}$ compatible with exact sequences.

Means $\phi'|_{\mathbb{C}} : \mathbb{C} \to \breve{\mathbf{G}}$ (one-param nilp alg subgp), and $\phi'$ descends to inclusion $W_k \hookrightarrow \Gamma$.

Loc Langlands conj: $\phi' \leadsto$ finite L-pkt $\Pi(\phi') \subset \hat{G}(k)$.

More conjecture:

1. L-packets partition $\hat{G}(k)$;
2. $\Pi(\phi')$ depends only $\breve{\mathbf{G}}$-conj class of $\phi'$;
3. if $G(k)$ quasisplit, then $\Pi(\phi') \neq \emptyset$. 
Repn theory and algebraic geometry

Want to translate problems about reps of $G(k)$ to alg geom problems about parameters in $L^*G$.

Infl char of $\phi'$ is $\phi = \phi'|_{W_k}$. Each infl char $\phi: W_k \to L^*G$ extends in finitely many ways to $\phi': W'_k \to L^*G$: the parameters of infl char $\phi$.

Since $I_k$ compact, $^\forall G^{\phi(I_k)} = \text{centralizer in } ^\forall G$ of $\phi(I_k)$ is reductive algebraic in $^\forall G$.

Preimage $\widetilde{\text{Frob}}$ in $W_k$ defines $\phi(\widetilde{\text{Frob}}) \in L^*G$, so semisimple alg aut (indep of $\widetilde{\text{Frob}}$) $\sigma_\phi = \text{Ad}(\phi(\widetilde{\text{Frob}})) \in \text{Aut}(^\forall G^{\phi(I_k)})$.

$\sigma_\phi$ defines $^\forall G^\phi = (^\forall G^{\phi(I_k)})^{\sigma_\phi}$, twisted pseudolevi of $^\forall G^{\phi(I_k)}$.

$\eta(\phi) = \text{def } q$-eigenspace of $\sigma_\phi$ on $^\forall g^{\phi(I_k)}$, a vector space of nilpotent Lie algebra elements on which $^\forall G^\phi$ acts.

The algebraic geom we want is $^\forall G^\phi$ orbits on $\eta(\phi)$.

$\eta(\phi)$ is prehomogeneous for $^\forall G^\phi$: finitely many orbits.
Could you repeat that?

Start with a Langlands parameter $\phi': W'_k \to ^L G$.

Restriction $\phi_I$ of $\phi'$ to inertia $I_k \subset \text{Gal}(\bar{k}/k)$ is arithmetic; image is profinite (compact) subgroup of $^L G$:

$$Z_{^G}G(\phi(I_k)) = ^G\phi(I_k)$$

reductive algebraic.

An extension $\phi$ of $\phi_I$ to $W_k$ (called infinitesimal character) is given by a single element $\phi(\text{Frob})$ of $^L G$.

$\phi(\text{Frob})$ defines $\text{aut } \sigma_\phi$ of $^G\phi(I_k)$, fixed points $^G\phi$. $q$-eigspace of $d\sigma_\phi = \text{nilp subspace } \pi(\phi) \subset g^{\phi(I_k)}$.

$\pi(\phi)$ is prehomogeneous for $^G\phi$.

Parameters $\phi'$ of infl char $\phi \leftrightarrow ^G\phi$ orbits $O'$ on $\pi(\phi)$.

irreps of infl char $\phi \leftrightarrow ^G\phi$-eqvt perv sheaves on $\pi(\phi)$.

L-packet of $\phi' \leftrightarrow$ sheaves with support $O'$. 
What’s the plan?

L-group has short exact seq $$1 \rightarrow ^\lor G \rightarrow ^L G \rightarrow \Gamma \rightarrow 1$$.

L-subgroup is $$^L G \supset ^E H \rightarrow \Gamma$$, kernel $$^\lor H$$ reductive:

$$1 \rightarrow ^\lor G \rightarrow ^L G \rightarrow \Gamma \rightarrow 1$$

$$\cup \ \ \ \ | \ \ \ \ |$$

$$1 \rightarrow ^\lor H \rightarrow ^E H \rightarrow \Gamma \rightarrow 1$$

In this setting param $$\phi'_H$$ for $$^E H \sim\rightarrow$$ param $$\phi$$ for $$^L G$$;

$$\pi(\phi_H) \subset \pi(\phi), \quad ^\lor H^\phi \subset ^\lor G^\phi.$$ 

This is the geometric part of local Langlands functoriality.

So relating reps of $$G$$ to reps of $$H$$ amounts to relating perv sheaves on $$\pi(\phi)$$ to perv sheaves on $$\pi(\phi_H)$$.

To get strong theorems relating perverse sheaves to a subvariety, need strong hypotheses on the subvariety.

Example is Goresky-MacPherson Lefschetz formula.

Need subvariety = fixed points of an automorphism.
What’s an endoscopic group?

Langlands params are $^\vee G$ orbits on (algebraic variety).
So action of $s \in {^\vee G} \rightsquigarrow$ automorphism of params.

Endoscopic datum is

1. $s \in {^\vee G}$ semisimple;
2. L-subgroup $^E H \subset (^L G)^s \subset ^L G$, with
3. $^\vee H =$ identity component of $^\vee G^s$ reductive in $^\vee G$.

Root datum $\mathcal{R}(^\vee H)$ has dual root datum $\rightsquigarrow H/\overline{k}$.

$^E H \rightsquigarrow$ action of $\Gamma = \text{Gal}(\overline{k}/k)$ on root data,
$\rightsquigarrow$ inner class of $k$-forms of $H$.

Endoscopic group for $G = H/k$, any form in inner class.
Where’s the fixed point formula?

\[ s \in {}^\vee G \text{ semisimple, } L\text{-subgroup } {}^E H \subset ({}^L G)^s \subset {}^L G, \quad {}^\vee H = {}^\vee (G^s)_0. \]

Hypotheses imply \( {}^E H \) open in \(({}^L G)^s\).

Simplify by assuming \( {}^E H = ({}^L G)^s \). Then

(fixed pts of \( Ad(s) \) on params) = (params for \( {}^E H \)).

This equality allows application of a Lefschetz formula.

More precisely:

\[ \text{tr} (s \text{ action on perv cohom for } {}^L G) = \text{tr} (s \text{ action on perv cohom for } {}^E H). \]

Since \( s \) central in \( {}^E H \), right side is easy.

Equality seems to require \( s \) to centralize \( {}^E H \).

Generalization seems impossible...
Here’s how to generalize

Generalized endoscopic datum is

1. $s \in \check{\check{G}}$ semisimple;
2. $L$-subgroup $^E H \subset \check{L}G$ normalized by $s$;
3. $\check{\check{H}}$ = identity component of $\check{\check{G}}^s$ reductive in $\check{\check{G}}$;
4. quotient action of $\text{Ad}(s)$ on $\Gamma = ^E H/\check{\check{H}}$ is trivial.

As for endoscopic groups,

$^E H \rightsquigarrow$ Galois action on root datum for $\check{\check{H}}$

$\rightsquigarrow$ inner class of $k$-forms of $H$.

These $k$ forms are generalized endoscopic groups.

Define $\xi: ^E H \to \check{\check{H}}$ by $\xi(m) = sms^{-1}m^{-1}$ $(m \in ^E H)$.

Equivalently: $\text{Ad}(s)(m) = \xi(m)m$.

$\xi$ measures failure of $s$ to commute with $^E H$, or equivalently failure of $^E H$ to be endoscopic.

Then $\xi$ factors to $\Gamma = ^E H/\check{\check{H}}$, values in $Z(\check{\check{H}})$.

Precisely: $\xi$ is 1-cocycle of $\Gamma$ with values in $Z(\check{\check{H}})$. 
How do you generalize endoscopic transfer?

**Endoscopic transfer:** should correspond to map sheaves on $L G$ params $\rightsquigarrow$ sheaves on $E H$ params.

**Classical endoscopy:** $s$ acts by conjugation on $L G$ params; fixed points are $E H$ params.

Only $L G$-params in image are $^\vee G$-conj to $E H$-params.

**Generalized endoscopy:** $s$ still acts on $L G$-params, but does not fix $E H$ params: $\text{Ad}(s)(\phi_H(\gamma)) = \xi(\gamma)\phi_H(\gamma)$.

Try modify $\text{Ad}(s)$ by $\xi^{-1}$: $(s \circ_\xi \phi)(\gamma) = \xi^{-1}(\gamma)\text{Ad}(s)(\phi(\gamma))$. But this is not an action except on $E H$ params.

**Solution:** look only at params conjugate to $E H$ params:

$^\vee G \times_\nu_H(E H \text{ params}) \to (L G \text{ params}), \quad (g, \phi'_H) \mapsto \text{Ad}(g)\phi'_H$.

$s$ acts on left space by $s \circ_\xi (g, \phi'_H) = \text{Ad}(g)(\xi^{-1}\phi'_H)$.

Fixed points of $\circ_\xi$ are $E H$ params.
What’s that ★ action on parameters?

To make Langlands params $H (\forall H = \forall G^s)$ into fixed points, needed to compose action of $\text{Ad}(s)$ with $\text{mult}$ by a 1-cocycle.

Following very special case may shed some light. Result stated is Theorem for $k$ archimedean, and in various $p$-adic cases where local Langlands conj is proven.

Desideratum (Langlands); see Borel, Corvallis volume 2.

$$\phi': W'_k \rightarrow ^LG \rightsquigarrow L\text{-packet } \Pi(\phi') = \{\pi_\tau\}.$$  

$\pi_\tau$ is irrep of an inner $k$-form of $G$. Suppose that

$$\xi: W'_k \rightarrow Z(G^\forall)$$

is a 1-cocycle. Define

$$\xi \cdot \phi': W'_k \rightarrow ^LG, \quad (\xi \cdot \phi')(w) = \xi(w)\phi'(w).$$

1. The 1-cocycle condition means $\xi \cdot \phi'$ is also a group homomorphism, new Langlands parameter.

2. $\overline{\xi} \in H^1(W'_k, Z(\forall G)) \rightsquigarrow \text{smooth char } \gamma_{\overline{\xi}}$ of $G(k)$.

3. $\Pi(\xi \cdot \phi') = \{\gamma_{\overline{\xi}} \otimes \pi_\tau\}$.

Mult param by $Z(\forall G)$ cocycle tensors $G$ reps with 1-diml rep.
Classical endoscopic groups

Suppose $G/k$ reductive and $P = MN$ parabolic over $k$.
Put $X^*(M) =$ ratl chars of $M$, a $\Gamma$-fixed sublattice of $X^*(G)$.
$\leadsto \Gamma$-fixed sub $\subset X^*(^\vee G) \leadsto \Gamma$-fixed torus $^\vee A \subset ^\vee G$.
$^\vee M \overset{\text{def}}{=} ^\vee G^A$ is $\Gamma$-stable, dual to $M$: $^L M \simeq ^\vee M \rtimes \Gamma$.
Generic $s \in ^\vee A \leadsto (^L G)^s = ^L M \leadsto $ endoscopic group $M$.
Endoscopic transfer (reps of $M) \leadsto $ (reps of $G$) is $\text{Ind}_{MN}^G$.
Endoscopy is more powerful than parabolic induction.
Allows $Z_{^L G}(\Gamma$-fixed element), not just $\Gamma$-fixed torus.
But endoscopy also misses a lot of interesting subgroups.
Rational Cartan subgrp of $G$ is almost never endoscopic.
Generalized endoscopic groups

Suppose $L$ any rational Levi subgroup of $G \hookrightarrow \Gamma$ action on root datum of $L$.

If $G$ simply connected, easy to find $L \subset L \subset G$.

In general, get extended group $E L \subset L \subset G$.

$s \in Z(\vee L)$ generic $\mapsto (s, E L)$ gen endoscopic datum $\mapsto L$ generalized endoscopic for $G$.

Endoscopic transfer from general ratl Levi $L$ should be important generalization of parabolic induction.

Over $\mathbb{R}$, this is Zuckerman’s cohomological induction.

Over a $p$-adic field, this is still a mystery.

Harish-Chandra would tell us to get to work.