What does the unitary dual look like?

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Real Reductive Groups and the Theta Correspondence

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Outline

Introduction

Your friend the Weyl group

My friend the affine Weyl group

What do we know now about $\widehat{G(\mathbb{R})}_u$?

The fundamental parallelepiped

The FPP conjecture

Slides eventually at
`http://www-math.mit.edu/~dav/paper.html`

What's this about really?

$G(\mathbb{R})$ real reductive algebraic group.

$\widehat{G(\mathbb{R})}_u$ = (equiv classes of) irr unitary reps of $G(\mathbb{R})$.

I'll assume that studying this set (unitary dual) is the most world's best problem.

How can you approach it?

I'll start by answering the question in the title.

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

Example of $SL(2,\mathbb{R})$

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$$SL(2,\mathbb{R}) \rightsquigarrow \left\{ \begin{array}{l} (\text{point}, \mathbb{R}^1, \{0\}) \longleftrightarrow \text{spherical unitary princ series} \\ (\text{point}, \mathbb{R}^1, \{0\}) \longleftrightarrow \text{nonsph unitary princ series} \\ (\text{point}, \mathbb{R}^0, \mathbb{N}) \longleftrightarrow \text{holomorphic discrete series} \\ (\text{point}, \mathbb{R}^0, \mathbb{N}) \longleftrightarrow \text{antihol discrete series} \\ ([0,1], \mathbb{R}^0, \{0\}) \longleftrightarrow \text{complementary series} \end{array} \right\}$$

This is two lines, two half lattices, and one interval.

Picture for $SL(2,\mathbb{R})$ found by Valentine Bargmann in 1947.

For those with OCD or PhD: more words are needed to make this precise. Example: nonsph princ ser at 0 is sum of two irreps: $\text{nonsph}(\text{pt},0,0) = \text{hol ds}(\text{pt},0,0) + \text{antihol ds}(\text{pt},0,0)$.

That the picture works for any real reductive $G(\mathbb{R})$ comes from Harish-Chandra, Langlands, Knapp, Zuckerman about 1985.

So what do we need to do?

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j \; U_j \times V_j \times C_j.$$

Describe $\widehat{G(\mathbb{R})}_u \leftrightsquigarrow$ describe cpt polyhedra U_j.

Vec space V_j, cone-in-lattice C_j important but easy.

Main question today: what do cpt polyhed U_j look like?

Answer: U_j is finite union of product of simplices.

Goals for today:

1. say what kinds of simplices are allowed
2. recall work of Barbasch, (Barbasch and his friends) giving beautiful precise list of simplices in many cases
3. say how `atlas` software computes ugly precise list of simplices in all cases

Remind me about the Weyl group...

G cplx conn red alg group $\supset B$ Borel $\supset H$ max torus.

(X^* alg chars of H) $\supset$ (R roots) $\supset$ (Π simple roots).

(X_* alg cochars) $\supset$ ($R^\vee$ coroots) $\supset$ ($\Pi^\vee$ simple coroots).

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

$\mathfrak{h}^*_{\mathbb{R}}$ is the real vector space where the classical root system lives.

Root hyperplanes are $E_\alpha = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) = 0\}$ (each α in R).

Each root α defines simple reflection: $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$,

$$s_\alpha(\gamma) = \gamma - \gamma(\alpha^\vee) \cdot \alpha, \quad s_\alpha(\alpha) = -\alpha, \quad s_\alpha = \text{identity on } E_\alpha.$$

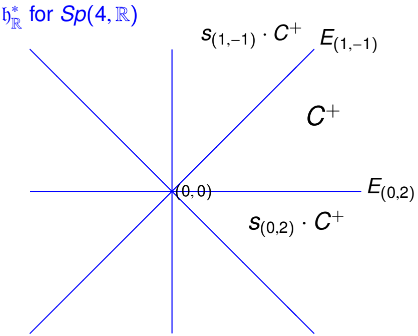
Weyl group of G is $W =$ group generated by all s_α.

The open positive Weyl chamber is the open simplicial cone

$$C^+ = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} | \gamma(\alpha^\vee) > 0 \quad (\alpha \in \Pi \text{ simple})\}.$$

A Weyl chamber in $\mathfrak{h}^*_{\mathbb{R}}$ is a subset $w \cdot C^+$ (some $w \in W$).

What do Weyl chambers look like?



$\overline{C}^+$ is fundamental domain for W action on $\mathfrak{h}^*_{\mathbb{R}}$.

Action of W on Weyl chambers is simply transitive

dominant faces of $\overline{C}^+$ of codim $d \longleftrightarrow$ cardinality d subsets of Π

any face of $\mathfrak{h}^*_{\mathbb{R}}$ is in $W \cdot$(unique dom face)

And the affine Weyl group?

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

Aff coroots are $R^{\vee,\mathrm{aff}} = \{(\alpha^\vee, m) \mid \alpha^\vee \in R^\vee, m \in \mathbb{Z}\}$.

Pos aff coroots are $R^{\vee,\mathrm{aff},+} = \{(\alpha^\vee, m) \mid m > 0 \text{ or } \alpha^\vee \in R^{\vee,+}, m = 0\}$.

Write $\alpha_0^\vee$ = lowest coroot (unique since G simple).

Simple aff coroots are $\Pi^{\vee,\mathrm{aff}} = \{(\alpha^\vee, 0) \mid \alpha^\vee \in \Pi^\vee\} \cup \{(\alpha_0^\vee, 1)\}$.

Aff hyperplanes $E_{\alpha,m} = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) + m = 0\}$.

aff coroot $\rightsquigarrow$ simple aff reflection: $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$,

$$s_{\alpha^\vee,m}(\gamma) = \gamma - (\gamma(\alpha^\vee) + m) \cdot \alpha, \quad s_{\alpha^\vee,m} = \text{id on } E_{\alpha^\vee,m}.$$

Affine Weyl group of G is W^{aff} = group generated by all $s_{\alpha^\vee,m}$.

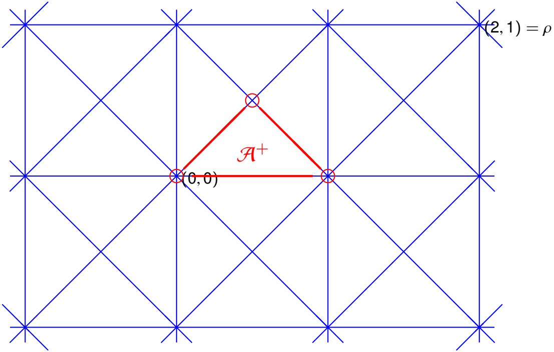
The open fundamental alcove is the open simplex

$$\mathcal{A}^+ = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} | \gamma(\alpha^\vee) + m > 0 \quad ((\alpha^\vee, m) \in \Pi^{\vee,\mathrm{aff}} \text{ simple})\}$$
$$= \{\gamma \in C^+ \mid \gamma(\alpha_0^\vee) < 1\}.$$

An alcove in $\mathfrak{h}^*_{\mathbb{R}}$ is a subset $w \cdot \mathcal{A}^+$ (some $w \in W^{\mathrm{aff}}$).

What do alcoves look like?

$\mathfrak{h}^*_{\mathbb{R}}$ for $Sp(4,\mathbb{R})$



$\overline{\mathcal{A}}^+$ is fundamental domain for W action on $\mathfrak{h}^*_{\mathbb{R}}$.

Action of W^{aff} on alcoves is simply transitive

fund faces of $\overline{\mathcal{A}}^+$ of codim $d \longleftrightarrow$ order d subsets of $\Pi^{\vee,\text{aff}}$

any face of $\mathfrak{h}^*_{\mathbb{R}}$ is in $W^{\text{aff}} \cdot$(unique fundamental face)

What good are all these faces?

Langlands classif: irrs of real infl character indexed by

1. discrete parameter $(x, \lambda) \approx$ lowest K-type
2. continuous parameter $\gamma =$ infinitesimal character.

Here $x =$ KGB element: orbit of $K(\mathbb{C})$ on Borels in $G(\mathbb{C})$.

Finite # of x: 3 for $SL(2, R)$, 201 for $Sp(8, \mathbb{R})$, 320206 for split E_8.

Given x, set of allowed λ is finite # of cones in lattices

Given (x, λ), set of allowed γ is affine space $V_{\mathbb{R}}(x, \lambda) \subset \mathfrak{h}^*_{\mathbb{R}}$.

Therefore $V_{\mathbb{R}}(x, \lambda)$ is disjoint union of faces.

Theorem (Speh-V) Fix discrete parameter (x, λ).

1. If $\gamma_1, \gamma_2 \in$ same face of $V_{\mathbb{R}}(x, \lambda)$, then irr reps $J(x, \lambda, \gamma_1)$ and $J(x, \lambda, \gamma_2)$ are both unitary or both nonunitary.
2. Set of unitary γ is a compact polyhedron $U(x, \lambda) \subset V_{\mathbb{R}}(x, \lambda)$, a finite union of faces.

What does that say about the unitary dual?

Corollary Set $\widehat{G(\mathbb{R})}_{u,\text{real}}$ = unitary reps of real infl char. Then

$$\widehat{G(\mathbb{R})}_{u,\text{real}} = \bigcup_{x \in KGB} \bigcup_{\substack{\lambda \text{ allowed} \\ \text{for } x}} U(x,\lambda)$$

Claim in introduction:

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j \; U_j \times V_j \times C_j.$$

Polyhedra $U(x,\lambda)$ are the U_j in the introduction.

Extending Cor to all infl chars gives real vector spaces V_j.

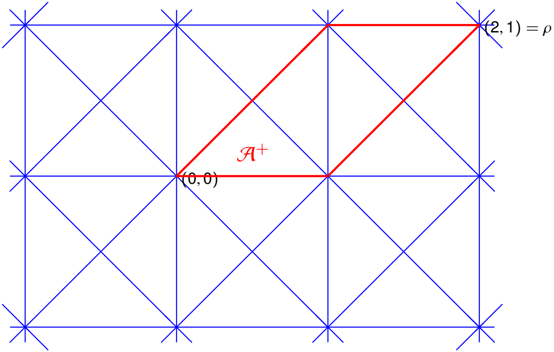
Given x, λ's are finite union of cones in lattices C_j.

To prove Claim, need to show $U(x,\lambda)$ is nearly independent of λ.

To describe unitary dual, need to compute all $U(x,\lambda)$.

What's the FPP...

FPP $\subset \mathfrak{h}^*_{\mathbb{R}}$ for $Sp(4, \mathbb{R})$



fundamental parallelepiped $= \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \leq \gamma(\alpha^\vee) \leq 1 \mid (\alpha \in \Pi)\}$

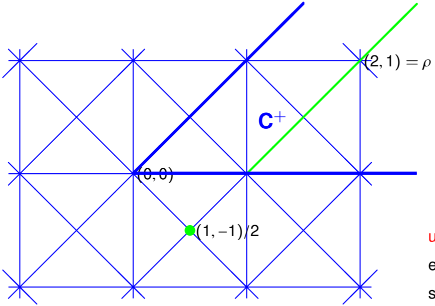
Union of $\#W/\#Z(G_{sc})$ alcoves.

$G(\mathbb{R})$	# alcoves	# faces
$SL(2, \mathbb{R})$	1	3
$Sp(4, \mathbb{R})$	4	19
split E_8	696729600	2416970476

. . . and how does it help the unitary dual?

Real Langlands parameter (x, λ, γ) defines

1. Cartan involution $\theta = \theta(x)$ acting on $\mathfrak{h}^*_{\mathbb{R}}$
2. Cartan decomp $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{t}^*_{\mathbb{R}} + \mathfrak{a}^*_{\mathbb{R}}$ (± 1 eigenspaces)
3. differential of λ $d\lambda \in \mathfrak{t}^*_{\mathbb{R}}$
4. "A-parameter $\nu = \gamma(x, \lambda, \gamma) = \gamma - d\lambda$
5. Definition of param $\rightsquigarrow$ $\gamma \in \overline{C^+}$ is dominant.



(x, λ) first disc series, Siegel par

$$\theta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \; d\lambda = (1/2, -1/2)$$

$$\mathfrak{a}^*_{\mathbb{R}} = \{(t, t)\}$$

green line is allowed infl chars γ.

unitary part is vertices $(2 + m_0, m_0)/2$,

edges $\{(1 + t, t) \mid t \in (1 + m_1/2, 1 + (m_1 + 1)/2)\}$,

some m_0 and m_1 in $\mathbb{N}$

Define $U_{FPP}(x, \lambda) = \{\gamma \in FPP \mid J(x, \lambda, \gamma) \text{ is unitary}\}$.

The FPP conjecture

Suppose (x, λ, γ) is a real Langlands parameter of infinitesimal character γ.

FPP conjecture is distilled from work of Dan Barbasch.

Define $S(\gamma) = \{\alpha \in \Pi \mid \gamma(\alpha^\vee) \leq 1\}$, a set of simple roots,

$\mathfrak{q} = \mathfrak{q}(\gamma) = \mathfrak{l} + \mathfrak{u}$ parabolic with Levi generated by $S(\gamma)$.

1. γ belongs to the FPP if and only if $\mathfrak{q} = \mathfrak{g}$.
2. Conjecture If $J(x, \lambda, \nu)$ is unitary, then $\mathfrak{q}$ is θ-stable.
3. If $\mathfrak{q}$ is θ-stable, then $J(x, \lambda, \gamma)$ is good range cohomologically induced from $J(x_L, \lambda_L, \gamma_L)$ on L.
   Here $\lambda_L = \lambda - \rho(\mathfrak{u})$, $\gamma_L = \gamma - \rho(\mathfrak{u})$, $\gamma_L \in FPP(L)$.

Assuming this conjecture,

$$U(x, \lambda) = \bigcup_{\theta\text{-stable } \mathfrak{q}} U_{FPP}(x_L, \lambda_L) + \rho(\mathfrak{u})$$

Conclusion: assuming conjecture, unitary dual is known if we compute (finitely many) $U_{FPP}(x, \lambda)$, the FPP infl characters for unitary reps in the series (x, λ).