18.781 Problem Set 7 solutions

Due Monday, April 8 in class.

This problem set is about continued fractions. To fix the notation, I'll write here a little of what's written in the text. The starting point is two integers

$$u_0, u_1, \qquad u_1 \geq 1.$$

The algorithm for computing the continued fraction expansion is very much like the Euclidean algorithm: repeated division with remainder

$$\begin{aligned} u_0 &= u_1 a_0 + u_2, \qquad (0 \leq u_2 < u_1)\\ u_1 &= u_2 a_1 + u_3, \qquad (0 \leq u_3 < u_2)\\ &\vdots\\ u_{n-1} &= u_n a_{n-1} + u_{n+1} \qquad (0 \leq u_{n+1} < u_n)\\ u_n &= u_{n+1} a_n. \end{aligned}$$

Then

$$\begin{aligned} \frac{u_0}{u_1} &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{1}{a_n}}}}\\ &=_{\text{def}} \langle a_0, \ldots, a_n \rangle. \end{aligned}$$

1a. The text says that you should start with a fraction u_0/u_1 in lowest terms; that is, with the property that u_0 and u_1 have no common factor. If you do that, what is the value of u_{n+1}?

The calculation is exactly the Euclidian algorithm for calculating

$$\gcd(u_0, u_1) = \gcd(u_1, u_2) = \cdots = \gcd(u_{n+1}, 0) = u_{n+1}.$$

The assumption is that $\gcd(u_0, u_1) = 1$, so $u_{n+1} = 1$.

1b. Explain what happens in the algorithm above if you start with a fraction u_0/u_1 that is not in lowest terms.

The algorithm still calculates $\gcd(u_0, u_1) = u_{n+1}$; so u_{n+1} must divide every u_j. The calculation with these not-relatively-prime u_0 and u_1 is just u_{n+1} times the calculation with u_0/u_{n+1} and u_1/u_{n+1}. The a_j appearing are exactly the same: the conclusion is

$$u_0/u_1 = \langle a_0, a_1, \ldots, a_n \rangle = (u_0/u_{n+1})/(u_1/u_{n+1}).$$

That is, the algorithm for calculating the continued fraction expansion is still exactly correct even if u_0/u_1 is **not** in lowest terms.

2. Define

$$A_j = \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},$$

and define

$$\begin{pmatrix} P_j & P_{j-1} \\ Q_j & Q_{j-1} \end{pmatrix} = A_0 A_1 \cdots A_j. \qquad (n \geq j \geq 0$$

Prove that for all $0 \leq j \leq n$

$$\langle a_0, \ldots, a_j \rangle = P_j / Q_j,$$

$$P_j Q_{j-1} - P_{j-1} Q_j = (-1)^{j+1},$$

$$Q_0 = 1, \qquad Q_j \geq Q_{j-1},$$

with equality only if $j = 1$ **and** $a_1 = 1$**. Finally, for** $1 \leq j \leq n$**,**

$$\frac{P_j}{Q_j} - \frac{P_{j-1}}{Q_{j-1}} = \frac{(-1)^{j+1}}{Q_j Q_{j-1}}.$$

I'll prove each of these formulas by induction on j. For $j = 0$ we have

$$P_0 = a_0, \quad Q_0 = 1, \qquad P_{-1} = 1, \quad Q_{-1} = 0.$$

Therefore

$$P_0/Q_0 = a_0 = \langle a_0 \rangle, P_0 Q_{-1} - P_{-1} Q_0 = a_0 \cdot 0 - 1 \cdot 1 = -1,$$

proving the first two formulas. The third is also clear. For any value of j, the fourth formula arises by dividing the second by $Q_j Q_{j-1}$. According to the third formula, this division is legal for $1 \leq j \leq n$; so I'll say no more about the fourth formula.

Now for the induction step: we suppose that $1 \leq j \leq n$, and that the first three formulas are known for $j - 1$; we want to prove them for j. Notice first of all that by definition

$$\begin{aligned} \begin{pmatrix} P_j & P_{j-1} \\ Q_j & Q_{j-1} \end{pmatrix} &= A_0 A_1 \cdots A_{j-1} \begin{pmatrix} a_j & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_{j-1} & P_{j-2} \\ Q_{j-1} & Q_{j-2} \end{pmatrix} \begin{pmatrix} a_j & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_{j-1} a_j + P_{j-2} & P_{j-1} \\ Q_{j-1} a_j + Q_{j-2} & Q_{j-1} \end{pmatrix}. \end{aligned}$$

Multiplying matrices gives recursion formulas

$$P_j = P_{j-1} a_j + P_{j-2}, \qquad Q_j = Q_{j-1} a_j + Q_{j-2}.$$

For the first formula, I will cheat and use the result of Problem 3. (You can check that the solution of Problem 3 won't use the result of Problem 2, so this is not really cheating.) We get

$$\langle a_0, \ldots, a_j \rangle = \frac{P_{j-1} a_j + P_{j-2}}{Q_{j-1} a_j + Q_{j-2}} = \frac{P_j}{Q_j},$$

where at the end I used the recursion formula established above.

Once we know these four matrix entries, the second formula is the determinant:

$$P_jQ_{j-1} - P_{j-1}Q_j = \det(A_0A_1\cdots A_j) = \det(A_0)\det(A_1)\cdots\det(A_j).$$

Each factor A_i clearly has determinant -1, so the product is $(-1)^{j+1}$..

For the third formula, we already know that $Q_0 = 1$ and $Q_{-1} = 0$. The recursion above is

$$Q_j = Q_{j-1}a_j + Q_{j-2};$$

the first term is greater than or equal to Q_{j-1} since $a_j > 0$, and the second is nonnegative. This proves that $Q_j \geq Q_{j-1}$. Equality can hold only if $Q_{j-2} = 0$ (so that $j = 1$) and $a_j = 1$.

3. If $x > 0$ is any real number, define

$$\langle a_0, \ldots, a_n, x\rangle =_{\mathbf{def}}= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{x}}}}}$$

(If x is a positive integer, this is consistent with our notation for continued fractions.) Using the notation of Problem 2, prove that

$$\langle a_0, \ldots, a_n, x\rangle = \frac{P_nx + P_{n-1}}{Q_nx + Q_{n-1}}.$$

We prove this by induction on n. If $n = 0$, the desired formula is

$$a_0 + \frac{1}{x} = \frac{a_0x + 1}{1 \cdot x + 0},$$

which is true. If $n \geq 1$, then (by inductive hypothesis)

$$\begin{aligned}
\langle a_0, \ldots, a_{n-1}, a_n, x\rangle &= \langle a_0, \ldots, a_{n-1}, a_n + \frac{1}{x}\rangle \\
&= \frac{P_{n-1}(a_n + \frac{1}{x}) + P_{n-2}}{Q_{n-1}(a_n + \frac{1}{x}) + Q_{n-2}} \\
&= \frac{x(P_{n-1}a_n + P_{n-2}) + P_{n-1}}{x(Q_{n-1}a_n + Q_{n-2}) + Q_{n-1}} \\
&= \frac{P_nx + P_{n-1}}{Q_nx + Q_{n-1}}
\end{aligned}$$

Here we first multiplied numerator and denominator by x to clear fractions, then used the recursion formulas for P_n and Q_n from Problem 2.

4. Find an explicit formula (something like $4 - 2\sqrt{3}$) for the periodic continued fraction

$$\langle 1, 2, 3, 1, 2, 3, 1, 2, 3, \dots \rangle = \langle \overline{1, 2, 3} \rangle.$$

(Hint: if you use the previous problems, you can make most of the arithmetic into multiplying some 2×2 matrices.)

Write $x = \langle \overline{1, 2, 3} \rangle$. Then by definition

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{x}}}$$

Matrix multiplication gives

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 7 & 2 \end{pmatrix}.$$

According to Problem 2, the definition written above simplifies to

$$x = \frac{10x + 3}{7x + 2}, \qquad 7x^2 + 2x = 10x + 3, \qquad 7x^2 - 8x - 3 = 0.$$

The roots of this last equation are $\dfrac{4 \pm \sqrt{37}}{7}$. Our x is clearly greater than 1, so we need the positive square root:

$$x = \frac{4 \pm \sqrt{37}}{7} = 1.44039464\dots$$

This is consistent with the first few continued fraction approximations computed above:

$$1, \quad \frac{3}{2} = 1.5, \quad \frac{10}{7} = 1.42857\dots.$$

The next two are

$$\frac{13}{9} = 1.444\dots, \quad \frac{36}{25} = 1.44.$$

Notice that the approximations are alternately above and below the limit.