18.781 Problem Set 6

Due Monday, October 24 in class.

1. Suppose that

$$Q(x,y) = Ax^2 + Bxy + Cy^2$$

is a binary integral quadratic form. Recall from class that the discriminant of Q is defined to be

$$D = B^2 - 4AC.$$

The form Q is said to *represent zero* if there are integers x and y, not both zero, such that $Q(x,y) = 0$.

1(a). Suppose that either A or C is zero. Show that then Q represents zero, and the discriminant is a perfect square.

1(b). Suppose that A and C are both non-zero, $(x,y) \neq (0,0)$, but $Q(x,y) = 0$. Explain why x and y are *both* non-zero.

1(c). Suppose that A and C are both non-zero, $(x,y) \neq (0,0)$, but $Q(x,y) = 0$. Prove that $(2Ax + By)/y$ is a (rational) square root of D. Explain why it follows that D must be a perfect square.

1(d). Suppose that A and C are both non-zero, and that D is a perfect square. Prove that Q represents zero.

1(e). Suppose that Q represents zero. Prove that there is a new coordinate system

$$u = px + qy, \qquad v = rx + sy$$

(with p, q, r, s integers satisfying $ps - qr = 1$) so that in the new coordinates

$$Q(u,v) = A'u^2 + B'uv,$$

with B' a square root of D. (Hint: you can quote results from the exercises for section 5.7. This problem is still a bit more than I should really expect you to be able to do, so don't lose sleep over it.)

2. I proved in class Dirichlet's theorem that if ξ is any irrational number, then there are infinitely many rational numbers p/q so that

$$|\xi - p/q| < 1/q^2.$$

The point of this problem is to show that you can't do much better than this in general. Define

$$\xi_0 = (1 + \sqrt{5})/2,$$

the larger root of the equation

$$f(x) = x^2 - x - 1.$$

This is the Golden Ratio, about which you can read in art history classes as well as in mathematics. I'll talk about this equation using the binary quadratic form

$$Q(p,q) = p^2 - pq - q^2.$$

2(a). Calculate the discriminant of the quadratic form Q. Explain why Q is indefinite and does not represent zero.

2(b). List the pairs of numbers appearing on successive edges of John Conway's river for the quadratic form Q. (The sequence of pairs eventually repeats; you can list the pairs in one period, then put a bar over it.)

2(c). Show that if p and q are any integers with q not zero, then

$$|f(p/q)| \geq 1/q^2.$$

2(d). Suppose that

$$1/2 \leq x \leq \xi_0.$$

Prove that

$$|f(x)| \leq (\xi_0 - x)(2\xi_0 - 1) = (\xi_0 - x)(\sqrt{5}).$$

Deduce that

$$|\xi_0 - x| \geq f(x)/\sqrt{5}.$$

2(e). Suppose that p/q is a rational approximation to ξ_0, and that $p/q < \xi_0$. Prove that

$$|\xi_0 - p/q| > 1/\sqrt{5}q^2.$$

Remark. This problem says that ξ_0 can't be very well approximated by rational numbers from below; for example, you can't get something like Dirichlet's theorem with $1/3q^2$ in place of $1/q^2$. (It's also true that you can't get better-than-$1/\sqrt{5}q^2$ approximations from above, but the inequalities are not quite so simple as in 2(d) above.) The number $1/\sqrt{5}$ is best possible: a theorem of Hurwitz says that if ξ is any irrational, then there are infinitely many rationals p/q with

$$|\xi - p/q| < 1/\sqrt{5}q^2.$$

Do you see why this doesn't contradict 2(e)?

3. Write down a specific irrational number ξ_1 with the property that for every positive integer k, there are infinitely many rational numbers p/q such that

$$|\xi_1 - p/q| < 1/q^k.$$

(Hint: taking a finite number of terms in the decimal expansion of ξ gives a rational approximation to ξ. Usually the error in this approximation is terrible (like $1/q$), but sometimes it's much smaller.)