# 18.781 Problem Set 5

Due Monday, October 17 in class.

**1.** You might remember from calculus Newton's method for finding roots of the equation $f(t) = 0$. The idea is to begin with an approximation $t_0$ to a root; to write down the linear approximation

$$f(t) \approx f(t_0) + f'(t_0)(t - t_0),$$

and then to choose as a (hopefully better) approximation to a root of $f$ a root $t_1$ of the linear equation

$$0 = f(t_0) + f'(t_0)(t_1 - t_0);$$

that is, to define

$$t_1 = t_0 - f(t_0)/f'(t_0).$$

Now you can repeat the process starting with $t_1$ in place of $t_0$. This makes sense as long as $f'(t_i) \neq 0$, and under favorable conditions the sequence $\{t_i\}$ converges to a root of $f$. In this problem I'll look at the function $f(t) = t^2 - N$, with $N$ a positive integer; approximating roots of $f$ means approximating $\sqrt{N}$.

**1(a).** Show that if $t_0$ is any positive number (regarded as an approximation to $\sqrt{N}$) then Newton's method leads to the new approximation

$$t_1 = \frac{1}{2}(t_0 + \frac{N}{t_0}).$$

(You can think of this as saying that if $t_0$ is (for instance) a little smaller than $\sqrt{N}$, then $N/t_0$ is roughly the same amount *larger* than $\sqrt{N}$, so the average of these two numbers is quite close to $\sqrt{N}$.)

**1(b).** Show that if $x_0$ and $y_0$ are positive integers, and we think of $t_0 = x_0/y_0$ as a rational approximation to $\sqrt{N}$, then Newton's method leads to a new rational approximation

$$t_1 = x_1/y_1, \qquad x_1 = x_0^2 + Ny_0^2, \quad y_1 = 2x_0y_0.$$

**1(c).** Suppose that $(x_0, y_0)$ is a solution of Pell's equation $x^2 - Ny^2 = 1$. Prove that the pair $(x_1, y_1)$ defined in (b) is also a solution.

**1(d).** You may notice that the new solutions to Pell's equation provided by (c) all have $y$ even. Suppose that $N$ is an even integer not divisible by 4. Prove that if $(x, y)$ is *any* solution of Pell's equation $x^2 - Ny^2 = 1$, then $y$ must be even.

**2.** I'll be proving in class that if $\xi$ is any irrational real number, then there are infinitely many rational numbers $p/q$ such that

$$|\xi - p/q| < 1/q^2.$$

Prove that this assertion is *false* for every rational number: that is, that if $r$ is a rational number, then there are only finitely many distinct rational numbers $p/q$ such that

$$|r - p/q| < 1/q^2.$$

**3.** Problem 5.4.5 in the text on page 83.