

18.781 Problem Set 10

Due Monday, December 5 in class.

1. Find a monic polynomial $f(x)$ with integer coefficients such that

$$f(\sqrt{3} + \sqrt{5}) = 0.$$

2. Suppose that α is an algebraic integer, and that

$$f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$$

is the monic polynomial of lowest degree satisfied by x . Prove that α^{-1} is an algebraic integer if and only if $a_0 = \pm 1$.

3. An algebraic integer α is said to have *degree at most m* if it satisfies a monic polynomial with integer coefficients having degree m . Suppose that α has degree at most m , and g is any polynomial with integer coefficients. Prove that $g(\alpha)$ is an algebraic integer of degree at most m .

4. (Denominators for algebraic numbers.) Suppose that β is a complex number, f is a non-zero polynomial with rational coefficients, and that $f(\beta) = 0$. Prove that there is a positive integer d such that $d\beta$ is an algebraic integer. (That is, any algebraic number is an algebraic integer divided by a positive ordinary integer.) Explain how to find a possible d by inspection of the polynomial f .

5. In the axioms for a ring given on page 182 of the text, the axiom $a \times 0 = 0$ is listed. Prove that this fact is a consequence of the other axioms (and therefore it should not be listed as an axiom).

6. Give an example of a non-commutative ring having exactly eight elements. (Hint: use matrices with entries in the field $\mathbb{Z}/2\mathbb{Z}$ of two elements. I think that rings having a smaller number of elements have to be commutative, but I would be entertained to be shown wrong.)