Course outline for 18.755

March 29, 2020

1 Introduction

The class "Lie groups and Lie algebras II" is going to be on Zoom from now on, at `https://zoom.us/j/910271961` I'm trying to learn to use a digital tablet, which is what I'm planning to use for most classes. You can listen live, which I recommend if it's at all possible; but the recording will be available at

`https://video.odl.mit.edu/collections/`

There is a recording of the class Wednesday March 11 already there.

My experience with many Zoom meetings this week has been that seeing and hearing familiar people is extraordinarily valuable. I have therefore set up the class Zoom meeting to enable your cameras. If you don't want to appear to the class, you can easily turn off your camera; a camera icon and a microphone icon appear in the lower left of the Zoom screen, and you can click on them. If you are OK with a picture but not live, you can add one in your Zoom profile `https://zoom.us/profile` It will appear whenever your camera is muted.

I think that the lecture will work best if your microphones are usually muted, so they will be muted when you join the meeting. I strongly encourage you to ask questions whenever you have them, or if anything is unclear. You can unmute your microphone temporarily by holding the space bar on your keyboard, or for a longer time by clicking the microphone icon.

Office hours will take place at a different Zoom address

`https://zoom.us/j/535168014`

(Different because for this one your microphone will be automatically unmuted, and there will be no recording.) They are scheduled Monday 4-5:30 (I'll start the office hour Zoom immediately upon ending the class) and Tuesday 2:30-3:30. If these hours are inconvenient, or you have a question later in the week, please send me an email suggesting some possible times; I'll get back to you as soon as possible to set up a meeting.

2 Topics and schedule

This section is an outline of the mathematics to be covered in the rest of the class. I will update it frequently as we go along, posting versions (`755scheduleC`,...)

to the web page. Bird's eye view: I want to describe in some detail the classification of compact connected Lie groups, and to say as much as possible about proofs.

One reason for this focus is that compact Lie groups are central to all kinds of mathematics: the Gram-Schmidt process is something that (almost) everybody at MIT learns, and it's about the structure of the (compact) orthogonal group.

Another reason for the focus is one of those happy "coincidences" with which mathematics is seasoned: the **answer** for the classification of compact connected Lie groups is extremely close to the answer to a variety of rather different classification problems: finite simple groups, and rational surface singularities, to name two. It's also the case that many of the ideas in the proof are relevant in these different settings.

Monday March 30: homotopy Long exact sequence in homotopy groups for $H \to G \to G/H$. Inductive proof that

$$\pi_1(SU(n)) = \{1\}, \qquad \pi_1(Sp(n)) = \{1\}$$
$$\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z} \quad (n \geq 3).$$

Wednesday April 1: Frobenius State Frobenius theorem relating integrable distributions and foliations on manifolds. Deduce existence of a subgroup for every Lie subalgebra.

Friday April 3: exp(maps) If H and G are Lie groups with H connected and simply connected, then Lie group homomorphisms from H to G are in one-to-one correspondence with Lie algebra homomorphisms from $\mathfrak{h}$ to $\mathfrak{g}$. (If H is *not* simply connected, then there may be more Lie algebra homomorphisms than Lie group homomorphisms.)

Monday April 6: tori A *torus* is a compact abelian Lie group. A *lattice* is a finitely generated torsion-free abelian group. Describe two equivalences of categories from tori to lattices (*covariant* $T \to X_*(T)$ and *contravariant* $T \to X^*(T)$). Definitions are

$$X_*(T) = \mathrm{Hom}(U(1), T), \qquad X^*(T) = \mathrm{Hom}(T, U(1)).$$

Here $U(1)$ is the circle group.

Wednesday April 8: weights Theorem: Representations of a torus T—that is, Lie group homomorphisms $\pi\colon T \to GL(N, \mathbb{C})$ to matrices, modulo conjugation by $GL(N)$—are in one-to-one correspondence with N-tuples of weights

$$(\xi_1, \ldots, \xi_N) \in [X^*(T)]^N$$

modulo permutation.

Friday April 10: maximal tori and roots If G is a compact Lie group and $T \subset G$ is a maximal torus, then the *roots of T in G* are the nonzero weights of the complexified adjoint representation:

$$R(G, T) \subset X^*(T) \backslash \{0\}.$$

Define root spaces, relate to the Lie bracket:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

This is an *enormous* amount of information about the structure of $\mathfrak{g}$: if $\alpha+\beta$ is *not* a root, then the two root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ must *commute* with each other.

Monday April 13: root $\mathbf{SU(2)}$, coroots Each root $\alpha \in R(G,T)$ gives rise to a Lie group homomorphism

$$\phi_\alpha \colon SU(2) \to G$$

defined up to conjugation by T. Restricting ϕ_α to $U(1) \subset SU(2)$ defines

$$\alpha^\vee \colon U(1) \to T, \qquad \alpha^\vee \in X_*(T).$$

These are the *coroots*

$$R^\vee(G,T) \subset X_*(T) \backslash \{0\}.$$

Wednesday April 15: root data for classical groups A maximal torus in $G = U(n)$ is the diagonal subgroup $T = U(1)^n$, so

$$X^*(T) \simeq X_*(T) = \mathbb{Z}^n.$$

If we write e_i for the standard basis of $\mathbb{Z}^n$, then

$$R(G,T) = \{e_i - e_j \mid 1 \le i \ne j \le n\} \simeq R^\vee(G,T).$$

Same calculation for $Sp(n)$, $SO(n)$.

Friday April 17: Weyl group $\mathbf{W(G,T)}$ For each root α,

$$\sigma_\alpha =_{\text{def}} \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N_G(T)$$

defines an automorphism s_α of the root datum. Find formulas

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \qquad (\lambda \in X^*(T)$$

$$s_\alpha(\ell) = \ell - \langle \alpha, \ell \rangle \alpha^\vee \qquad (\ell \in X_*(T).$$

Compute this explicitly for G classical.

Monday April 20 Patriots' Day Holiday: no class

Wednesday April 22: axioms for root data. State correspondence compact connected Lie groups $\leftrightarrow$ root data

Friday April 24: exceptional root data

Monday April 27: simple roots and Dynkin diagram

Wednesday April 29: Coxeter graphs. Classification of root systems

Friday May 1: From root systems to root data

Monday May 4: affine Weyl group and conjugacy classes in G

Wednesday May 6: automorphisms: new root data from old

Friday May 8: Langlands dual group

Monday May 11 LAST CLASS: is there really any chance I can cover all the stuff above without an overflow day?