18.755 ninth problems, due on Gradescope Wednesday, April 15, 2020, 16:00 Eastern time

I described in class how to find all connected groups with a given Lie algebra from a simply connected group and knowledge of its center. So this problem set is about finding centers. *Calculate the center* means identify the group elements in the center, and say what the group law is. In some cases you may not be able to give a complete answer; say as much as you can.

Recall that $SO(n)$ is the group of $n \times n$ real orthogonal matrices of determinant 1. We proved in class that $SO(n)$ is connected for all $n \geq 1$, and that

$$\pi_1(SO(n)) = \begin{cases} 0 & (n = 1) \\ \mathbb{Z} & (n = 2) \\ \mathbb{Z}/2\mathbb{Z} & (n \geq 3). \end{cases}$$

This fundamental group has a unique quotient of order 2 for $n \geq 2$, and therefore there is a unique connected double cover

$$\mathrm{Spin}(n) \quad (n \geq 2)$$

with a short exact sequence of Lie groups

$$1 \to \{1, \epsilon\} \to \mathrm{Spin}(n) \to SO(n) \to 1.$$

1. Calculate the center of $SO(n)$ for all $n \geq 1$.

2. Calculate the center of $\mathrm{Spin}(n)$ for all $n \geq 2$.

3. Calculate the center of the real Lie group $SU(n)$ (consisting of $n \times n$ complex unitary matrices of determinant 1) for all $n \geq 1$.

4. Calculate the center of $Sp(n)$ (consisting of $n \times n$ quaternionic matrices preserving the standard Hermitian form on $\mathbb{H}^n$) for all $n \geq 1$.

5. You now have four infinite families of compact connected Lie groups $\mathrm{Spin}(n_1)$, $SU(n_2)$, and $Sp(n_3)$. Find some (or preferably all!) examples of pairs (G_1, G_2) of groups on these lists satisfying

$$\dim G_1 = \dim G_2, \qquad Z(G_1) \simeq Z(G_2).$$

You can use the formulas from class

$$\dim SO(n_1) = n_1(n_1 - 1)/2, \qquad \dim SU(n_2) = n_2^2 - 1, \qquad \dim Sp(n_3) = 2n_3^2 + n_3.$$