18.755 8th problems; due on Gradescope **Wednesday April 8, 2020 by 16:00 EST**

These problems concern the Lie group

$$G = SL(2, \mathbb{R}) = 2 \times 2 \text{ real matrices of determinant } 1.$$

This group has a subgroup

$$K = SO(2) = \{r(\theta) =_{\text{def}} \begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix} \mid \theta \in \mathbb{R}/\mathbb{Z}\}.$$

The Lie algebra of G is

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = 2 \times 2 \text{ real matrices of trace } 0.$$

A basis is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying the relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

The three complex matrices

$$Z = iH, \qquad X = E - F, \qquad Y = iE + iF$$

satisfy the bracket relations

$$[Z, X] = 2Y, \qquad [Y, Z] = 2X, \qquad [X, Y] = 2Z.$$

They span the Lie algebra

$$\mathfrak{su}(2) = 2 \times 2 \text{ complex skew-Hermitian matrices of trace } 0,$$

which is the Lie algebra of $SU(2)$.

An n-dimensional **real** representation π of $\mathfrak{sl}(2, \mathbb{R})$ consists of three $n \times n$ real matrices H_π, E_π, F_π subject to the relations

$$[H_\pi, E_\pi] = 2E_\pi, \qquad [H_\pi, F_\pi] = -2F_\pi, \qquad [E_\pi, F_\pi] = H_\pi.$$

Any n-diml real representation π of $\mathfrak{sl}(2, \mathbb{R})$ gives an n-diml **complex** representation π_c of $\mathfrak{su}(2)$:

$$Z_{\pi_c} = iH_\pi, \qquad X_{\pi_c} = E_\pi - F_\pi, \qquad Y_{\pi_c} = iE_\pi + iF_\pi,$$

automatically satisfying the bracket relations

$$[Z_\pi, X_\pi] = 2Y_\pi, \qquad [Y_\pi, Z_\pi] = -2X_\pi, \qquad [X_\pi, Y_\pi] = 2Z_\pi.$$

The subalgebra/subgroup correspondence we just proved attaches to π_c a group homomorphism

$$\Pi_c\colon SU(2) \to GL(n,\mathbb{C}), \qquad \Pi_c(\exp(aX + bY + cZ)) = \exp(aX_{\pi_c} + bY_{\pi_c} + cZ_{\pi_c}).$$

(Do you see why this is true?)

1. Suppose that π is an n-dimensional real representation of $\mathfrak{sl}(2,\mathbb{R})$. **Prove that the matrix H_π is diagonalizable with integer eigenvalues.**

2. Easy linear algebra says that every element $g \in SL(2,\mathbb{R})$ has an expression

$$g = r(\theta_1)\exp(tH)r(\theta_2) \qquad (\theta_i \in \mathbb{R}, t \geq 0).$$

The constant t is uniquely determined by g, and depends continuously on g.

(1) **Prove that any Lie algebra homomorphism $\pi\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ is the differential of a Lie group homomorphism $\Pi\colon SL(2,\mathbb{R}) \to GL(n,\mathbb{R})$.**

(2) **Prove that if $\Pi\colon SL(2,\mathbb{R}) \to GL(n,\mathbb{R})$ is any Lie group homomorphism, then Π has closed image.**

3. Of course you know that $\pi_1(SO(2)) = \mathbb{Z}$. You may assume (essentially we did this in class) that the inclusion of K defines an isomorphism

$$\pi_1(K) \simeq \pi_1(G).$$

This means that the universal cover $\widetilde{G}$ of G contains the universal cover $\widetilde{K} = \mathbb{R}$; it's natural to write

$$\widetilde{K} = \{\widetilde{r}(\theta) \mid \theta \in \mathbb{R}\}.$$

Find an example of a Lie group M with Lie algebra $\mathfrak{m}$, and an inclusion

$$\pi\colon \mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{m}$$

so that the corresponding Lie subgroup of M is *not* closed.

Hint: this is hard. The simplest example I know has

$$\mathfrak{m} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}.$$