18.755 fifth problems, due in class Wednesday, March 11, 2020

1. Let V be the vector space $C_c^\infty(\mathbb{R})$ of compactly supported smooth functions on the real line. Calculus has a lot to say about two families of linear transformations on V: *translation by t*

$$(T_t f)(x) = f(x-t) \qquad (t \in \mathbb{R})$$

and *multiplication by exponentials*

$$(M_\xi f)(x) = e^{ix\xi} f(x).$$

It's very easy to check that each of these families is a group under composition of linear operators:

$$T_t T_{t'} = T_{t+t'}, \qquad T_0 = \text{identity},$$

and similarly for M. In this way it's natural to regard each of $\{T_t \mid t \in \mathbb{R}\}$ and $\{M_\xi \mid \xi \in \mathbb{R}\}$ as a one-dimensional Lie group, isomorphic to $\mathbb{R}$. You may assume all that.

Now let G be the group of linear transformations of V generated by all the T_t and M_ξ.

(1) Prove that G is in a natural way a Lie group.
(2) Calculate $\pi_0(G)$ and $\pi_1(G)$.
(3) Is G diffeomorphic to a group of matrices?

Suppose (M, m_0) is a connected manifold with a base point m_0, and (G, e) is a connected Lie group with (natural) base point the identity. I hope by Monday 3/9 to have defined fundamental groups and universal covering spaces; for this problem set you can take the definitions to be

$$\widetilde{M} =_{\text{def}} \{\text{homotopy classes of paths in } M \text{ starting at } m_0\},$$

and in particular

$$\widetilde{G} =_{\text{def}} \{\text{homotopy classes of paths in } G \text{ starting at } e\}.$$

A convenient notation for paths is

$$\mu\colon [0,1] \to M, \quad \mu(0) = m_0, \qquad \gamma\colon [0,1] \to G, \quad \gamma(0) = e.$$

The covering maps are

$$\pi_M\colon \widetilde{M} \to M, \quad \pi_M(\mu) = \mu(1)$$

and similarly for $\widetilde{G}$. The group structure on $\widetilde{G}$ is defined by the group multiplication in G, applied pointwise to two paths.

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2. With notation as above, suppose that G acts (smoothly) on M. Explain how to define a natural action of $\widetilde{G}$ on $\widetilde{M}$. Explain exactly what you need to check to see that your definition makes sense. Write carefully some details of this checking (enough to show convincingly that you understand it).

3. Suppose that G is the circle group

$$G = \{\exp(2\pi i\theta) \mid \theta \in \mathbb{R}\} \simeq \mathbb{R}/\mathbb{Z}.$$

You may assume that every path starting at the origin in G is homotopic to a (unique) path

$$\gamma_\theta(t) = \exp(2\pi i t\theta) \qquad (0 \leq t \leq 1),$$

so that the universal covering group is

$$\widetilde{G} = \{\gamma_\theta \mid \theta \in \mathbb{R}\} \simeq \mathbb{R}.$$

(1) Find an action of G on a manifold M so that the action of $\widetilde{G}$ on $\widetilde{M}$ is *faithful*: that no nontrivial element of $\widetilde{G}$ acts trivially on $\widetilde{M}$.
(2) Find a *faithful* action of G on a manifold N so that the action of $\widetilde{G}$ on $\widetilde{N}$ descends to G: that is, that every element of $\mathbb{Z} \subset \widetilde{G}$ acts trivially on $\widetilde{N}$.
(3) Is it possible for the action of G on M in the first part *not* to be faithful?