18.755 fourth problems, due in class Wednesday, March 4, 2020

It turns out that most of the interesting Lie groups can be described as groups of linear transformations preserving various bilinear forms. This problem set is therefore about some of those groups. We can say a lot of things over any field k, and that's worth doing. Recall that a *bilinear form* on a k vector space V is a map

$$\beta: V \times V \to k$$

satisfying

$$\beta(au + bv, w) = a\beta(u, w) + b\beta(v, w), \qquad \beta(x, cy + dz) = c\beta(x, y) + d\beta(x, z);$$

here $u, v, w, x, y, z \in V$ are vectors and $a, b, c, d \in k$ are scalars.

Bilinear forms may be identified with linear maps

$$B_\beta \in \mathrm{Hom}(V, V^*), \qquad \beta(v, w) = [B_\beta(v)](w).$$

The form is called *symmetric* if

$$\beta(v, w) = \beta(w, v), \quad \text{equivalently} \quad {}^tB_\beta = B_\beta,$$

and *skew-symmetric* if

$$\beta(v, v) = 0.$$

(Applying the definition of skew-symmetric to $v + w$ and using bilinearity shows that a skew-symmetric form must also have the perhaps more familiar property

$$\beta(v, w) = -\beta(w, v), \quad \text{equivalently} \quad {}^tB_\beta = -B_\beta.$$

If the characteristic of k is not 2, this familiar property is easily seen to be *equivalent* to the first. But in characteristic 2 it is the condition $\beta(v, v) = 0$ that is more interesting. We will mostly not worry about characteristic 2.) The form is *nondegenerate* if for every nonzero $v \in V$ there is a $w \in V$ so that $\beta(v, w) \neq 0$, and for every nonzero w' there is a v' so that $\beta(v', w') \neq 0$. If V is finite-dimensional, this is equivalent to B_β being invertible.

If $V = k^n$, then $\mathrm{Hom}(V, V^*)$ can be identified with $n \times n$ matrices. The identification of forms is

$$\beta(v, w) = {}^twB_\beta v, \qquad (B_\beta)_{ij} = \beta(e_j, e_i).$$

Clearly symmetric (respectively skew-symmetric, except in characteristic 2) forms correspond to symmetric (respectively skew-symmetric) matrices. Nondegenerate forms correspond to invertible matrices.

The group $GL(V)$ of invertible linear transformations acts on bilinear forms by change of variable

$$(g \cdot \beta)(v, w) = \beta(g^{-1} \cdot v, g^{-1} \cdot w).$$

The corresponding action of $GL(V)$ on linear maps is

$$g \cdot B = {}^tg^{-1}Bg^{-1}.$$

This action preserves the property of being symmetric or skew symmetric.

1. (This problem is over any field k.) If β is a symmetric bilinear form on V, the *orthogonal group of* β is by definition

$$O_\beta = \{g \in GL(V) \mid g \cdot \beta = \beta\}.$$

(If the characteristic of k is two, this definition still makes sense but it is not correct—that is, it is not what is called an orthogonal group in characteristic two. Just ignore that for this problem.) Prove that if $x \in GL(V)$, then

$$O_{x\cdot\beta} = xO_\beta x^{-1}.$$

That is, *equivalence classes of symmetric forms give rise to conjugacy classes of subgroups of* $GL(V)$.

2. Describe all possible symmetric bilinear forms on $\mathbb{R}^n$ up to the action of $GL(n, \mathbb{R})$. That is, describe the orbits of $GL(n, \mathbb{R})$ on symmetric $n \times n$ matrices. (Hint: there are $(n+1)(n+2)/2$ orbits. You can if you like say "by XXX's theorem, the answer is ... " but in this case you should cite some place where the theorem you use is stated.)

3. If $V = \mathbb{R}^n$, and β corresponds to a symmetric matrix B, then the definition says that

$$O_\beta = \{g \in GL(n, \mathbb{R}) \mid {}^t g^{-1} B g^{-1} = B\}.$$

Find a simple description for the set of matrices X in the Lie algebra $\mathfrak{o}_\beta$ of O_β. (Hint: you can use the fact that the Lie algebra of a Lie subgroup H of G consists of all $X \in \mathfrak{g}$ such that $\exp(tX) \in H$ for every real t.)

4. Describe the Lie algebra $\mathfrak{o}(p, q)$ by saying exactly which matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (A\ p \times p, \quad B\ p \times q, \quad C\ q \times p, \quad D\ q \times q)$$

belong to the Lie algebra. (This means writing some conditions on and relations among the four matrices A, B, C, and D. A good answer is one that can be understood just by knowing about matrices, without knowing about bilinear forms.) Use your description to calculate the dimension of $O(p, q)$.

5. Describe all possible skew-symmetric bilinear forms on $\mathbb{R}^{2n}$ up to the action of $GL(2n, \mathbb{R})$. That is, describe the orbits of $GL(2n, \mathbb{R})$ on skew-symmetric $2n \times 2n$ matrices. (Hint: there are $n+1$ orbits. You can if you like say "by XXX's theorem, the answer is ... " but in this case you should cite some place where the theorem you use is stated.)