**18.755 Problem Set 3 due 2/26/20 in class**

1. Suppose that $V$ is a finite-dimensional real vector space, and that

$$\alpha\colon \mathbb{R} \times V \to V$$

is a *continuous* (not necessarily smooth) action of $\mathbb{R}$ on $V$ by *linear* transformations. It is equivalent to assume that

$$A\colon \mathbb{R} \to GL(V), \qquad A(t)v = \alpha(t, v)$$

is a *continuous* group homomorphism. **Prove that there is a linear map $T \in \mathrm{Hom}(V, V)$ with the property that**

$$A(t) = \exp(tT).$$

**Hint.** The main point here is to show that $A$ is a smooth map; once you know that, you can find a differential equation that it satisfies and solve the problem. To prove that the (linear transformation-valued) function $s \mapsto A(s)$ is smooth is the same as proving that for every vector $w \in V$, the (vector-valued) function $s \mapsto A(s)w$ is smooth. (If you use that fact, you should explain why it's true.) The main hint is to look for a proof of

**Lemma.** *Suppose $\phi \in C_c^\infty(\mathbb{R})$ is a compactly supported smooth function, and $v \in V$. Define*

$$w = \int_{-\infty}^{\infty} \phi(t)A(t)v\, dt.$$

*Then $s \mapsto A(s)w$ is smooth.*

(If you don't manage to prove the lemma, you can get some credit just for using it to solve the problem.)

2. Suppose $M$ is a manifold, $X$ is a smooth vector field on $M$, and

$$\phi\colon (a, b) \to M$$

is an integral curve for $X$. The definition of integral curve is that $\phi$ is smooth, and its differential satisfies

$$d\phi(t) = X_{\phi(t)} \qquad (t \in (a, b)) :$$

the right side of the equation is the value of the vector field $X$ at the point $\phi(t) \in M$.

Suppose now that $F \in C^\infty(M)$ is a smooth function, and assume that $X \cdot F = 0$. **Prove that $F$ is constant on $\phi$**; that is, that the function $F \circ \phi\colon (a, b) \to \mathbb{R}$ is constant.

3. We have seen that if $V$ is a finite-dimensional vector space and $U$ is an open set in $V$, then there is a natural isomorphism $T_m(U) \simeq V$ for every $m \in U$. In particular, a vector field on $U$ is exactly the same thing as a smooth function

$$X\colon U \to V.$$

Look at the case

$$V = n \times n \text{ matrices } = \mathfrak{gl}(n, \mathbb{R}), \qquad U = GL(n, \mathbb{R}).$$

The conclusion is that a vector field on $GL(n, R)$ is exactly the same thing as a smooth map

$$X\colon GL(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}).$$

**Which of these smooth maps are *left-invariant* vector fields?**

**Hint.** Left-invariant vector fields $X$ on a Lie group $G$ were defined in class on February 21. The definition was this: for every $g$ in $G$, we defined *left translation by* $g$

$$\lambda(g)\colon G \to G, \qquad \lambda(g)(x) = gx.$$

Then $X$ is left-invariant if for every $g$ and $x$ in $G$,

$$X_{\lambda(g)(x)} = d\lambda(g)(X_x).$$