18.755 second problems, due in class 2/19/20

1. Suppose A is an $n \times n$ real matrix. Prove that

$$\exp(A) = \lim_{N\to\infty} (I + \frac{1}{N}A)^N.$$

Prove also that

$$\det(\exp(A)) = \exp(\operatorname{tr} A)$$

(with $\operatorname{tr} A$ the sum of the diagonal entries of A).

2. Define

$$GL^+(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det(A) > 0\}.$$

It follows from Problem 1 that

$$\exp\colon \mathfrak{gl}(n,\mathbb{R}) \to GL^+(n,\mathbb{R}).$$

Is this map surjective? (Here $\mathfrak{gl}(n,\mathbb{R})$ means all $n \times n$ real matrices.)

3. The "standard symplectic form" on

$$\mathbb{R}^{2n} = \{(x,y) \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}^n\}$$

is

$$\omega((x,y),(x',y')) = x \cdot y' - y \cdot x'.$$

The *real symplectic group* is

$$Sp(2n,\mathbb{R}) = \{g \in GL(2n,\mathbb{R}) \mid \omega(g\cdot(x,y), g\cdot(x',y')) = \omega((x,y),(x',y'))\}.$$

Prove that $Sp(2n,\mathbb{R})$ is a Lie group, and calculate its dimension.

4. Find a natural inclusion $\phi\colon GL(n,\mathbb{R}) \hookrightarrow Sp(2n,\mathbb{R})$.