18.755 twelfth and last problems, due on Gradescope Wednesday, May 6, 4:00

This problem set is about the classification of root data. In order to do it, you'll need to read some parts of Sections 6 and 7 of the notes `roots.pdf` on the class web site.

The general setting for the problem is a root datum

$$\mathcal{R} = (X^*, R, X_*, R^\vee).$$

As usual this means that X^* and X_* are dual lattices (for instance the character lattice and cocharacter lattice of maximal torus T in a compact Lie group K), the roots are a finite subset $R \subset X^*$, and the coroots $R^\vee \subset X_*$ are in bijection with the roots.

In this problem $\mathcal{R}$ will be assumed to be **simply laced**, meaning that if α and β are any two roots, then

$$\langle \alpha, \beta^\vee \rangle = \text{ 0 or } \pm 1.$$

We will also assume that $\mathcal{R}$ is **simple**, meaning that for any roots α and β in R, we can find a finite sequence of roots

$$\alpha = \gamma_0,\ \gamma_1,\ \dots,\ \gamma_n = \beta$$

with the property that

$$\langle \gamma_{p-1}, \gamma_p^\vee \rangle \neq 0 \qquad (1 \le p \le n).$$

Before continuing with the assumptions, we pause for the first problem.

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1. Assume that the root datum $\mathcal{R}$ is simply laced and simple. Prove that the Weyl group acts transitively on R.

Back to the assumptions. We assume that $\mathcal{R}$ is **adjoint**, meaning that the lattice X^* is generated by R (every element of X^* is an integer combination of roots). Finally we fix a choice of **positive roots** R^+ as in Definition 6.2 of the notes, and write

$$\Pi = \Pi(R^+)$$

for the corresponding set of **simple roots**. You may assume (as is proven in Corollary 6.7 of the notes) that Π **is a basis of the lattice** X^*.

Recall from Definition 6.4 of the notes the definition of the **Dynkin diagram** of $\mathcal{R}$. This is a connected graph $\Gamma = \Gamma(\mathcal{R})$ with vertex set Π.

Finally, assume that **the Dynkin diagram** Γ **has a nontrivial automorphism** τ, having the additional property that **the vertex** α **is never adjacent to the vertex** $\tau(\alpha)$. (We allow $\tau(\alpha) = \alpha$ *sometimes*, just not *all the time.*)

In the table in Section 3 of the notes, the diagrams A, D, and E are simply laced. The ones admitting nontrivial automorphisms τ as above are A_{2m-1} ($m \ge 2$), D_n ($n \ge 4$), and E_6. (You should figure out what these graph automorphisms are in each case.)

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2. Prove that the permutation τ of Π extends to an automorphism (still called τ) of the root datum $\mathcal{R}$.

3. Prove that α is never adjacent to $\tau(\alpha)$ for any root $\alpha \in R$. More precisely, prove that

$$\langle \alpha, \tau(\alpha)^\vee \rangle \neq 0 \implies \alpha = \tau(\alpha).$$

4. Suppose τ has order m. Define $X(\tau)^* = (X^*)^\tau$, the fixed points of τ on X^*, and

$$R(\tau) = \{\beta \in R \mid \tau(\beta) = \beta\} \cup \{\gamma + \tau(\gamma) + \ldots + \tau^{m-1}(\gamma) \mid \gamma \in R, \tau(\gamma) \neq \gamma\},$$

a finite subset of $X(\tau)^*$. Show how to complete this definition to a root datum

$$\mathcal{R}(\tau) = (X(\tau)^*, R(\tau), X(\tau)_*, R(\tau)^\vee)$$

having simple roots

$$\Pi(\tau) = \{\beta \in \Pi \mid \tau(\beta) = \beta\} \cup \{\gamma + \tau(\gamma) + \ldots + \tau^{m-1}(\gamma) \mid \gamma \in \Pi, \tau(\gamma) \neq \gamma\}.$$

5. Show how to construct the Dynkin diagram $\Gamma(\tau)$ of $\mathcal{R}(\tau)$ from the diagram Γ of $\mathcal{R}$.

This problem set explains how to construct a root datum that is **not** simply laced from one that is. One can show that the method actually constructs **all** root data that are not simply laced. For that reason, the problem of classifying root data is more or less reduced to the simply laced case. That case I hope to discuss in class.

Very similar ideas show how to construct a compact Lie group corresponding to a non-simply laced root datum as a subgroup of a simply laced one.