18.755 problems due 2/12/20 in class (or electronically earlier)

1. Suppose V is a vector space over a field F, and $U \subset V$ is a subspace. Define

$$GL(U,V) = \{g \in GL(V) \mid gU = U\}.$$

Prove that there is a short exact sequence

$$1 \to N \to GL(U,V) \to GL(U) \times GL(V/U) \to 1,$$

and that the normal subgroup N satisfies $N \simeq \mathrm{Hom}_F(V/U, U)$ (where the group operation on the right is addition of linear maps).

A group S is called *solvable* if there is a collection of subgroups

$$\{e\} = N_0 \subset N_1 \subset \cdots \subset N_r = S$$

so that N_{i-1} **normal** in N_i (written $N_{i-1} \triangleleft N_i$) and N_i/N_{i-1} is **abelian** ($1 \le i \le r$).

Subgroups H_1 and H_2 are *conjugate* if there is $g \in G$ such that $gH_1g^{-1} = H_2$.

2. Suppose $n \geq 1$ is an integer. Define $G = GL(n, \mathbb{C})$ to be the group of all $n \times n$ invertible complex matrices, and

$$B = \{g = (g_{ij}) \in G \mid i > j \implies g_{ij} = 0\}$$

the subgroup of upper triangular matrices.

(1) **Prove that B is solvable.**
(2) **Prove or give a counterexample: every element $g \in G$ is conjugate to an element of B.**
(3) **Prove or give a counterexample: if S is a solvable subgroup of G, then S is conjugate to a subgroup of B.**

3. Let $G = SL(2, \mathbb{R})$, the group of 2×2 real matrices of determinant 1.

(1) **Prove that the subgroups**

$$H_1 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}, \qquad H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

are conjugate.

(2) **Find as many *non-conjugate connected* subgroups $H \subset G$ as you can. You should *prove* that your subgroups are not conjugate.**